What is a Coxeter group?

Jad Abou Yassin

Institut Denis Poisson, Université de Tours

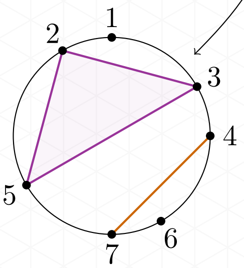
Journées des Jeunes de l'IDP, Orléans – 26 au 28 mai 2026

My thesis' subject

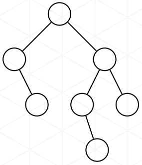
« Noncrossing partitions and generalization of c-sortable elements in affine Coxeter groups »

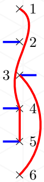
My thesis' subject

« Noncrossing partitions and generalization of c-sortable elements in affine COXETER groups »

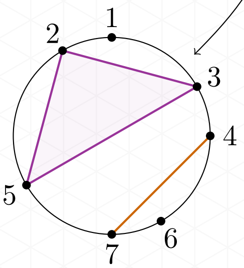


My thesis' subject

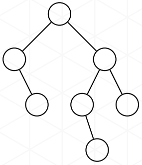


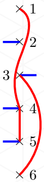


« Noncrossing partitions and generalization of c-sortable elements in affine Coxeter groups »

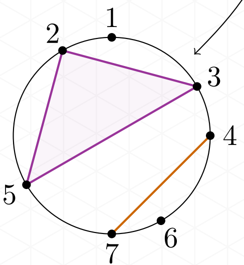


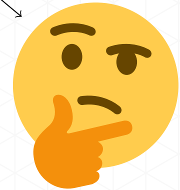
My thesis' subject





« Noncrossing partitions and generalization of c-sortable elements in affine COXETER groups »



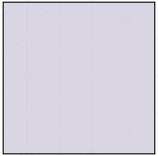


Part II

Symmetries of polytopes

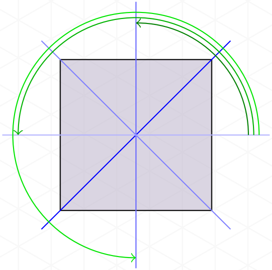
Symmetries of the n-cube

- A symmetry of a figure in $\mathbb{R}^d$ is an isometric linear transformation of $\mathbb{R}^d$.



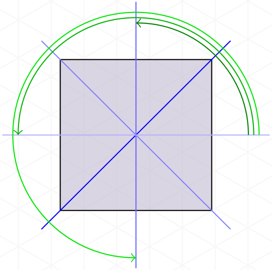
Symmetries of the n-cube

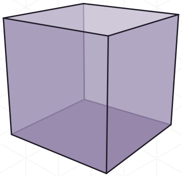
- A symmetry of a figure in $\mathbb{R}^d$ is an isometric linear transformation of $\mathbb{R}^d$.
- There are 8 symmetries of a square (2-cube): 4 reflections and 4 rotations (including the rotation of angle 0).



Symmetries of the n-cube

- A symmetry of a figure in $\mathbb{R}^d$ is an isometric linear transformation of $\mathbb{R}^d$.
- There are 8 symmetries of a square (2-cube): 4 reflections and 4 rotations (including the rotation of angle 0).

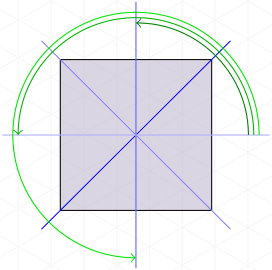


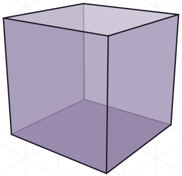


- For a (3-)cube, it already is more complicated to visualize and count them: there are 48 symmetries.

Symmetries of the n-cube

- A symmetry of a figure in $\mathbb{R}^d$ is an isometric linear transformation of $\mathbb{R}^d$.
- There are 8 symmetries of a square (2-cube): 4 reflections and 4 rotations (including the rotation of angle 0).

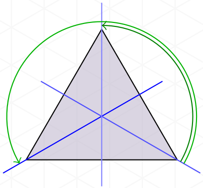




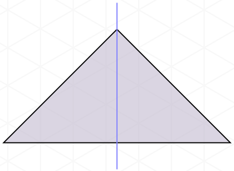
- For a (3-)cube, it already is more complicated to visualize and count them: there are 48 symmetries.
- In higher dimension, it seems very complicated (answer: $2^n n!$ symmetries of a hypercube of dimension n).

Regular polygons

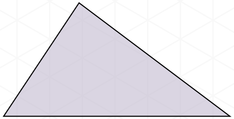
▷ When an polygon is regular, it has the highest number of symmetries.



Equilateral : 6 symmetries



Isosceles : 2 symmetries



Generic : 1 symmetry

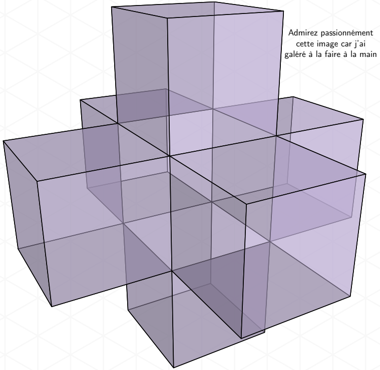
▷ We will only consider symmetries of regular figures.

Regular polytopes

- What about regular polytopes in higher dimension?

Regular polytopes

▷ What about regular polytopes in higher dimension?

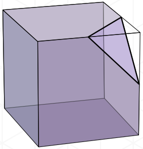


Regular polytopes

▷ What about regular polytopes in higher dimension?

▷ In general, a regular polytope of dimension n is defined recursively:

- its faces are identical and regular of dimension $n-1$
- its vertex figures are identical and regular of dimension $n-1$

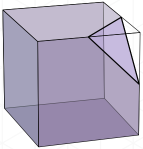


Regular polytopes

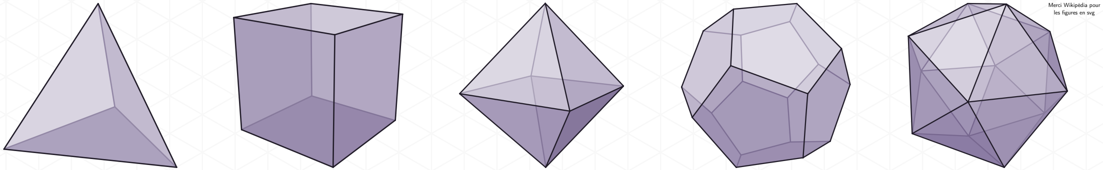
▷ What about regular polytopes in higher dimension?

▷ In general, a regular polytope of dimension n is defined recursively:

- its faces are identical and regular of dimension $n-1$
- its vertex figures are identical and regular of dimension $n-1$



▷ In 3 dimensions, there are 5 regular polyhedra, famously known as the Platonic solids.



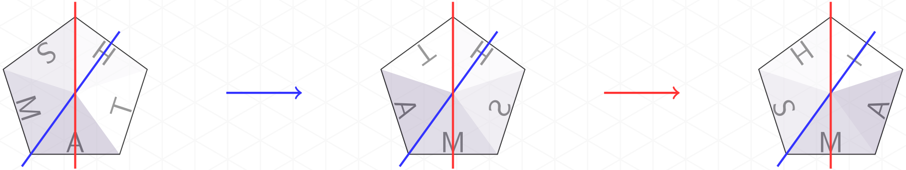
Tetrahedron Cube Octahedron Dodecahedron Icosahedron

Symmetries from reflections

- It seems difficult to understand the symmetries of a regular polytope

Symmetries from reflections

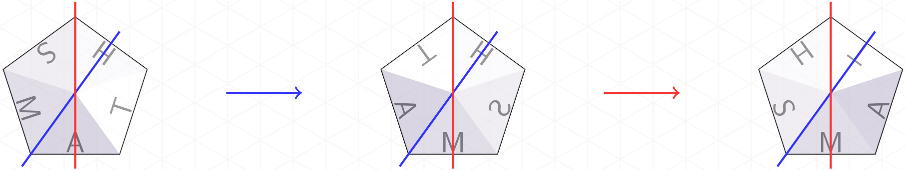
- ▷ It seems difficult to understand the symmetries of a regular polytope
- ▷ In fact, every symmetry of a regular polytope can be expressed as a succession of reflections.



Symmetries from reflections

- It seems difficult to understand the symmetries of a regular polytope

- In fact, every symmetry of a regular polytope can be expressed as a succession of reflections.



- We only need to study the reflections.

Part III

Reflection groups

Group structure

▷ Among the symmetries:

- There is always the identity,
- Applying two symmetries consecutively has the same effect as applying another symmetry once,
- A symmetry can always be undone by applying another symmetry.

Group structure

▷ Among the symmetries:

- There is always the identity,
- Applying two symmetries consecutively has the same effect as applying another symmetry once,
- A symmetry can always be undone by applying another symmetry.

▷ These properties give the set of symmetries of a regular polytope a group structure.

Group structure

▷ Among the symmetries:

- There is always the identity,
- Applying two symmetries consecutively has the same effect as applying another symmetry once,
- A symmetry can always be undone by applying another symmetry.

▷ These properties give the set of symmetries of a regular polytope a group structure.

▷ What we saw before shows that these groups are generated by the reflections.

Reflection groups

- A reflection group is a subgroup of $\mathrm{Isom}(\mathbb{R}^d)$ generated by reflections.

- Not every element is a reflection, but all can be obtained as a product of some reflections.

Reflection groups

- A reflection group is a subgroup of $\mathrm{Isom}(\mathbb{R}^d)$ generated by reflections.

- Not every element is a reflection, but all can be obtained as a product of some reflections.

- Since symmetry groups of regular polytopes are also generated by reflections, they are reflection groups
  $\Rightarrow$ the study of reflection groups is a more general context in which we can (hopefully) find results useful to the study of the symmetries of polytopes.

Reflection groups

- A reflection group is a subgroup of $\mathrm{Isom}(\mathbb{R}^d)$ generated by reflections.

- Not every element is a reflection, but all can be obtained as a product of some reflections.

- Since symmetry groups of regular polytopes are also generated by reflections, they are reflection groups
  $\Rightarrow$ the study of reflection groups is a more general context in which we can (hopefully) find results useful to the study of the symmetries of polytopes.

- We could study the general case, but we limit ourselves to the cases of finite reflection groups.

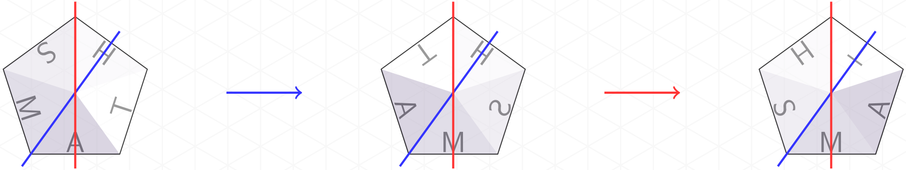
Some properties

- If G is a reflection group, generated by the reflections $r_1, \ldots, r_n$, then for all i, $r_i^2 = 1$.

Some properties

▷ If G is a reflection group, generated by the reflections $r_1, \dots, r_n$, then for all i, $r_i^2 = 1$.

▷ If $r_i \neq r_j$, then $r_i r_j$ is a rotation of some angle α in the orthogonal plane of the intersection of their reflection hyperplanes. Thus, $(r_i r_j)^2$ is a rotation in the same plane of angle 2α, and so on. Since the reflection group is finite, α is a rational multiple of π, and thus there exists an integer $m_{i,j} \in \mathbb{N}^*$ such that $(r_i r_j)^{m_{i,j}} = 1$.



Part IV

Coxeter groups

An abstraction

- We define an abstract group that satisfies the two properties of reflection groups we saw before.

An abstraction

- We define an abstract group that satisfies the two properties of reflection groups we saw before.

- Consider $S = \{s_1, \ldots, s_n\}$ a set of letters, and for each pair (s_i, s_j), define an integer $m_{i,j} = m_{j,i} \in \mathbb{N}^*$ such that $m_{i,i} = 1$.

An abstraction

- We define an abstract group that satisfies the two properties of reflection groups we saw before.
- Consider $S = \{s_1, \dots, s_n\}$ a set of letters, and for each pair (s_i, s_j), define an integer $m_{i,j} = m_{j,i} \in \mathbb{N}^*$ such that $m_{i,i} = 1$.
- If w is a word on the alphabet S, we can transform it using the relations $(s_i s_j)^{m_{i,j}} = 1$ (the empty word).

An abstraction

- We define an abstract group that satisfies the two properties of reflection groups we saw before.
- Consider $S = \{s_1, \ldots, s_n\}$ a set of letters, and for each pair (s_i, s_j), define an integer $m_{i,j} = m_{j,i} \in \mathbb{N}^*$ such that $m_{i,i} = 1$.
- If w is a word on the alphabet S, we can transform it using the relations $(s_i s_j)^{m_{i,j}} = 1$ (the empty word).
- Example: $S = \{s_1, s_2\}$, $(s_1 s_1)^1 = (s_2 s_2)^1 = 1$ and $(s_1 s_2)^3 = 1$. Then we can transform $s_1 s_2 s_1$ into $s_2 s_1 s_2$:

$$s_1 s_2 s_1 \rightarrow s_1 s_2 \underbrace{s_1 s_1}_{1} s_2 s_1 s_2 s_1 s_2 \rightarrow s_1 \underbrace{s_2 s_2}_{1} s_1 s_2 s_1 s_2 \rightarrow \underbrace{s_1 s_1}_{1} s_2 s_1 s_2 \rightarrow s_2 s_1 s_2$$

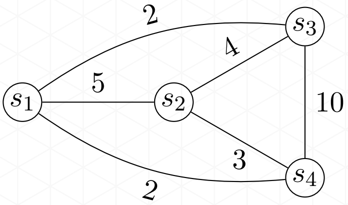
Coxeter graphs

- We encode the data of the alphabet S and the integers $m_{i,j}$ using a weighted graph.
- The set of vertices is S and two vertices s_i, s_j are linked by an edge of weight $m_{i,j}$.

Coxeter graphs

- We encode the data of the alphabet S and the integers $m_{i,j}$ using a weighted graph.
- The set of vertices is S and two vertices s_i, s_j are linked by an edge of weight $m_{i,j}$.
- Example : the following system is represented by this graph.

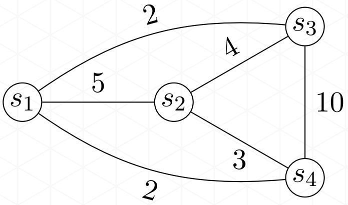
$$\begin{cases} S = \{s_1, s_2, s_3, s_4\} \\ m_{1,1} = m_{2,2} = m_{3,3} = m_{4,4} = 1 \\ m_{1,2} = 5 \\ m_{1,3} = m_{1,4} = 2 \\ m_{2,3} = 4 \\ m_{2,4} = 3 \\ m_{3,4} = 10 \end{cases} \longleftrightarrow$$



Coxeter graphs

- We encode the data of the alphabet S and the integers $m_{i,j}$ using a weighted graph.
- The set of vertices is S and two vertices s_i, s_j are linked by an edge of weight $m_{i,j}$.
- For simplicity, we do not represent the edges of weight 2
- Example : the following system is represented by this graph.

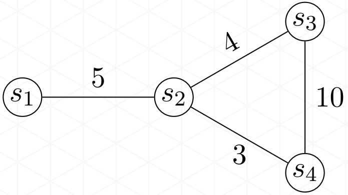
$$\begin{cases} S = \{s_1, s_2, s_3, s_4\} \\ m_{1,1} = m_{2,2} = m_{3,3} = m_{4,4} = 1 \\ m_{1,2} = 5 \\ m_{1,3} = m_{1,4} = 2 \\ m_{2,3} = 4 \\ m_{2,4} = 3 \\ m_{3,4} = 10 \end{cases} \quad \longleftrightarrow$$



Coxeter graphs

- We encode the data of the alphabet S and the integers $m_{i,j}$ using a weighted graph.
- The set of vertices is S and two vertices s_i, s_j are linked by an edge of weight $m_{i,j}$.
- For simplicity, we do not represent the edges of weight 2
- Example : the following system is represented by this graph.

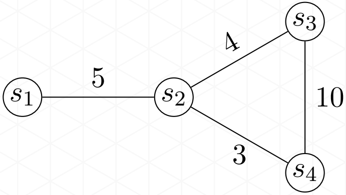
$$\begin{cases} S = \{s_1, s_2, s_3, s_4\} \\ m_{1,1} = m_{2,2} = m_{3,3} = m_{4,4} = 1 \\ m_{1,2} = 5 \\ m_{1,3} = m_{1,4} = 2 \\ m_{2,3} = 4 \\ m_{2,4} = 3 \\ m_{3,4} = 10 \end{cases} \longleftrightarrow$$



Coxeter graphs

- ▷ We encode the data of the alphabet S and the integers $m_{i,j}$ using a weighted graph.
- ▷ The set of vertices is S and two vertices s_i, s_j are linked by an edge of weight $m_{i,j}$.
- ▷ For simplicity, we do not represent the edges of weight 2, and we omit the weight of the edges of weight 3.
- ▷ Example : the following system is represented by this graph.

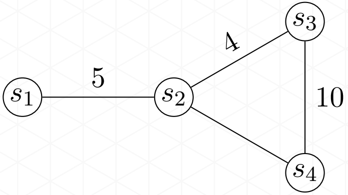
$$\begin{cases} S = \{s_1, s_2, s_3, s_4\} \\ m_{1,1} = m_{2,2} = m_{3,3} = m_{4,4} = 1 \\ m_{1,2} = 5 \\ m_{1,3} = m_{1,4} = 2 \\ m_{2,3} = 4 \\ m_{2,4} = 3 \\ m_{3,4} = 10 \end{cases} \longleftrightarrow$$



Coxeter graphs

- We encode the data of the alphabet S and the integers $m_{i,j}$ using a weighted graph.
- The set of vertices is S and two vertices s_i, s_j are linked by an edge of weight $m_{i,j}$.
- For simplicity, we do not represent the edges of weight 2, and we omit the weight of the edges of weight 3.
- Example : the following system is represented by this graph.

$$\begin{cases} S = \{s_1, s_2, s_3, s_4\} \\ m_{1,1} = m_{2,2} = m_{3,3} = m_{4,4} = 1 \\ m_{1,2} = 5 \\ m_{1,3} = m_{1,4} = 2 \\ m_{2,3} = 4 \\ m_{2,4} = 3 \\ m_{3,4} = 10 \end{cases} \longleftrightarrow$$



Coxeter group

- Given a Coxeter graph, a **Coxeter group** is the set of all words on the alphabet S, modulo the identification of two words that can be transformed into each other by the relations $(s_i s_j)^{m_{i,j}} = 1$.

Coxeter group

- Given a Coxeter graph, a Coxeter group is the set of all words on the alphabet S, modulo the identification of two words that can be transformed into each other by the relations $(s_i s_j)^{m_{i,j}} = 1$.

- For example, we have $s_1 s_2 s_1 = s_2 s_1 s_2$ in the Coxeter group of graph s_1 — s_2.

Coxeter group

▷ Given a Coxeter graph, a Coxeter group is the set of all words on the alphabet S, modulo the identification of two words that can be transformed into each other by the relations $(s_i s_j)^{m_{i,j}} = 1$.

▷ For example, we have $s_1 s_2 s_1 = s_2 s_1 s_2$ in the Coxeter group of graph s_1 — s_2.

▷ We write such a group using a group presentation:

$$\langle s_1, \dots, s_n \mid (s_i s_j)^{m_{i,j}} = 1 \; \forall i, j \rangle.$$

generators ← $s_1, \dots, s_n$

$(s_i s_j)^{m_{i,j}} = 1 \; \forall i, j$ → relations

Finite reflection groups and Coxeter groups

▷ Finite reflections groups are abstractly described by Coxeter groups.

▷ Question: were we too general?

Finite reflection groups and Coxeter groups

- Finite reflections groups are abstractly described by Coxeter groups.
- Question: were we too general?

Theorem 1 (Coxeter, 1935).

Every finite reflection group can be presented as a Coxeter group, and every finite Coxeter group has a faithful representation as a finite reflection group.

Finite reflection groups and Coxeter groups

▷ Finite reflections groups are abstractly described by Coxeter groups.

▷ Question: were we too general?

Theorem 1 (Coxeter, 1935).

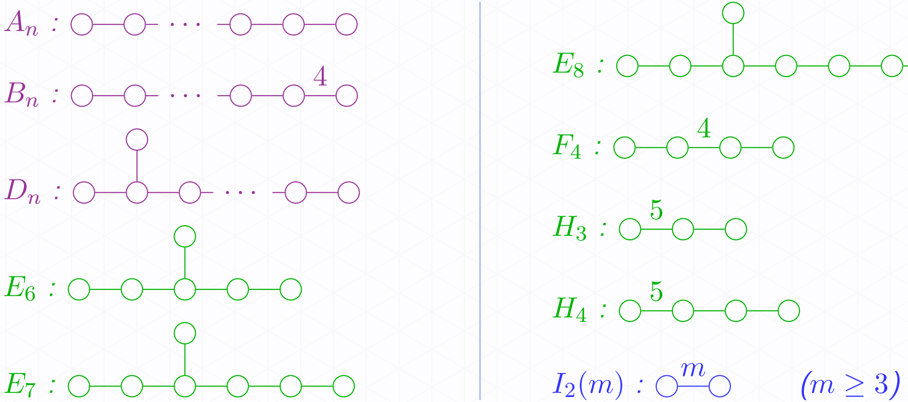
Every finite reflection group can be presented as a Coxeter group, and every finite Coxeter group has a faithful representation as a finite reflection group.

▷ Moreover, the finite Coxeter groups were classified using their diagrams.

Classification of finite irreducible COXETER groups

Theorem 2 (Classification of finite COXETER groups).

The following list represents all the finite irreducible COXETER groups. Any finite COXETER group can be obtained as a direct product of irreducible finite COXETER group.



Part V

Returning to regular polytopes

How does the theory of Coxeter groups help?

▷ Abstraction layers : regular polytopes $\to$ reflection groups $\to$ Coxeter groups

How does the theory of Coxeter groups help?

▷ Abstraction layers : regular polytopes → reflection groups → Coxeter groups

Theorem 3.

Regular polytopes are in bijection with Coxeter groups with a linear diagram.

How does the theory of Coxeter groups help?

▷ Abstraction layers : regular polytopes → reflection groups → Coxeter groups

Theorem 3.

Regular polytopes are in bijection with Coxeter groups with a linear diagram.

How does the theory of COXETER groups help?

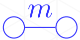
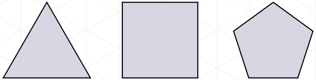
- ▷ Abstraction layers : regular polytopes → reflection groups → COXETER groups

Theorem 3.

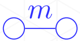
Regular polytopes are in bijection with COXETER groups with a linear diagram.

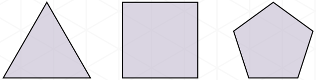
- ▷ In this case, the number of nodes represent the dimension of the polytope

Classification of regular polytopes

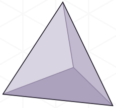
- **dimension 2:** $I_2(m)$  $\longrightarrow$ regular m-gon  $\cdots$

Classification of regular polytopes

▷ **dimension 2:** $I_2(m)$ 

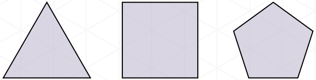
$\longrightarrow$ regular m-gon  $\cdots$

▷ **dimension 3:**

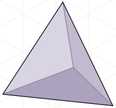
A_3 :  $\longrightarrow$ tetrahedron 

Classification of regular polytopes

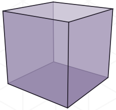
▷ **dimension 2:** $I_2(m)$ 

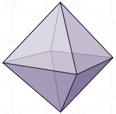
$\longrightarrow$ regular m-gon  $\cdots$

▷ **dimension 3:**

A_3 :  $\longrightarrow$ tetrahedron 

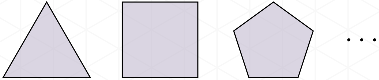
B_3 : 

$\longrightarrow$ cube  , 

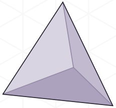
$\longrightarrow$ octahedron 

Classification of regular polytopes

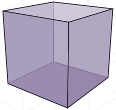
▷ **dimension 2:** $I_2(m)$ 

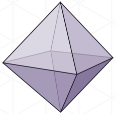
$\longrightarrow$ regular m-gon 

▷ **dimension 3:**

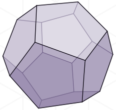
A_3 :  $\longrightarrow$ tetrahedron 

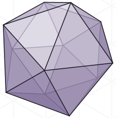
B_3 : 

$\longrightarrow$ cube  , 

$\longrightarrow$ octahedron 

H_3 : 

$\longrightarrow$ dodecahedron  , 

$\longrightarrow$ icosahedron 

In higher dimensions

- **dimension 4:**

  A_4 :  $\longrightarrow$ 4-simplex (tetrahedron)

In higher dimensions

- **dimension 4:**

A_4 :  $\longrightarrow$ 4-simplex (tetrahedron)

B_4 : 

$\longrightarrow$ 4-cube, 

$\longrightarrow$ 4-orthoplex (octahedron).

In higher dimensions

- **dimension 4:**

A_4 :  $\longrightarrow$ 4-simplex (tetrahedron)

B_4 : 

$\longrightarrow$ 4-cube, 

$\longrightarrow$ 4-orthoplex (octahedron).

H_4 : 

$\longrightarrow$ 120-cell (dodecahedron), 

$\longrightarrow$ 600-cell (icosahedron).

In higher dimensions

- **dimension 4:**

A_4 :  $\longrightarrow$ 4-simplex (tetrahedron)

B_4 :  $\longrightarrow$ 4-cube,  $\longrightarrow$ 4-orthoplex (octahedron).

H_4 :  $\longrightarrow$ 120-cell (dodecahedron),  $\longrightarrow$ 600-cell (icosahedron).

F_4 :  $\longrightarrow$ 24-cell : no equivalent in 3 dimensions !

In higher dimensions

▷ **dimension 4:**

A_4 : ○–○–○–○ $\longrightarrow$ 4-simplex (tetrahedron)

B_4 : ○–4–○–○–○ $\longrightarrow$ 4-cube, ○–○–○–4–○ $\longrightarrow$ 4-orthoplex (octahedron).

H_4 : ○–5–○–○–○ $\longrightarrow$ 120-cell (dodecahedron), ○–○–○–5–○ $\longrightarrow$ 600-cell (icosahedron).

F_4 : ○–○–4–○–○ $\longrightarrow$ 24-cell : no equivalent in 3 dimensions !

▷ **dimension $n \geq 5$:**

A_n : ○–○– ⋯ –○–○–○ $\longrightarrow$ n-simplex (tetrahedron)

B_n : ○–4–○– ⋯ –○–○–○ $\longrightarrow$ n-cube, ○–○– ⋯ –○–○–4–○ $\longrightarrow$ n-orthoplex (octahedron).

Part VI

Conclusion

What about my thesis?

- I study combinatorial objects that appear within the Coxeter groups.
- The combinatorics of these objects have many links to other fields: Coxeter-Catalan combinatorics.

What about my thesis?

- I study combinatorial objects that appear within the Coxeter groups.
- The combinatorics of these objects have many links to other fields: Coxeter-Catalan combinatorics.
- I also study infinite Coxeter groups, which do not always have a representation as a reflection group of $\mathbb{R}^d$.
- My goal is to generalize some results true in finite Coxeter groups.

Thank you very much for your attention!