Noncrossing arc diagrams and permutations

Jad Abou Yassin

Institut Denis Poisson, University of Tours

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Summary

1. Noncrossing arc diagrams
   - Definition
   - Enumeration for small values of n
2. Permutations
   - Definition
   - A bijective map from permutations to noncrossing arc diagrams
3. Adding restrictions
   - c-noncrossing arc diagrams
   - Noncrossing partitions

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× 1

× 2

× 3

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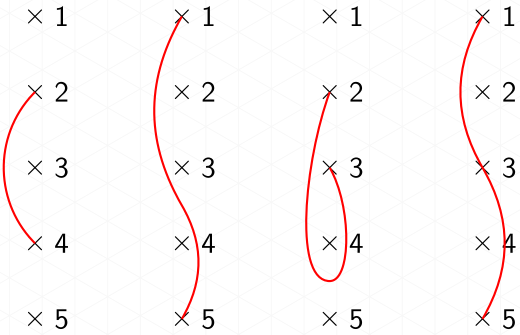

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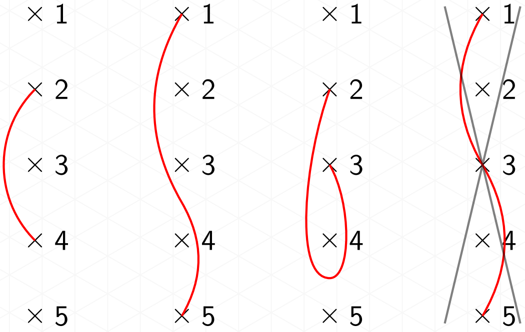

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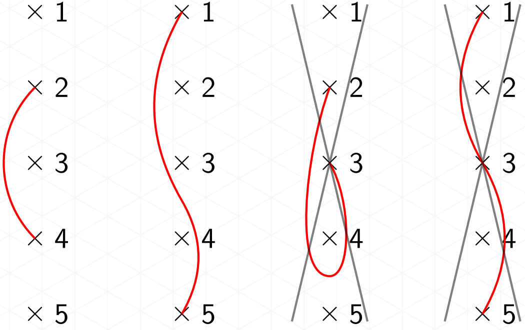

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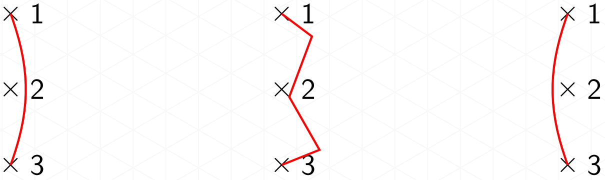

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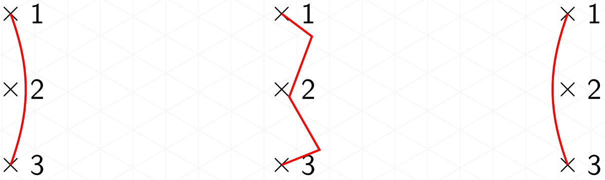

A combinatorical definition of an arc: a tuple (p, q, L, R) where p is the initial point, q the final point, L the numbers on the left of the arc and R the ones on the right of the arc.

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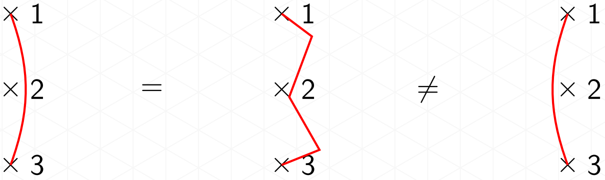
$(1, 3, \{2\}, \emptyset) =$



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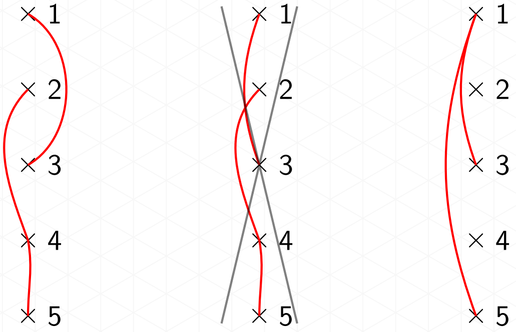

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Let $n \in \mathbb{N}^*$. A noncrossing arc diagram on n points is a collection of arcs that do not cross, except possibly at their endpoints, and they can't share the same initial point or final point.

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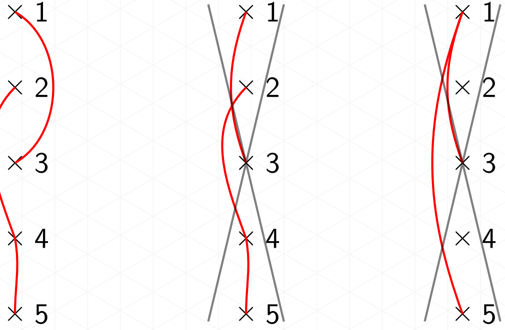

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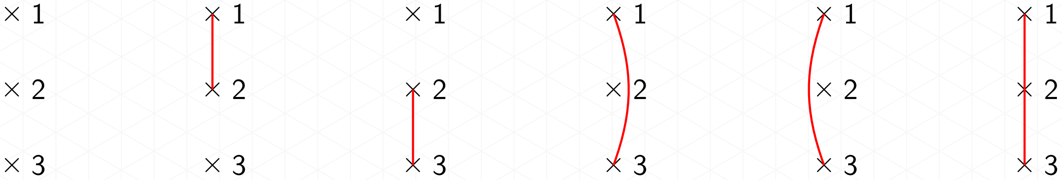

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2. Permutations
   - Definition
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Permutations

Let $n \in \mathbb{N}^*$. A bijective map from the set $[\![1, n]\!]$ into itself is called a permutation. We note $\mathfrak{S}_n$ the set of all permutations of $[\![1, n]\!]$.

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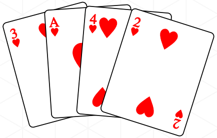
Example: the permutation  is represented as 3142.

Inversions and cover reflections

Let $\sigma \in \mathfrak{S}_n$ be a permutation. An inversion of σ is a pair of integers (i, j) such that $i < j$ and j appears before i in the one line notation of σ.

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Example : $\sigma = 3142$. The inversions of σ are $(1, 3)$, $(2, 3)$ and $(2, 4)$. Among them, only $(1, 3)$ and $(2, 4)$ are cover reflections.

Arc associated to a cover reflection of a permutation

Let $\sigma \in \mathfrak{S}_n$. If (p, q) is a cover reflection of σ, we define an arc (p, q, L, R) where L (resp. R) is the set of all integers between p and q that appear on the left (resp. right) of p and q in the one line notation of σ.

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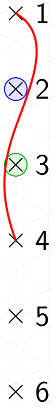

A map from permutations to noncrossing arc diagrams

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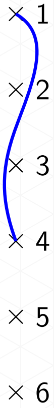

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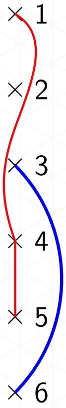

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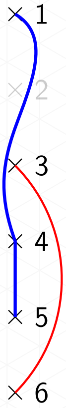

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Consequence

Theorem

The previous constructions define a bijective map from the set of noncrossing arc diagrams on n points to the set of permutations on $[\![1, n]\!]$.

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Corollary

There are $n!$ noncrossing arc diagrams on n points.

c-noncrossing arc diagrams

Let $c = L_c \sqcup R_c$ be a partition of $[\![2, n-1]\!]$ in two parts. For example, with $n = 5$, $c = \{3\} \sqcup \{2, 4\}$.

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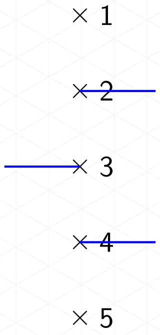
Let $c = L_c \sqcup R_c$ be a partition of $[\![2, n-1]\!]$ in two parts. For example, with $n = 5$, $c = \{3\} \sqcup \{2, 4\}$.

We add horizontal half-lines starting at each point $i \in [\![2, n-1]\!]$ and going to the left (resp. right) if $i \in L_c$ (resp. $i \in R_c$).



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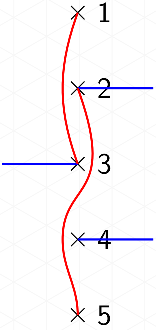
A c-noncrossing arc diagram is a noncrossing arc diagram that do not cross the horizontal added lines.



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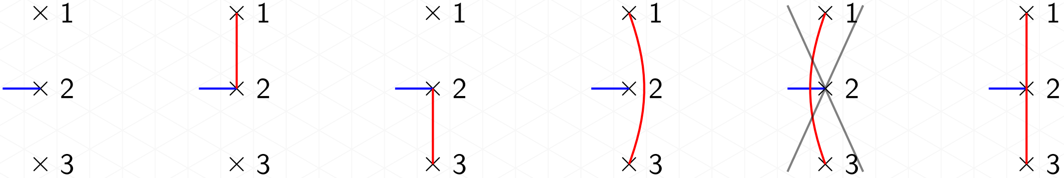

Correspondence with noncrossing partitions

A noncrossing partition is a set of noncrossing polygons with verticies on a line. We can transform a c-noncrossing arc diagram into a noncrossing partition, and vice versa.

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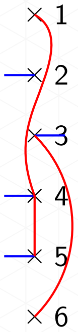

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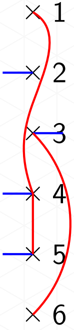
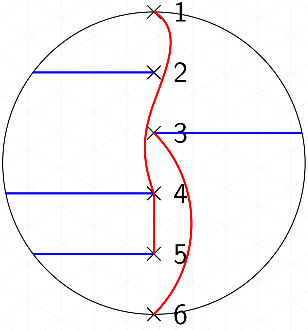


$\longleftrightarrow$



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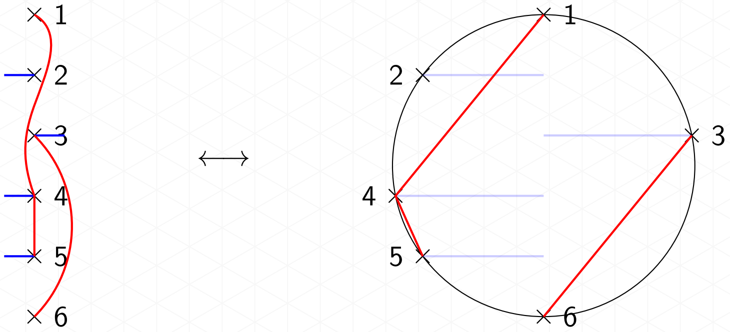

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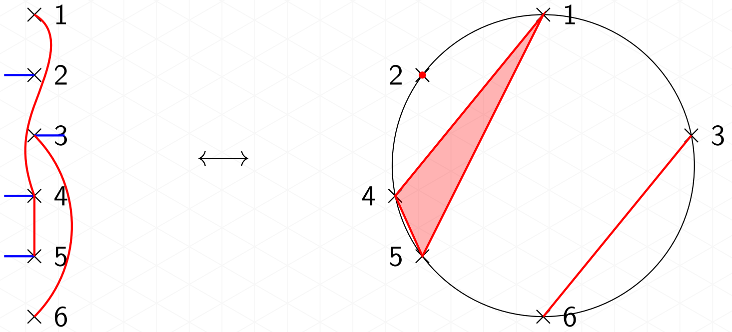

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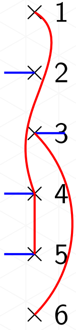

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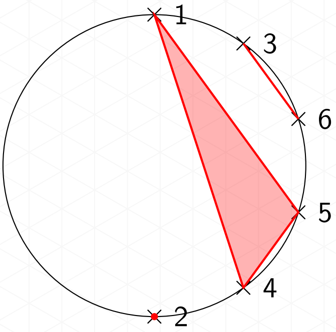

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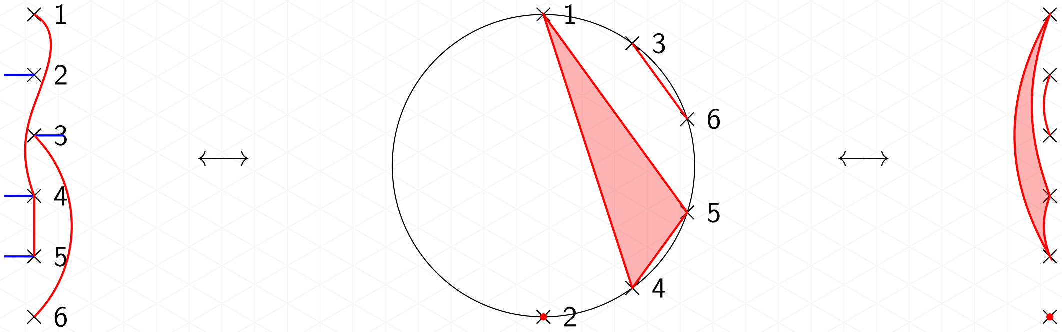

Consequence

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The permutations associated with c-noncrossing arc diagrams are called c-sortable elements (of $\mathfrak{S}_n$). We now know there also are enumerated by the CATALAN numbers.

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My thesis' title : « Noncrossing partitions and generalization of c-sortable elements in affine Coxeter groups ».

Thank you for your attention!