A slice of Catalan combinatorics from the perspective of Coxeter sortable elements of the symmetric group

Jad Abou Yassin

Institut Denis Poisson, University of Tours

Le Croisic - June 9, 2025

Summary

1. Coxeter sortable elements of the symmetric group
   - Symmetric group
   - Coxeter sortable elements
2. Non crossing partitions
   - Geometric non crossing partitions
   - Bijection with the c-sortable elements
3. Binary trees
   - Binary trees and SSA algorithm
   - Link with c-sortable elements

1. COXETER sortable elements of the symmetric group
   - Symmetric group
   - COXETER sortable elements

2. Non crossing partitions

3. Binary trees

Notations

Let $n \in \mathbb{N}^*$. We note $\mathfrak{S}_n$ the symmetric group of order n.

An element $\sigma \in \mathfrak{S}_n$ is represented by its one line notation : $\sigma(1)\sigma(2)\ldots\sigma(n)$.
Example : $\mathfrak{S}_3 = \{1\,2\,3,\ 2\,1\,3,\ 2\,3\,1,\ 1\,3\,2,\ 3\,1\,2,\ 3\,2\,1\}$.

Notations

Let $n \in \mathbb{N}^*$. We note $\mathfrak{S}_n$ the symmetric group of order n.

An element $\sigma \in \mathfrak{S}_n$ is represented by its one line notation : $\sigma(1)\sigma(2)\ldots\sigma(n)$.

Example : $\mathfrak{S}_3 = \{1\,2\,3,\ 2\,1\,3,\ 2\,3\,1,\ 1\,3\,2,\ 3\,1\,2,\ 3\,2\,1\}$.

COXETER group : $\mathfrak{S}_n \simeq \left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i^2 = 1 & \forall 1 \leq i \leq n-1 \\ s_i s_j = s_j s_i & \forall |i-j| > 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \forall 1 \leq i \leq n-2 \end{array} \right\rangle$

COXETER elements

A COXETER element of a COXETER group W is an element of W that is the product of all the generators exactly once.

Examples : $s_1 s_2 \dots s_{n-1}$, $s_1 s_3 \dots s_{n-1} \dots s_4 s_2$.

COXETER elements

A COXETER element of a COXETER group W is an element of W that is the product of all the generators exactly once.

Examples : $s_1 s_2 \dots s_{n-1}$, $s_1 s_3 \dots s_{n-1} \dots s_4 s_2$.

A COXETER element of $\mathfrak{S}_n$ is permutation c of $\mathfrak{S}_n$ that is a great cycle of the form $(1, a_1, \dots, a_k, n, b_l, \dots, b_1)$ where $k + l = n - 2$, $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_l$.

Coxeter elements

A Coxeter element of a Coxeter group W is an element of W that is the product of all the generators exactly once.

Examples : $s_1 s_2 \dots s_{n-1}$, $s_1 s_3 \dots s_{n-1} \dots s_4 s_2$.

A Coxeter element of $\mathfrak{S}_n$ is permutation c of $\mathfrak{S}_n$ that is a great cycle of the form $(1, a_1, \dots, a_k, n, b_l, \dots, b_1)$ where $k + l = n - 2$, $a_1 < \dots < a_k$ and $b_1 < \dots < b_l$.

It is characterized by a partition of $\{2, \dots, n-1\} = \underbrace{\{a_1, \dots, a_k\}}_{L_c} \sqcup \underbrace{\{b_1, \dots, b_l\}}_{R_c}$.

Examples : let $n = 6$, $(1, 3, 4, 6, 5, 2)$, $(1, 6, 5, 4, 3, 2)$, $(1, 2, 3, 4, 5, 6)$, $\cancel{(1, 5, 2, 6, 3, 4)}$.

COXETER sortable elements

Let c be a COXETER element of $\mathfrak{S}_n$. A permutation $\sigma \in \mathfrak{S}_n$ is c-sortable [Rea05] if its one line notation avoids the following patterns :

- $ki \ldots j$ for $i < j < k$ and $j \in L_c$
- $j \ldots ki$ for $i < j < k$ and $j \in R_c$.

COXETER sortable elements

Let c be a COXETER element of $\mathfrak{S}_n$. A permutation $\sigma \in \mathfrak{S}_n$ is c-sortable [Rea05] if its one line notation avoids the following patterns :

- $ki \ldots j$ for $i < j < k$ and $j \in L_c$
- $j \ldots ki$ for $i < j < k$ and $j \in R_c$.

Examples : let $n = 7$ and $c = (1, 3, 4, 6, 7, 5, 2)$.

- 7416325 is not c-sortable because it contains a pattern $ki \ldots j$ with $j = 6 \in L_c$.
- 6521743 is not c-sortable because it contains a pattern $j \ldots ki$ with $j = 5 \in R_c$.
- 3167425 is c-sortable because it avoids all the patterns.

CATALAN

Let $n \in \mathbb{N}^*$. For any COXETER element c of $\mathfrak{S}_n$, there are $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ c-sortable elements.

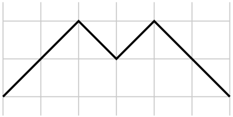
CATALAN

Let $n \in \mathbb{N}^*$. For any COXETER element c of $\mathfrak{S}_n$, there are $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ c-sortable elements.

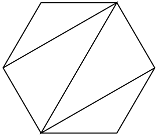
This means the c-sortable elements are in bijection with all the objects enumerated by the CATALAN numbers.

Examples :

DYCK paths of length $2n$



Triangulations of $(n+2)$-gons



Well parenthesized expressions of $n+1$ factors

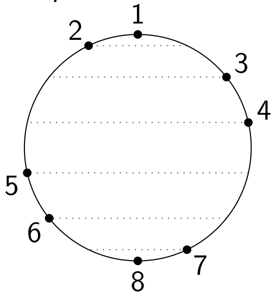
$$((a(bc))d)(ef)$$

Labeling the circle

Let $n \in \mathbb{N}^*$ and place on a circle the numbers from 1 to n such that :

- 1 is at the highest point and n is at the lowest point,
- no two numbers are on the same height,
- reading from top to bottom the numbers are increasing.

Example : $n = 8$

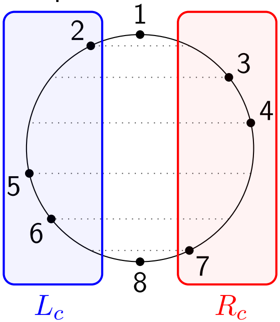


Labeling the circle

Let $n \in \mathbb{N}^*$ and place on a circle the numbers from 1 to n such that :

- 1 is at the highest point and n is at the lowest point,
- no two numbers are on the same height,
- reading from top to bottom the numbers are increasing.

Example : $n = 8$



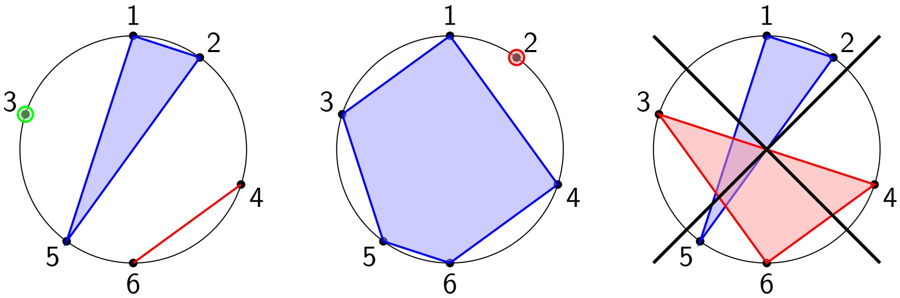
$c = (1, 2, 5, 6, 8, 7, 4, 3)$

A c-labeling is a labeling such that on the left are the elements of L_c and on the right the elements of R_c

Non crossing partitions

A c-non crossing partition is a set of non crossing polygons with vertices the marked points of a c-labeled circle. Single points and segments are considered as polygons.

Examples : $n = 6$ and $c = (1, 3, 5, 6, 4, 2)$



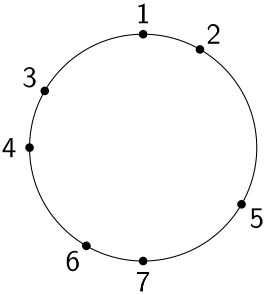
Bijection with the c-sortable elements

Theorem (READING [Rea05])

The set of c-sortable elements is in bijection with the set of c-non crossing partitions via an explicit map called nc_c.

Example : let $n = 7$, $c = (1, 3, 4, 6, 7, 5, 2)$ and $\sigma = 3167425$. It is a c-sortable element.

$$\sigma = 3167425 \longrightarrow$$



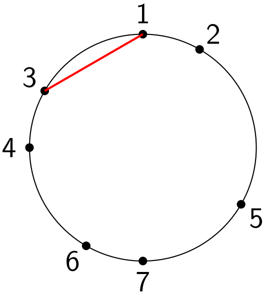
Bijection with the c-sortable elements

Theorem (Reading [Rea05])

The set of c-sortable elements is in bijection with the set of c-non crossing partitions via an explicit map called nc_c.

Example : let $n = 7$, $c = (1, 3, 4, 6, 7, 5, 2)$ and $\sigma = 3167425$. It is a c-sortable element.

$\sigma = 3167425 \longrightarrow$



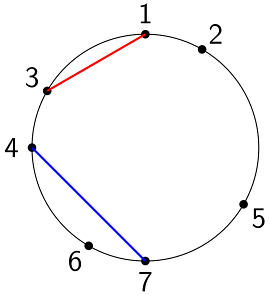
Bijection with the c-sortable elements

Theorem (Reading [Rea05])

The set of c-sortable elements is in bijection with the set of c-non crossing partitions via an explicit map called nc_c.

Example : let $n = 7$, $c = (1, 3, 4, 6, 7, 5, 2)$ and $\sigma = 3167425$. It is a c-sortable element.

$\sigma = 3167425 \longrightarrow$



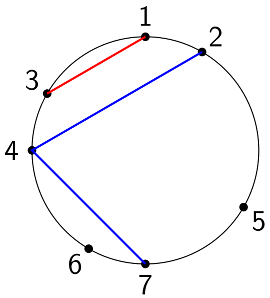
Bijection with the c-sortable elements

Theorem (READING [Rea05])

The set of c-sortable elements is in bijection with the set of c-non crossing partitions via an explicit map called nc_c.

Example : let $n = 7$, $c = (1, 3, 4, 6, 7, 5, 2)$ and $\sigma = 3167425$. It is a c-sortable element.

$$\sigma = 3167425 \longrightarrow$$

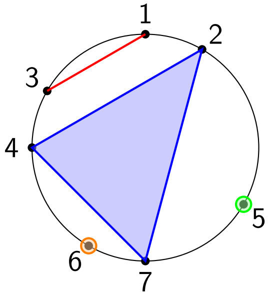


Bijection with the c-sortable elements

Theorem (READING [Rea05])

The set of c-sortable elements is in bijection with the set of c-non crossing partitions via an explicit map called nc_c.

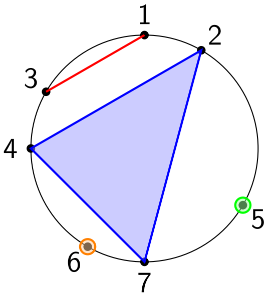
Example : let $n = 7$, $c = (1, 3, 4, 6, 7, 5, 2)$ and $\sigma = 3167425$. It is a c-sortable element.

$\sigma = 3167425 \longrightarrow$  $= nc_c(\sigma)$.

Inverse map

The inverse map of nc_c can be computed by selecting the polygons of a c-non crossing partition in a specific order [Gob18].

Example : Let's use the c-non crossing partition we computed in the previous slide.

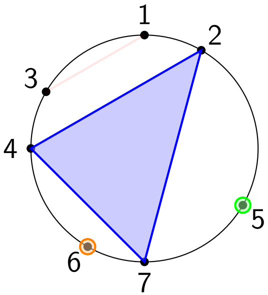


$\longrightarrow$

Inverse map

The inverse map of nc_c can be computed by selecting the polygons of a c-non crossing partition in a specific order [Gob18].

Example : Let's use the c-non crossing partition we computed in the previous slide.

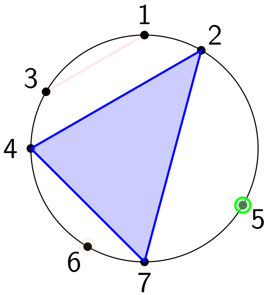


$\longrightarrow \{3, 1\},$

Inverse map

The inverse map of nc_c can be computed by selecting the polygons of a c-non crossing partition in a specific order [Gob18].

Example : Let's use the c-non crossing partition we computed in the previous slide.

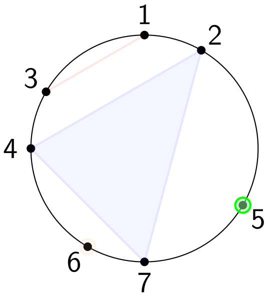


$\longrightarrow \{3, 1\}, \{6\},$

Inverse map

The inverse map of nc_c can be computed by selecting the polygons of a c-non crossing partition in a specific order [Gob18].

Example : Let's use the c-non crossing partition we computed in the previous slide.

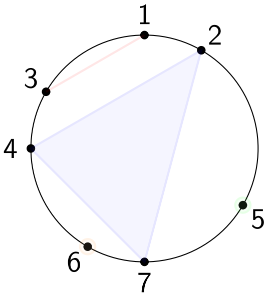


$\longrightarrow \{3, 1\}, \{6\}, \{7, 4, 2\},$

Inverse map

The inverse map of nc_c can be computed by selecting the polygons of a c-non crossing partition in a specific order [Gob18].

Example : Let's use the c-non crossing partition we computed in the previous slide.

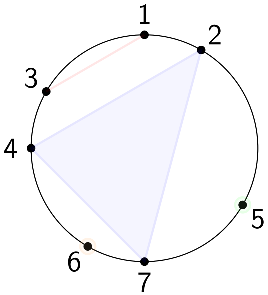


$\longrightarrow \{3,1\},\{6\},\{7,4,2\},\{5\}$

Inverse map

The inverse map of nc_c can be computed by selecting the polygons of a c-non crossing partition in a specific order [Gob18].

Example : Let's use the c-non crossing partition we computed in the previous slide.



$$\longrightarrow \{3,1\},\{6\},\{7,4,2\},\{5\} \longrightarrow 3167425 = \sigma$$

1. COXETER sortable elements of the symmetric group

2. Non crossing partitions

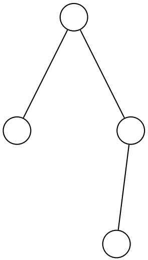
3. Binary trees

- Binary trees and SSA algorithm
- Link with c-sortable elements

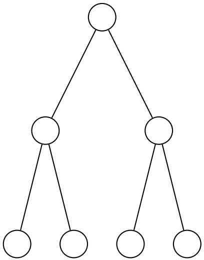
Binary trees

A binary tree is either an empty tree or a node with exactly one left child and one right child that are binary trees. The size of a binary tree is the number of nodes in the tree.

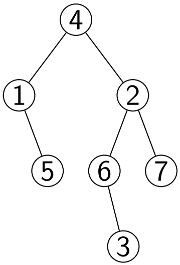
Examples :



A binary tree.



A complete binary tree.

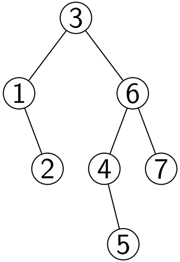


A labeled binary tree.

Binary search trees, descending trees

A binary search tree is a labeled binary tree such that the label of each node is larger than the labels of its left child and smaller than the labels of its right child.

Examples :



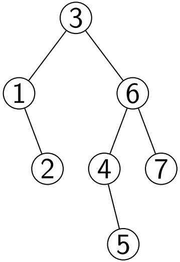
A binary search tree

Binary search trees, descending trees

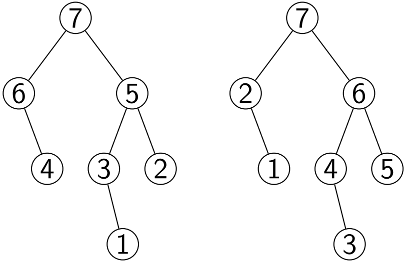
A binary search tree is a labeled binary tree such that the label of each node is larger than the labels of its left child and smaller than the labels of its right child.

A descending tree is such that the label of each node is larger than the labels of its descendants.

Examples :



A binary search tree



Two descending trees of the same shape

SSA algorithm

Theorem (SSA algorithm [HNT04])

There is an explicit bijection

$$\mathfrak{S}_n \simeq \left\{ (T,Q) \;\middle|\; \begin{array}{l} T \text{ is a binary search tree of size } n \text{ and} \\ Q \text{ is a descending tree of the same shape as } T \end{array} \right\}$$

Example : Let $n = 7$ and $\sigma = 2154763$.

$T(\sigma) =$ $Q(\sigma) =$

SSA algorithm

Theorem (SSA algorithm [HNT04])

There is an explicit bijection

$$\mathfrak{S}_n \simeq \left\{ (T,Q) \;\middle|\; \begin{array}{l} T \textit{ is a binary search tree of size } n \textit{ and} \\ Q \textit{ is a descending tree of the same shape as } T \end{array} \right\}$$

Example : Let $n = 7$ and $\sigma = 2154763$. (position $= 7$)

$T(\sigma) =$ 3 $\qquad Q(\sigma) =$ 7

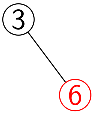
SSA algorithm

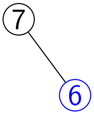
Theorem (SSA algorithm [HNT04])

There is an explicit bijection

$$\mathfrak{S}_n \simeq \left\{ (T,Q) \;\middle|\; \begin{array}{l} T \text{ is a binary search tree of size } n \text{ and} \\ Q \text{ is a descending tree of the same shape as } T \end{array} \right\}$$

Example : Let $n = 7$ and $\sigma = 2154763$. (position $= 6$)

$T(\sigma) =$ 

$Q(\sigma) =$ 

SSA algorithm

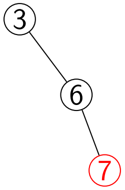
Theorem (SSA algorithm [HNT04])

There is an explicit bijection

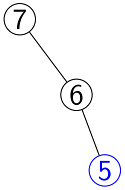
$$\mathfrak{S}_n \simeq \left\{ (T, Q) \;\middle|\; \begin{array}{l} T \text{ is a binary search tree of size } n \text{ and} \\ Q \text{ is a descending tree of the same shape as } T \end{array} \right\}$$

Example : Let $n = 7$ and $\sigma = 2154763$. (position $= 5$)

$T(\sigma) =$



$Q(\sigma) =$



SSA algorithm

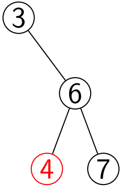
Theorem (SSA algorithm [HNT04])

There is an explicit bijection

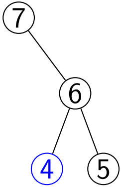
$$\mathfrak{S}_n \simeq \left\{ (T,Q) \;\middle|\; \begin{array}{l} T \text{ is a binary search tree of size } n \text{ and} \\ Q \text{ is a descending tree of the same shape as } T \end{array} \right\}$$

Example : Let $n = 7$ and $\sigma = 215\textcolor{red}{4}763$. (position $= \textcolor{blue}{4}$)

$T(\sigma) =$



$Q(\sigma) =$



SSA algorithm

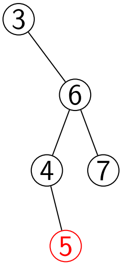
Theorem (SSA algorithm [HNT04])

There is an explicit bijection

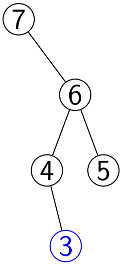
$$\mathfrak{S}_n \simeq \left\{ (T,Q) \;\middle|\; \begin{array}{l} T \textit{ is a binary search tree of size } n \textit{ and} \\ Q \textit{ is a descending tree of the same shape as } T \end{array} \right\}$$

Example : Let $n = 7$ and $\sigma = 2154763$. (position $= 3$)

$T(\sigma) =$



$Q(\sigma) =$



SSA algorithm

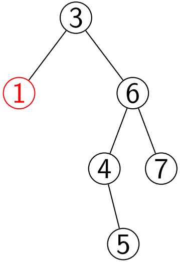
Theorem (SSA algorithm [HNT04])

There is an explicit bijection

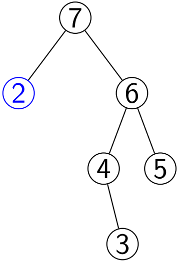
$$\mathfrak{S}_n \simeq \left\{ (T, Q) \;\middle|\; \begin{array}{l} T \textit{ is a binary search tree of size } n \textit{ and} \\ Q \textit{ is a descending tree of the same shape as } T \end{array} \right\}$$

Example : Let $n = 7$ and $\sigma = 2{\color{red}1}54763$. (position $= {\color{blue}2}$)

$T(\sigma) =$



$Q(\sigma) =$



SSA algorithm

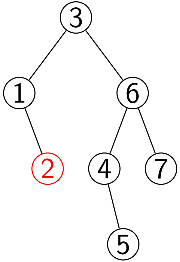
Theorem (SSA algorithm [HNT04])

There is an explicit bijection

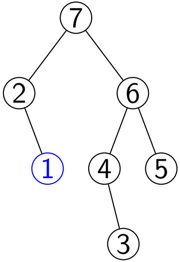
$$\mathfrak{S}_n \simeq \left\{ (T,Q) \;\middle|\; \begin{array}{l} T \textit{ is a binary search tree of size } n \textit{ and} \\ Q \textit{ is a descending tree of the same shape as } T \end{array} \right\}$$

Example : Let $n = 7$ and $\sigma = {\color{red}2}154763$. (position $= {\color{blue}1}$)

$T(\sigma) =$



$Q(\sigma) =$



SSA algorithm

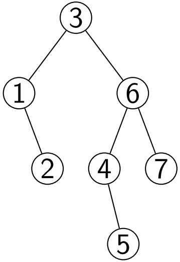
Theorem (SSA algorithm [HNT04])

There is an explicit bijection

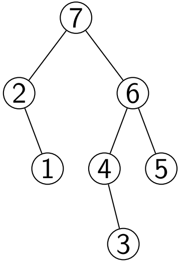
$$\mathfrak{S}_n \simeq \left\{ (T,Q) \;\middle|\; \begin{array}{l} T \text{ is a binary search tree of size } n \text{ and} \\ Q \text{ is a descending tree of the same shape as } T \end{array} \right\}$$

Example : Let $n = 7$ and $\sigma = 2154763$.

$T(\sigma) =$



$Q(\sigma) =$



Sylvester congruence

SSA algorithm : $\sigma \in \mathfrak{S}_n$ is encoded by $(T(\sigma), Q(\sigma))$.

What happens if we forget $Q(\sigma)$? Can we describe all $\sigma' \in \mathfrak{S}_n$ s.t. $T(\sigma) = T(\sigma')$?

Sylvester congruence

SSA algorithm : $\sigma \in \mathfrak{S}_n$ is encoded by $(T(\sigma), Q(\sigma))$.

What happens if we forget $Q(\sigma)$? Can we describe all $\sigma' \in \mathfrak{S}_n$ s.t. $T(\sigma) = T(\sigma')$?

Yes! $T(\sigma) = T(\sigma')$ iff σ' can be obtained from σ by a series of transformations of the form $ki \dots j \leftrightarrow ik \dots j$ with $i < j < k$ $\longrightarrow$ Sylvester congruence on $\mathfrak{S}_n$.

Example : $\sigma' = 5421763$ has the same binary search tree than $\sigma = 2154763$

$$2154763 \to 2514763 \to 2541763 \to 5241763 \to 5421763$$

Link with c-sortable elements

If $c = (1, 2, 3, \ldots, n-1, n)$, then we have the following one to one maps :

2143

σ with no $ki \ldots j$ pattern
$=$ a c-sortable element

Link with c-sortable elements

If $c = (1, 2, 3, \dots, n-1, n)$, then we have the following one to one maps :

$$2143 \quad \leftrightarrow \quad \left\{ \begin{array}{c} 2143 \\ 2413 \\ 4213 \end{array} \right\}$$

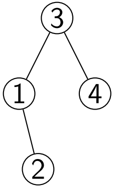
σ with no $ki\dots j$ pattern
$=$ a c-sortable element

A sylvester class

Link with c-sortable elements

If $c = (1, 2, 3, \ldots, n-1, n)$, then we have the following one to one maps :

$$2143 \quad \leftrightarrow \quad \left\{ \begin{array}{c} 2143 \\ 2413 \\ 4213 \end{array} \right\} \quad \leftrightarrow$$



σ with no $ki \ldots j$ pattern
$=$ a c-sortable element

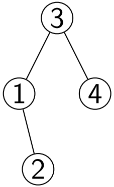
A sylvester class

A binary search tree

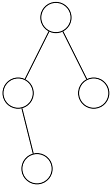
Link with c-sortable elements

If $c = (1, 2, 3, \ldots, n-1, n)$, then we have the following one to one maps :

$$2143 \quad \leftrightarrow \quad \left\{ \begin{array}{c} 2143 \\ 2413 \\ 4213 \end{array} \right\} \quad \leftrightarrow$$



$\leftrightarrow$



σ with no $ki \ldots j$ pattern $=$ a c-sortable element

A sylvester class

A binary search tree

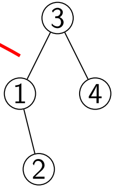
A binary tree

Link with c-sortable elements

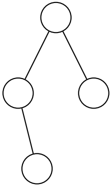
If $c = (1, {\color{blue}2}, {\color{blue}3}, {\color{blue}\dots}, {\color{blue}n-1}, n)$, then we have the following one to one maps :

postfix reading

$$2143 \quad \leftrightarrow \quad \left\{ \begin{array}{c} 2143 \\ 2413 \\ 4213 \end{array} \right\} \quad \leftrightarrow$$



$\leftrightarrow$



σ with no $ki \dots j$ pattern
$=$ a c-sortable element

A sylvester class

A binary search tree

A binary tree

The map from binary trees to c-sortable elements can be directly obtained with a postfix reading of the associated binary search tree.

What I do

Reading's map nc_c : $\{c$-sortable elements$\} \to \{c$-non crossing partitions$\}$ is well defined for any (finite rank) Coxeter group, and is a bijection in all finite Coxeter groups.

In infinite Coxeter groups, it is only injective, but never surjective. For example, in type $\widetilde{A}_1$ and $c = st$, all reflections are c-non crossing but all the following ones are not in the image of nc_c : tst, $tstst$, $tststst$, ...

My goal is to define a generalized notion of c-sortable elements, at least in the affine types, such that Reading's map can be naturally extended to be a bijection.

Thank you for your attention!

Further questions

- What about other Coxeter elements? $\longrightarrow$ another tree structure : Cambrian trees [CP17].

Further questions

- What about other Coxeter elements? $\longrightarrow$ another tree structure : Cambrian trees [CP17].

- What about other finite Coxeter groups? $\longrightarrow$ W-Catalan numbers [Rei97].

Further questions

- What about other COXETER elements? $\longrightarrow$ another tree structure : Cambrian trees [CP17].
- What about other finite COXETER groups? $\longrightarrow$ W-CATALAN numbers [Rei97].
- What about infinite COXETER groups? $\longrightarrow$ It's complicated... Non crossing partitions are well defined (in a more algebraic way) as well as READING bijection, but it never is surjective. In the affine type, there's a hope to generalize the c-sortable elements to a larger family that becomes in bijection with the non-crossing partitions $\longrightarrow$ my thesis.

Grégory Chatel and Vincent Pilaud.
Cambrian hopf algebras.
Advances in Mathematics, 311:598–633, April 2017.

Thomas Gobet.
Dual garside structures and coxeter sortable elements, 2018.

F. Hivert, J. C. Novelli, and J. Y. Thibon.
The algebra of binary search trees, 2004.

Nathan Reading.
Clusters, coxeter-sortable elements and noncrossing partitions, 2005.

Victor Reiner.
Non-crossing partitions for classical reflection groups.
Discrete Mathematics, 177(1):195–222, 1997.