Noncrossing arc diagrams and COXETER sortable elements

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Summary

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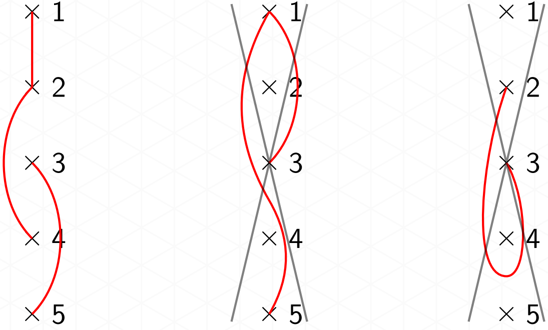
Part I

Noncrossing arc diagrams and a bijection with the symmetric group

Arc diagrams

An arc is a continuous curve joining two points $p < q$ without crossing any other point and moving vertically.

A noncrossing arc diagram is a collection of arcs on the same diagram that do not cross, except possibly at their endpoints, and they cannot share the same starting and ending point.

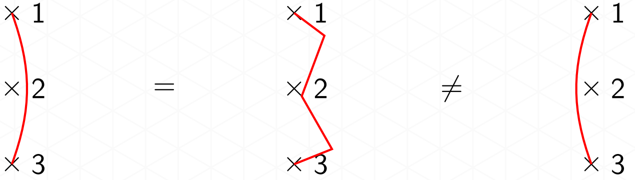


Combinatorial description

Two arcs are equal if one can be continuously transformed into the other without changing their endpoints and crossing any point.

For example, the two first arcs are equal, but they differ from the third.

$(1,3,\{2\},\emptyset) =$



$= (1,3,\emptyset,\{2\})$

Definition 1 (Combinatorial definition of an arc).

An arc is a tuple (p,q,L,R) where p is the initial point, q the final point, L the numbers on the left of the arc and R the numbers on the right of the arc.

Symmetric group: Notation

Let $n \in \mathbb{N}^*$. A bijective map from the set $[\![1, n]\!]$ into itself is called a permutation. The symmetric group, denoted $\mathfrak{S}_n$, is the set of all permutations of $[\![1, n]\!]$.

A permutation $\sigma \in \mathfrak{S}_n$ will be represented by its one-line notation: it is the sequence of integers $\sigma(1)\sigma(2)\ldots\sigma(n)$.

Example: $\sigma = 3142 \in \mathfrak{S}_4$. has $(1,3)$, $(2,3)$ and $(2,4)$ as inversions., but only $(1,3)$ and $(2,4)$ are cover reflections.

Definition 2 (Inversion).

An inversion of σ is a pair of integers (i, j) such that $i < j$ and $\sigma^{-1}(j) < \sigma^{-1}(i)$.

Definition 3 (Cover reflection).

A cover reflection of σ is an inversion (i, j) of σ such that i and j are consecutive in σ.

A map from permutations to noncrossing arc diagrams

Let $\sigma \in \mathfrak{S}_n$. For each cover reflection (p, q) of σ, we can define an arc (p, q, L, R) where L (resp. R) is the set of all integers between p and q that appear on the left (resp. right) of p and q in the one-line notation of σ.

Example : $\sigma = 254163$. The cover reflections of σ are $(1, 4)$, $(4, 5)$ and $(3, 6)$. We obtain the following noncrossing arc diagram.

Theorem 4 (Reading, 2015).

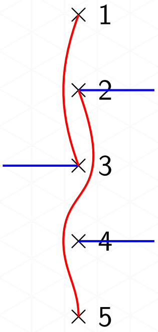
The previous construction defines a bijective map from $\mathfrak{S}_n$ to the set of noncrossing arc diagrams on n points.

Part II

Adding restrictions

c-noncrossing arc diagrams

At each point between 2 and $n-1$, we add a horizontal half-line going either to the left or to the right. This defines a partition of $[\![2, n-1]\!]$ in two parts, denoted $c = (L_c, R_c)$.



$c = (\{3\}, \{2, 4\})$

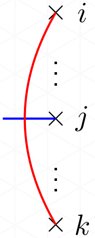
Definition 5 (c-noncrossing arc diagram).

A c-noncrossing arc diagram is a noncrossing arc diagram that do not cross the horizontal added lines.

Permutations associated to c-noncrossing arc diagrams

Question: Which permutations of $\mathfrak{S}_n$ correspond to c-noncrossing arc diagrams?

Suppose that σ doesn't correspond to a c-noncrossing arc diagram. There exists $i < j < k$ such that the arc $i \to k$ crosses the wall coming from j. If $j \in L_c$, then j appears on the right of i and k in the one-line notation of σ. Similarly, if $j \in R_c$, then j appears on the left of i and k.



Theorem 6.

A permutation $\sigma \in \mathfrak{S}_n$ corresponds to a c-noncrossing arc diagram if and only if for all $1 \leq i < j < k \leq n$, its one-line notation avoids the patterns $ki \ldots j$ with $j \in L_c$ and $j \ldots ki$ with $j \in R_c$.

COXETER sortable elements and some context

These permutations are called c-sortable elements of $\mathfrak{S}_n$.

In fact, c-sortable elements can be defined in any COXETER group. They were introduced by N. READING in 2005 as an intermediary object to construct a bijection between noncrossing partitions and clusters in a finite COXETER group.

For an infinite COXETER group, like the affine symmetric group, the map between c-sortable elements and noncrossing partitions is only injective and not surjective.

My thesis: « Noncrossing partitions and generalization of c-sortable elements in affine COXETER groups ».

Part III

Affine permutations and cyclic c-noncrossing arc diagrams

Affine permutations

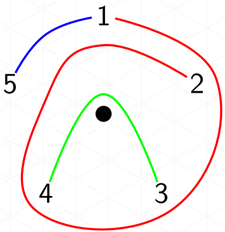
The affine symmetric group, denoted $\widehat{\mathfrak{S}}_n$, is the set of all permutations σ of $\mathbb{Z}$ such that for all $i, p \in \mathbb{Z}$, $\sigma(i + pn) = \sigma(i) + pn$, and $\sum_{i=1}^{n} \sigma(i) = \frac{n(n+1)}{2}$. Example for $n = 3$:

$\cdots$	-4	-3	-2	-1	0	1	2	3	4	5	6	7	$\cdots$
$\cdots$	-2	-7	0	1	-4	3	4	-1	6	7	2	9	$\cdots$

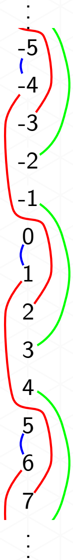
The one-line notation of such a permutation is infinite, but the notions of inversion and cover reflection are still well defined.

Cyclic noncrossing arc diagrams

A cyclic noncrossing arc diagram is a noncrossing arc diagram on an infinite number of points indexed by $\mathbb{Z}$ such that $\alpha = (p, q, L, R)$ is an arc if and only if $\alpha + n = (p + n, q + n, L + n, R + n)$ is an arc.



We can also « roll up » the diagram into a circle labeled by the elements of $\mathbb{Z}/n\mathbb{Z}$. In this case, all the copies $\alpha + kn$ are merged into one cyclic arc.



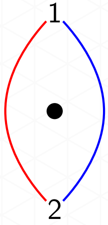
Non surjective map

Unlike the case of the symmetric group, in this case there are cyclic noncrossing arc diagrams that cannot be obtained from an affine permutation.

Example: an affine permutation that is associated to the following cyclic noncrossing arc diagram would have a one-line notation of

$$\dots\ 4\ \ 3\ \ 2\ \ 1\ \ 0\ \ -1\ \ -2\ \ -3\ \ -4\ \dots$$

which is impossible.



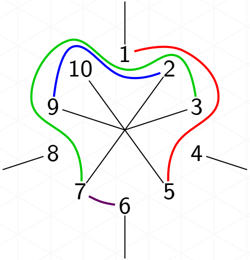
Theorem 7 (Barkley, 2025).

The set of cyclic noncrossing arc diagrams is in bijection with the set of widely generated elements of $\mathrm{WO}(\mathrm{TTot}_n)$.

Adding restriction

We add radial segments going from each number i, either towards the center or outwards. This defines a partition of $[\![1, n]\!]$ in two (nonempty) parts $c = (L_c, R_c)$.

Example : $n = 10$, $c = \{1, 4, 6, 8\} \sqcup \{2, 3, 5, 7, 9, 10\}$



Definition 8 (Cyclic c-noncrossing arc diagram).

A cyclic c-noncrossing arc diagram is a cyclic noncrossing arc diagram such that every arc $\alpha = (p, q, L, R)$ satisfies $L \subset L_c + n\mathbb{Z}$ and $R \subset R_c + n\mathbb{Z}$.

Current work

Using the bijection established by BARKLEY, is there a way to describe the preimages of the cyclic c-noncrossing arc diagrams?

Yes! It's a family of total orders on $\mathbb{Z}$ that satisfies some very explicit properties. Moreover, among the affine permutations, only the c-sortable elements belongs to this family of preimages.

Finally, on the other hand, there is also a bijection between cyclic c-noncrossing arc diagrams and c-noncrossing partitions of the affine symmetric group.

This "solves" the case for the affine symmetric group: « Noncrossing partitions and generalization of c-sortable elements in affine COXETER groups ».

Thank you for your attention!

Bibliography

Grant T. Barkley.

Extended weak order for the affine symmetric group, 2025.

Nathan Reading.

Clusters, coxeter-sortable elements and noncrossing partitions, 2005.

Nathan Reading.

Noncrossing arc diagrams and canonical join representations.

2015.

Nathan Reading and David E Speyer.

Sortable elements in infinite coxeter groups.

2010.