On the hard-core lattice gas model

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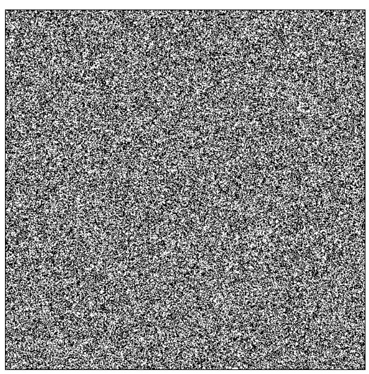
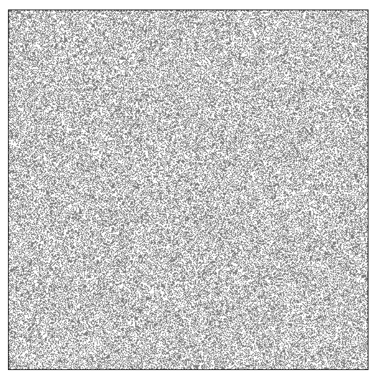
Introduction



(a) Ising model : About 1 over 2 sites are occupied.



(b) Hard-core model : About 227 over 1000 sites are occupied.

Figure 1: Typical behaviour for similar parameters of the Ising model (on the left) and of the hard-core model (on the right) on $[\![1, 500]\!]^2$ as a subgraph of $\mathbb{Z}^2$.

In statistical physics, the Ising-Lenz model (or Ising model, the reader can find a complete course on this subject in [FV17]) is the most common toy model used in order to model the behaviour of physics systems. For instance, we can define thanks to it the lattice gas model in which the particles of a gas can only place themselves on the nodes of a lattice and the probability that a particle places itself on a node depends on the fact that they are or not particles on the neighbouring nodes.

The fact that it is a Isinz-Lenz model gives us results on the behaviour of the gaz at high temperatures or low temperatures and on when phase transition takes place. However even though this model is very useful, we can ask ourselves what could possibly happens if the particles are hard and cannot allow having a neighbour.

To model this behaviour, we will define the hard-core lattice gas model (usually called the hard-core model) and the goal of this paper is to prove the results of the paper of Van den Berg and Steif [BS94] that is the existence of the model in an infinite graph and results on phase transition. We will often do comparison between the Ising-Lenz model and the hard-core model in order to enlighten some results.

In section 1, we will define the model by giving in subsection 1.1 some preleminary but necessary notions in order to define in subsection 1.2 the hard-core model in the finite graph case and in subsection 1.3 the hard-core model in the infinite graph case. Moreover, in subsection 1.4 we will see how to simulate the hard-core model on a computer thanks to a Markov Chain Monte Carlo method.

In section 2, we will prove necessary properties mostly given by percolation to prove the existence of the hard-core model in the infinite graph case and to prove criteria for uniqueness of measure for the hard-core model that will be proven in section 3.

Notations

In this whole paper, we will use the following notations :

- we denote $\mathbb{N}$ the set of non-negative integers, $\mathbb{R}_+^*$ the set of positive real numbers ;
- for all set X, we denote by $A \Subset X$ the fact that A is a finite subset of X ;
- for all sets X and Y, for all $u \in Y^X$, for all $x \in X$ and $A \subset X$, we denote u_x the element of u indexed by x and $u_A := (u_a)_{a \in A} \in Y^A$;
- for all sets X and Y, for all $A \subset X$, for all $u \in Y^A$, for all $v \in Y^{A^c}$, we denote uv the element $w \in Y^X$ such that for all $x \in A$, $w_a = u_a$ and for all $x \in A^c$, $w_a = v_a$.
- for $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, we denote the expectancy under $\mathbb{P}$ by $\langle \cdot \rangle_{\mathbb{P}}$ and the covariance under $\mathbb{P}$ by $\mathrm{Cov}_{\mathbb{P}}(\cdot)$.

Contents

1 Mathematical formulation of the hard-core model

The goal of this section is to define mathematically the hard-core model. In subsection 1.1, we will briefly recall some notions and define some objects that will be useful in order to model correctly the behaviour of such a hard-core lattice gas. In subsection 1.2 and subsection 1.3, we will define the hard-core model in the finite graph case and the infinite graph case respectively and give some properties in both cases. An important step in order to understand how the model works is to simulate it, we will explain in subsection 1.4 how to do so.

1.1 Some preliminary notions and notations

In this subsection, even though we suppose that the reader has basic notions of graph theory, we will fix some notations and define some notions in definition 1.1. In the whole paper, when we say graph, we mean *undirected* graph.

Definition 1.1 — Let $G = (V, E)$ be a graph.

- For all $v, w \in V$, we say that v and w are *adjacent* or *neighbours* and we denote it $v \sim w$ if and only if there is an edge between v and w, we call the set $N(v) = \{w \in V : v \sim w\}$ the *neighbourhood of* v and the *degree of* v is $d(v) = |N(v)|$. We say that G is *locally finite* if and only if for all $v \in V$, $d(v)$ is finite.
- For $\Lambda \subset V$, we denote $N(\Lambda) = \bigcup_{v \in \Lambda} N(v)$ the *neighbourhood of* Λ, $\overline{\Lambda} = \Lambda \cup N(\Lambda)$, and $\partial\Lambda = \overline{\Lambda} \setminus \Lambda$ the *boundary of* Λ.
- We say that G is *connected* if and only if for all $v, w \in V$, there is a path between v and w. We say that a path Π on G is *infinite* if and only if it goes through an infinite number of vertices.

Remark. For $A \subset V$, we will often mix up the subgraph of G induced by A and the set of vertices A.

The definition 1.2 is here to define essential notions for the hard-core model thanks to which we can do the exact definition of the model.

Definition 1.2 — Let $G = (V, E)$ be a finite or countably infinite, locally finite, connected graph.

- We call Ω_G the *set of configurations* the set defined by $\Omega_G = \{0, 1\}^G$.
- For all $\omega \in \Omega_G$, we say that ω is *feasible* if and only if ω has no two adjacent 1s (which also means that for all $v \in V$ and for all w neighbour of v, $\omega_v \omega_w = 0$).
- We call F_G the *set of feasible configurations* the set defined by :

$$\begin{aligned} F_G = \{\omega \in \Omega_G : \omega \text{ is feasible}\} &= \{\omega \in \Omega_G : \forall v \in V, \forall w \in N(v), \omega_v = 1 \Rightarrow \omega_w = 0\} \\ &= \{\omega \in \Omega_G : \forall v, w \in V, v \sim w \Rightarrow \omega_v \omega_w = 0\}. \end{aligned}$$

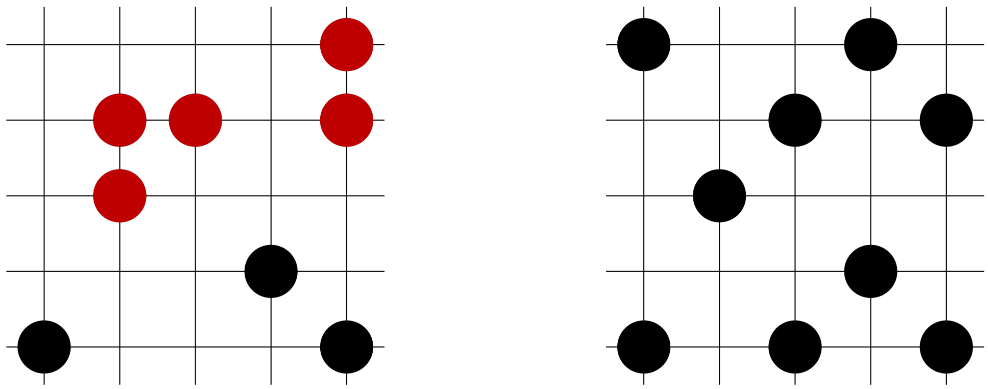


Figure 2: On the left, the configuration is non-feasible because the red balls are close ; on the right, it is a feasible configuration.

A configuration represents the placement of the particles on the graph : 1 if a particle is present on the vertex and 0 otherwise. Then a configuration is feasible if no two particles are neighbours.

Now that we have defined all that was necessary we can define the hard-core model on finite graphs (see subsection 1.2) or infinite graphs (see subsection 1.3).

1.2 The hard-core model on a finite graph

Let us firstly define the hard-core model on a finite graph in definition 1.3.

Definition 1.3 — Let $G = (V, E)$ be a finite connected graph and let $\mathbf{a} = (\mathbf{a}_v)_{v \in V}$ be a sequence of positive real numbers.

The *hard-core measure for G with activity* $\mathbf{a}$ is defined as the following probability measure $\mu_{G,\mathbf{a}}$ on Ω_G equipped with the discrete σ-algebra $\mathcal{P}(\Omega_G)$:

$$\begin{aligned} \forall \omega \in \Omega_G, \quad \mu_{G,\mathbf{a}}(\{\omega\}) &= \begin{cases} \frac{1}{Z_{G,\mathbf{a}}} \exp(-H_{G,\mathbf{a}}(\omega)) & \text{, if } \omega \text{ is feasible ;} \\ 0 & \text{, otherwise.} \end{cases} \\ &= \frac{1}{Z_{G,\mathbf{a}}} \mathbb{1}_F(\omega) \exp(-H_{G,\mathbf{a}}(\omega)), \end{aligned}$$

where $H_{G,\mathbf{a}}(\omega) = -\sum_{v \in V} \ln(\mathbf{a}_v)\omega_v$ is the *Hamilton function* of the system, $Z_{G,a} = \sum_{\omega \in F} \exp(-H_{G,\mathbf{a}}(\omega))$ is the normalizing constant also called the *partition function of the system* and F_G is the set of all feasible configurations.

Proof. We need to show that $\mu_{G,\mathbf{a}}$ is a probability measure which is really easy because Ω_G is finite and

$$\sum_{\omega \in \Omega_G} \frac{1}{Z_{G,\mathbf{a}}} \mathbb{1}_F(\omega) \exp(-H_{G,\mathbf{a}}(\omega)) = 1. \qquad \square$$

Remarks. For $\omega \in \Omega_G$, we will write $\mu_{G,\mathbf{a}}(\omega)$ instead of $\mu_{G,\mathbf{a}}(\{\omega\})$ and we can observe that

$$\mu_{G,\mathbf{a}}(\omega) = \frac{1}{Z_{G,\mathbf{a}}} \prod_{v \in V} \mathbf{a}_v^{\omega_v}.$$

If $\mathbf{a}$ is equal to $(a)_{v \in V}$ for $a > 0$, we will write a instead of $\mathbf{a}$ in all the notations and, in that case, for all $\omega \in \Omega_G$, $\mu_{G,a}(\omega) = \frac{1}{Z_{G,a}} a^{\sum_{v \in V} \omega_v}$.

Remark. When we compare to the Ising-Lenz model, we can wonder where appear the influence of the neighbours that is due to the product of the neighbouring values. However, the fact that only the feasible configurations are taken into account is linked to the fact that the hamiltonian could be written

$$H(\omega) = \infty \sum_{v \sim w} \omega_v \omega_w - \sum_{v \in V} \ln(\mathbf{a}_v) \omega_v,$$

where we have the convention $\infty \cdot 0 = 0$ and $\exp(-\infty) = 0$. In that case, the indicator function is not useful anymore.

Let us see how this model behaves on a particular case given by the graph of the figure 3 with activity a, we can wonder what the set of all non-feasible configurations F_G is. What is its cardinality ? What are the probabilities of the feasible configurations ? What happens if $a = 1$? if a is much smaller than 1 ? if a is much larger than 1 ?

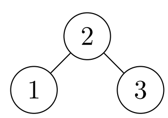


Figure 3: A finite connected graph of three nodes and two edges.

Let us answer these questions. The set of all feasible configurations is

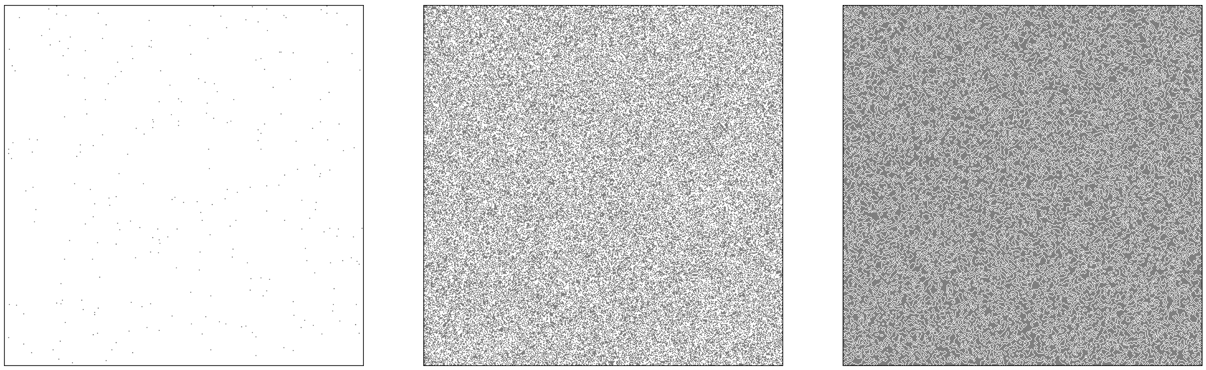
$$F_G = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,0,1)\},$$

it has a cardinality of 5 while Ω_G has a cardinality of 8. The probabilities of each configuration are put in the tabular of the figure 4, for the general case, for $a = 1$, for $a \ll 1$ and for $a \gg 1$.

$\omega \in F_G$	$\mu_{G,a}$ for $a > 0$	$\mu_{G,a}$ for $a = 1$	$\mu_{G,a}$ for $a \ll 1$	$\mu_{G,a}$ for $a \gg 1$
$(0,0,0)$	$\frac{1}{1+3a+a^2}$	$\frac{1}{5}$	$\simeq 1$	$\simeq 0$
$(1,0,0)$	$\frac{a}{1+3a+a^2}$	$\frac{1}{5}$	$\simeq 0$	$\simeq 0$
$(0,1,0)$	$\frac{a}{1+3a+a^2}$	$\frac{1}{5}$	$\simeq 0$	$\simeq 0$
$(0,0,1)$	$\frac{a}{1+3a+a^2}$	$\frac{1}{5}$	$\simeq 0$	$\simeq 0$
$(1,0,1)$	$\frac{a^2}{1+3a+a^2}$	$\frac{1}{5}$	$\simeq 0$	$\simeq 1$

Figure 4: The probabilities of the feasible configurations for the hard-core model on the graph of figure 3.

We can easily observe here that if $a = 1$, $\mu_{G,1}$ is the uniform distribution on F_G (it is the case for any finite connected graph), that if $a \gg 1$ we will have a maximum number of particles on the graph and that if $a \ll 1$ we will have a minimum number of particles on the graph. This behaviour is a general one as you can see on the figure 5 on a subgraph of $\mathbb{Z}^2$.



(a) $a = 0.001$: About 1 over 1000 sites are occupied.

(b) $a = 1.0$: About 227 over 1000 sites are occupied.

(c) $a = 1000.0$: About 365 over 1000 sites are occupied.

Figure 5: Three simulations of the behaviour of the hard-core model on $[\![1, 500]\!]^2$ as a subgraph of $\mathbb{Z}^2$ with different activities.

We will explain how to get the simulations of the figure 5 in the subsection 1.4. For now, let us prove some properties of the probability measure following the hard-core model. Firstly, let us recall the definition of Markov fields in definition 1.4 before proceeding with proposition 1.5.

Definition 1.4 — For all finite connected graph $G = (V, E)$, for all probability measures μ on Ω_G, we say that μ is a *Markov field* if and only if for all $v \in V$, for all $\eta \in \Omega_G$ such that $\mu(\{\omega \in \Omega_G : \omega_{\{v\}^c} = \eta_{\{v\}^c}\}) \neq 0$,

$$\mu(\cdot|\{\omega \in \Omega_G : \omega_{\{v\}^c} = \eta_{\{v\}^c}\}) = \mu(\cdot|\{\omega \in \Omega_G : \omega_{N(v)} = \eta_{N(v)}\}).$$

The Markov field property is an extension of the Markov chain property ("The probability of moving to the next state depends only on the present state and not on the previous states"). Indeed the probability that a vertex takes a certain value only depends on the values of the adjacent vertices.

Proposition 1.5 — For all finite connected graph $G = (V, E)$ and for all $\mathbf{a} = (\mathbf{a}_v)_{v \in V} \in (\mathbb{R}_+^*)^V$, $\mu_{G,\mathbf{a}}$ is a Markov field.

Proof. Let $G = (V, E)$ be a finite connected graph and let $\mathbf{a} \in (\mathbb{R}_+^*)^V$.

Let $v \in V$ and let $\eta \in \Omega_G$ such that $\mu(\{\omega \in \Omega_G : \omega_{\{v\}^c} = \eta_{\{v\}^c}\}) \neq 0$. Let $\omega' \in \Omega_G$,

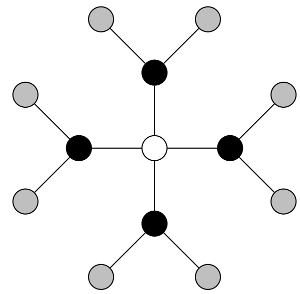


Figure 6: Illustration of the Markov field property : the value of the white vertex knowing the values of the other vertices only depends on the values of the black vertices and not the grey vertices.

$$\mu(\{\omega'\}|\{\omega \in \Omega_G : \omega_{\{v\}^c} = \eta_{\{v\}^c}\}) = \frac{\mathbf{a}_v \mathbb{1}_{F_G}(1_{\{v\}}\eta_{\{v\}^c})\delta_{1_{\{v\}}\eta_{\{v\}^c}}(\{\omega'\}) + \mathbb{1}_{F_G}(0_{\{v\}}\eta_{\{v\}^c})\delta_{0_{\{v\}}\eta_{\{v\}^c}}(\{\omega'\})}{\mathbf{a}_v \mathbb{1}_{F_G}(1_{\{v\}}\eta_{\{v\}^c}) + \mathbb{1}_{F_G}(0_{\{v\}}\eta_{\{v\}^c})}$$
$$= \mu(\{\omega'\}|\{\omega \in \Omega_G : \omega_{N(v)} = \eta_{N(v)}\}). \qquad \square$$

This property allows us to extend in a natural way the measure in the infinite graph case.

1.3 The hard-core model on an infinite graph

We want to model the same behaviour as in the finite graph case on a countably infinite, locally finite, connected graph $G = (V, E)$ with activity $\mathbf{a} \in (\mathbb{R}_+^*)^V$. The problem is the following : we cannot do the exact same model as previously because we would have to deal with infinite computation.

In order to model correctly this behaviour, we will need to use the notion of Gibbs measures (for more information on this matter, see [Geo11]) which appears as a natural extension of the finite graph case thanks to the fact that the previously defined probability measures are Markov fields. In our case, we will specifize a class of conditional probabilities that will caracterize the hard-core model. To do so, we will use the subgraphs of our graph and fix on the outside of the subgraphs exterior conditions as illustrated in figure 7.

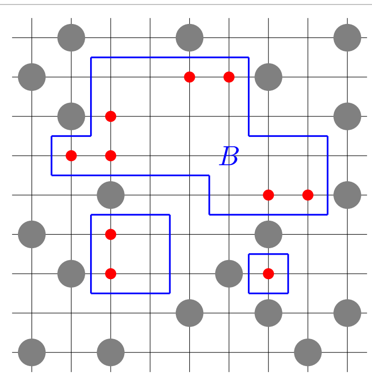


Figure 7: Representation of a condition for the infinite graph : the nodes in the area within the blue border are the vertices of B, the grey disks outside of B represent the exterior condition and the red dots correspond to the sites which cannot have particles because of the exterior condition.

Firstly, before defining the hard-core model on G, we need to make of Ω_G a measurable set. In order to do so, we will use the σ-algebra defined in definition 1.6.

Definition 1.6 — Let X and Y be two sets, we call the *obvious σ-algebra of X^Y* the σ-algebra generated by the sets $\{u \in X^Y : u_y = x\}$ for all $x \in X$ and $y \in Y$.

Now we can define the probability measure for the hard-core model in definition 1.7.

Definition 1.7 — Let $G = (V, E)$ be a countably infinite, locally finite, connected graph and let $\mathbf{a} = (\mathbf{a}_v)_{v \in V}$ be a sequence of real positive numbers.

We say that a probability measure μ on Ω_G equipped with the obvious σ-algebra is a *hard-core measure for G with activity* $\mathbf{a}$ if and only if for all $B \Subset V$, $\eta \in F_G$ and $\alpha = \{0,1\}^B$:

$$\begin{aligned}\mu(\omega_B = \alpha_B | \omega_{B^c} = \eta_{B^c}) &= \begin{cases} \frac{1}{Z^{\eta}_{B,\mathbf{a}}} \exp(-H^{\eta}_{B,\mathbf{a}}(\alpha)) & \text{, if } \alpha_B \eta_{B^c} \text{ is feasible ;} \\ 0 & \text{, otherwise.} \end{cases} \\ &= \frac{1}{Z^{\eta}_{B,\mathbf{a}}} \mathbb{1}_{F_G}(\alpha_B \eta_{B^c}) \exp(-H^{\eta}_{B,\mathbf{a}}(\alpha)),\end{aligned}$$

where $H^{\eta}_{B,\mathbf{a}}(\alpha) = -\sum_{v \in B} \ln(\mathbf{a}_v)\alpha_v$ is the Hamilton function of the system, $Z^{\eta}_{B,\mathbf{a}} = \sum_{\substack{\omega \in F_G \\ \omega_{B^c} = \eta_{B^c}}} \exp(H^{\eta}_{B,\mathbf{a}}(\omega))$ is the *partition function of the system* and F_G is the set of all feasible configurations.

Remark. We can observe that the hard-core model in the finite graph case also respect the condition in definition 1.7, that explains why it is a natural extension of the finite graph case.

The previous definition gives a specification of conditional probability measures and that is why a hard-core measure for G with activity $\mathbf{a}$ is a Gibbs measure. In such a case, two questions arise : Does such a measure exists ? On which condition(s) is there uniqueness ?

In the most general case, the first question is answered thanks to an argument of Georgii (see the chapter IV of [Geo11] especially p.400-401 where Georgii explains how to use the tools developped in the book to prove the existence of the hard-core model in the general case), however we cannot conclude for uniqueness.

In order to do so, we will study a special type of graph : the class of bipartite graphs. We will show in the section 3 results on the existence of at least one Gibbs measure for the hard-core model in the bipartite case and then show criteria for uniqueness.

1.4 Simulation of the hard-core model on a finite graph

In order to get a better understanding of how the hard-core model works on $G = (V, E)$ with activity $\mathbf{a}$, it can be useful to know how to simulate it. At first, we can think of constructing the set of feasible configurations F_G and then picking a configuration in it with respect to the probability measure for the hard-core model. However, in most cases, the order of magnitude of $|F_G|$ is exponential in $|V|$: for large $|V|$, the algorithm will have a too long time of execution. That is why we use a Markov Chain Monte Carlo method to simulate our model.

Before proceeding with the explanation of the methods, we will need to get some definitions and results around the Markov chains that we can find in the book of Häggström (see [Häg+02]) with their proofs which will not be given here.

First we need to define a topology on the probability measures with the total variation distance in definition 1.8.

Definition 1.8 — If ν and ν' are two probability distributions on $S = \{s_1, \dots, s_k\}$, then we define the *total variation distance* between ν and ν' as

$$d_{TV}(\nu, \nu') = \frac{1}{2} \sum_{i=1}^{k} |\nu(\{s_i\}) - \nu'(\{s_i\})| .$$

If $(\nu_n)_{n \in \mathbb{N}}$ is a sequence of probability measures on S and ν is a probability measure on S, then we say that $(\nu_n)_{n \in \mathbb{N}}$ *converges to ν in total variation as $n \to \infty$*, writing $\nu_n \overset{TV}{\to} \nu$, if and only if

$$\lim_{n \to \infty} d_{TV}(\nu_n, \nu) = 0.$$

We now recall the definition of a Markov chain and definitions useful for Markov chains in definition 1.9.

Definition 1.9 — Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $k \in \mathbb{N}^*$, let $P = (P_{i,j})_{i,j \in [\![1,k]\!]}$ be a matrix of $\mathcal{M}_k(\mathbb{R})$ and let $S = \{s_1, \dots, s_k\}$ be a finite set. Let π be a probability measure on S and $\tilde{\pi} = (\pi(\{s_1\}), \dots, \pi(\{s_k\}))$.

- A random process $(X_l)_{l \in \mathbb{N}}$ with finite state space S is said to be a *Markov chain with transition matrix P* if and only if for all $n \in \mathbb{N}$, for all $i, j \in [\![1, k]\!]$, for all $i_o, \dots, i_{n-1} \in [\![1, k]\!]$, we have

$$\mathbb{P}(X_{n+1} = s_j | X_0 = s_{i_0}, \dots, X_n = s_{i_n}) = \mathbb{P}(X_{n+1} = s_j | X_n = s_{i_n}) = P_{i,j}.$$

- A Markov chain $(X_l)_{l \in \mathbb{N}}$ with state space S and transition matrix P is said to be *irreducible* if and only if for all $s_i, s_j \in S$, we have $(P^n)_{i,j} > 0$. Otherwise, the chain is said to be *reducible*.

- A Markov chain $(X_l)_{l \in \mathbb{N}}$ with state space S and transition matrix P is said to be *aperiodic* if and only if for all $s_i \in S$, we have that $d(s_i) := \gcd\{n \geq 1 : (P^n)_{i,i} > 0\}$ is equal to 1. Otherwise, the chain is said to be *periodic*.

- Let $(X_l)_{l \in \mathbb{N}}$ be a Markov chain with state space S and transition matrix P. π is said to be a *stationary distribution* for the Markov chain if and only if $\tilde{\pi} P = \tilde{\pi}$ *i.e.* for $j \in [\![1, k]\!]$, $\sum_{i=1}^{k} \pi(\{s_i\}) P_{i,j} = \pi(\{s_j\})$.

- Let $(X_l)_{l\in\mathbb{N}}$ be a Markov chain with state space S and transition matrix P. π is said to be a *reversible* for the chain (or for the transition matrix P) if and only if for all $i, j \in [\![1, k]\!]$, we have $\pi(\{s_i\})P_{i,j} = \pi(\{s_j\})P_{j,i}$. The Markov chain is said to be *reversible* if and only if ther exists a reversible distribution for it.

The following results explain the need of the previous definitions in order to have a way to simulate the hard-core model.

Theorem 1.10 (The Markov chain convergence theorem) — Let $(X_l)_{l\in\mathbb{N}}$ be an irreducible aperiodic Markov chain with state space $S = \{s_1, \dots, s_k\}$ and transition matrix P. Then for any distribution π which is stationary for the transition matrix P, we have that the sequence $(\nu_l)_{l\in\mathbb{N}}$ of distributions of the Markov chain converges to π in total variation as $n \to \infty$.

With theorem 1.10, if we construct an irreducible aperiodic Markov chain with a stationary distribution, we can have a simulation of the stationary distribution by simulating the Markov chain long enough.

Theorem 1.11 (Uniquenes of the stationary distribution) — Any irreducible and aperiodic Markov chain has exactly one stationary distribution.

With theorem 1.11, we are sure to get only one distribution by simulating the Markov chain.

Theorem 1.12 — Let $(X_l)_{l\in\mathbb{N}}$ be a Markov chain with state space $S = \{s_1, \dots, s_k\}$ and transition matrix P. If π is a reversible distribution for the chain, then it is also a stationary distribution for the chain.

With theorem 1.12, we have another way to prove that a distribution is stationary.
Now that we have the needed tools in order to simulate the hard-core model, let us prove proposition 1.13.

Proposition 1.13 — Let $G = (V, E)$ be a finite connected graph and let $\mathbf{a} = (\mathbf{a}_v)_{v\in V}$.
Let us define the random process $(X^n)_{n\in\mathbb{N}}$ by :

- $X^0 = (0)_{v\in V}$;
- For each $n \geq 0$, X^{n+1} is defined by :
  1. Pick in V a vertex v at random with uniform distribution ;
  2. Draw a coin with probability $\mathbf{p}_v = \frac{\mathbf{a}_v}{1+\mathbf{a}_v}$ to have heads :
     - If the result is heads and all the neighbouring vertices of v are of value 0, $X_v^{n+1} = 1$;
     - Else, $X_v^{n+1} = 0$;
  3. For all vertices $w \neq v$, $X_w^{n+1} = X_w^n$.

Then (X^n) is a Markov chain, it is irreducible and aperiodic, $\mu_{G,\mathbf{a}}$ is reversible for the chain.
Hence, the distribution of the Markov chain converges to the hard-core measure for G with activity $\mathbf{a}$.

Proof. It is clear that (X^n) is a Markov chain. It is irreducible because there is a positive probability to go from one feasible configuration to another because for each feasible configuration ω with k 1s, there is a positive probability that we go in k steps to $(0_v)_{v\in V}$ and there is a positive probability that we go from $(0_v)_{v\in V}$ in k steps to ω. It is aperiodic because for all feasible configurations ω, there is a positive probability that we stay at ω if we were at ω at the step n.

Let us prove that $\mu_{G,\mathbf{a}}$ is reversible for the chain. Let us denote ω and ω' two feasible configurations and $P_{\omega,\omega'}$ (resp. $P_{\omega',\omega}$) the transition probability from ω to ω' (resp. from ω' to ω).

Let us denote $d = |\{v \in V : \omega_v \neq \omega'_v\}|$. We want to prove that $\mu_{G,a}(\omega)P_{\omega,\omega'} = \mu_{G,a}(\omega')P_{\omega',\omega}$.

- If $d = 0$, the equality is trivial.
- If $d \geq 2$, $P_{\omega,\omega'} = P_{\omega',\omega} = 0$ so the equality is trivial.
- If $d = 1$, there exists exactly on $v \in V$ such that $\omega_v \neq \omega'_v$. In that case, for all $w \in N(v)$, $\omega_v = \omega'_v = 0$. Let us suppose that $\omega_v = 1$, in that case

$$\mu_{G,\mathbf{a}}(\omega)\cdot P_{\omega,\omega'} = \frac{1}{Z_{G,\mathbf{a}}}\left(\prod_{w\neq v}\mathbf{a}_w^{\omega_v}\right)\mathbf{a}_v^1\cdot\frac{1}{|V|}\frac{1}{1+\mathbf{a}_v} \text{ and } \mu_{G,\mathbf{a}}(\omega')\cdot P_{\omega',\omega} = \frac{1}{Z_{G,\mathbf{a}}}\left(\prod_{w\neq v}\mathbf{a}_w^{\omega'_v}\right)\mathbf{a}_v^0\cdot\frac{1}{|V|}\frac{\mathbf{a}_v}{1+\mathbf{a}_v},$$

  hence $\mu_{G,a}(\omega)P_{\omega,\omega'} = \mu_{G,a}(\omega')P_{\omega',\omega}$.

We proved that $\mu_{G,\mathbf{a}}$ is reversible for the chain.

Thanks to the Markov chain convergence theorem, the uniqueness of the stationary distribution and theorem 1.12, we have the last conclusion. □

The consequence of proposition 1.13 is the fact that in order to simulate the hard-core model, we only need to simulate this Markov chain during a long enough time. The algorithm in appendix A automatically generates a jpeg image of a simulation of the hard-core model thanks to that method and was used to make the figure 5.

2 General results

Before proceeding with the hard-core model, we will need some tools of the theory of percolation.

In subsection 2.1, we will define what is percolation is and give some properties useful for the rest of the paper.

In subsection 2.2, we will prove some sufficient conditions for the uniqueness of Gibbs measures.

2.1 Percolation

The goal of this subsection is to define what percolation is and to give some results of percolation.

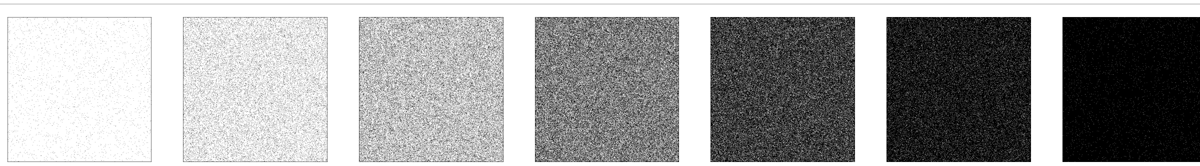
Definition 2.1 — Let $G = (V, E)$ be a finite or countably infinite, locally finite, connected graph and let $\mathbf{p} = (\mathbf{p}_v)_{v \in V}$ be a sequence of real numbers in $[0, 1]$. Let $\mathbb{P}_{G,\mathbf{p}}$ be the probability measure such that for a realisation ω under $\mathbb{P}_{G,\mathbf{p}}$, the ω_v are independant random variables and for all $v \in V$, ω_v follows the Bernouilli distribution with parameter $\mathbf{p}_v$.

We say that the vertex v is *open* (reps. *closed*) if and only if $\omega_v = 1$ (resp. $\omega_v = 0$). We say that a path on the graph is *open* if and only if all its vertices are open.

We say that *(independent site) percolation occurs* for G and $\mathbf{p}$ if and only if

$$\mathbb{P}_{G,\mathbf{p}}(\text{"There exists an infinite open path"}) > 0.$$

Remarks. In the case that all $\mathbf{p}_v$ are equal to $p \in [0, 1]$, we will write p instead of $\mathbf{p}$ in the previous notations. For all G finite connected graph, percolation never occurs. For all G countably infinite, locally finite, connected graph, if $\mathbb{P}_{G,\mathbf{p}}(\text{"There exists an infinite open path"}) > 0$, then, as "There exists an infinite open path" is a tail event, $\mathbb{P}_{G,\mathbf{p}}(\text{"There exists an infinite open path"}) = 1$.



(a) p = 0.01 (b) p = 0.1 (c) p = 0.25 (d) p = 0.5 (e) p = 0.75 (f) p = 0.9 (g) p = 0.99

Figure 8: Simulations of the behaviour of the $\mathbb{P}_p$ for different values of p

In definition 1.3, we defined the hard-core model on all the elementary events. However, we can define it in a different way thanks to percolation as we can see in 2.2.

Proposition 2.2 — Let $G = (V, E)$ be a finite connected graph and let $\mathbf{a} = (\mathbf{a}_v)_{v \in V}$ be a sequence of positive real numbers. Let $\mathbf{p}$ be defined as $\mathbf{p} = (\mathbf{p}_v)_{v \in V} = (\frac{\mathbf{a}_v}{1+\mathbf{a}_v})_{v \in V}$. Then $\mu_{G,\mathbf{a}}(\cdot) = \mathbb{P}_{G,\mathbf{p}}(\cdot\ |F_G)$ where F_G is the set of feasible configurations.

Proof. For all $\omega' \in \Omega_G$,

$$\mathbb{P}_{G,\mathbf{p}}(\{\omega'\}|F_G) = \mathbb{1}_{F_G}(\omega')\frac{\mathbb{P}_{G,\mathbf{p}}(\{\omega'\})}{\sum_{\omega\in F_G}\mathbb{P}_{G,\mathbf{p}}(\{\omega\})} = \mathbb{1}_{F_G}(\omega')\frac{\prod_{v\in V}\frac{\mathbf{a}_v^{\omega'_v}}{1+\mathbf{a}_v}}{\sum_{\omega\in F_G}\prod_{v\in V}\frac{\mathbf{a}_v^{\omega_v}}{1+\mathbf{a}_v}}$$
$$= \mathbb{1}_{F_G}(\omega')\frac{\prod_{v\in V}\mathbf{a}_v^{\omega'_v}}{\sum_{\omega\in F_G}\prod_{v\in V}\mathbf{a}_v^{\omega_v}} = \mu_{G,\mathbf{a}}(\omega'). \qquad \square$$

An important notion of percolation is the notion of critical probability defined in definition 2.3.

Definition 2.3 — Let $G = (V, E)$ be a countably infinite, locally finite, connected graph. Let us define *the critical probability for the graph G p^c_G* by :

$$p^c_G = \inf_{p\in[0,1]}\{\text{Percolation occurs for } G \text{ and } p\}.$$

Remark. One the main first goals of the theory of percolation was to prove that for a large class of graphs $p^c_G < 1$.

Similarly, we can define a critical activity for the hard-core model.

Definition 2.4 — Let $G = (V, E)$ be a countably infinite, locally finite, connected graph. Let us define *the critical activity for the graph G a^c_G* by :

$$a^c_G = \inf_{a>0}\{\text{The hard-core model for } G \text{ with activity } a \text{ has more than one Gibbs measure}\}.$$

If we define this notion, it's because we would like for $G = \mathbb{Z}^d$ $(d \geq 1)$ to separate the case of phase transition and ergodicity (the fact that there is no phase transition) by a simple criterion such as : if $a < a^c_{\mathbb{Z}^d}$, there is ergodicity. The problem is that, unless I am mistaken, that this is still an open question in $\mathbb{Z}^d$ to the contrary of the Ising model.

2.2 Uniqueness of Gibbs measures sufficient condition

We will present in this subsection some criteria on uniqueness of Gibbs measures. First we need definitions 2.5 and 2.6.

Definition 2.5 — Let $G = (V, E)$ be a graph, let S be a set, let $\omega, \omega' \in S^V$ let Π be a path on G, we say that Π *is a path of disagreement for* (ω, ω') if and only that for each vertices v in the path Π, $\omega_v \neq \omega'_v$.

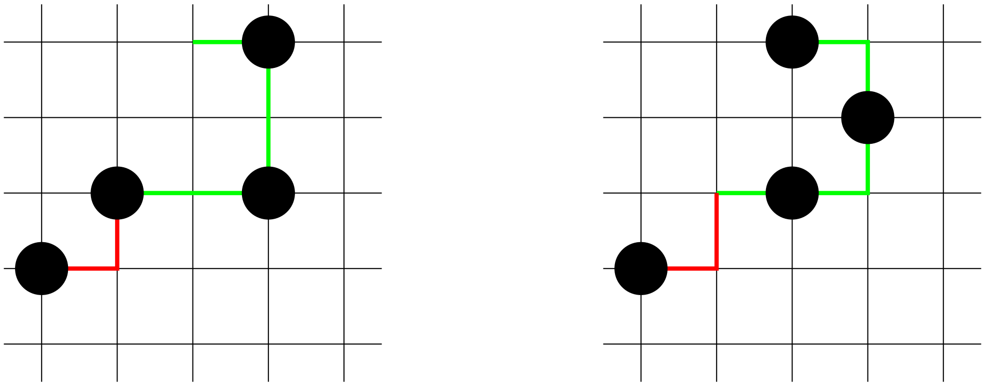


Figure 9: The green path is a path of disagreement for both configurations while the red one is not.

Definition 2.6 — Let $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ be two mesurable spaces and let $T : X_1 \to X_2$ be a measurable function. We say that T is a *measure preserving transformation* if and only if

$$\forall A \in \mathcal{B}_2, \mu_1 \left(\overset{-1}{T}(A) \right) = \mu_2(A).$$

Now we can prove theorem 2.7.

Theorem 2.7 ([Ber93]) — Let $G = (V, E)$ be a countably infinite graph, locally finite, connected graph, let S be a finite or countably infinite set and let $\Omega = S^V$. Let the probability measures μ and μ' on Ω equipped with the obvious σ-algebra be Markov fields with the same specification *i.e.* the same conditionnal probabilities. Consider two independent realisations ω under μ and ω' under μ'. If $(\mu \otimes \mu')("(\omega, \omega')$ has an infinite path of disagreement"$) = 0$, then $\mu = \mu'$.

Proof. Let us suppose that $(\mu \otimes \mu')("(\omega, \omega')$ has an infinite path of disagreement"$) = 0$. Let $A \Subset V$ and for all $a \in A$, $s_a \in S$. Let E be the event $\{\omega \in S^V : \forall a \in A, \omega_a = s_a\}$. We want to prove that $\mu(E) = \mu'(E)$.

For each pair $(\omega, \omega') \in \Omega \times \Omega$, we denote C_A the cluster of disagreement containing A *i.e.*

$$C_A = A \cup \{i \in V : \text{There is path of disagreement between } i \text{ and a vertex of } \partial A\}.$$

Because of the assumption, C_A is finite with probability 1.

Let $T : \Omega \times \Omega \to \Omega \times \Omega$ such that for each pair $(\omega, \omega') \in \Omega \times \Omega$, if we denote $(\sigma, \sigma') = T(\omega, \omega')$, $\sigma_{C_A} = \omega'_{C_A}$, $\sigma_{C_A^c} = \omega_{C_A^c}$, $\sigma'_{C_A} = \omega_{C_A}$ and $\sigma'_{C_A^c} = \omega'_{C_A^c}$. T is bijective and because μ and μ' are Markov fields with the same specification, T is measure preserving.

Hence, since E involves only vertices in $A \subset C_A$, we have

$$\mu(E) = (\mu \otimes \mu')(E \times \Omega) = (\mu \otimes \mu')(T(E \times \Omega)) = (\mu \otimes \mu')(\Omega \times E) = \mu'(E). \qquad \square$$

One of the consequences of theorem 2.7 is corollary 2.8.

Corollary 2.8 ([BS94]) — Let $G = (V, E)$ be a countably infinite graph, locally finite, connected graph, let S be a finite or countably infinite set and let $\Omega = S^V$. Let the probability measures μ and μ' on Ω equipped with the obvious σ-algebra be Markov fields with the same specification. Consider two independent realisations ω under μ and ω' under μ'. Let us define for each $v \in V$

$$\mathbf{p}_v = \sup_{\substack{\alpha,\alpha' \in S^{N(v)} \\ \alpha \neq \alpha'}} (\mu \otimes \mu')(\omega_v \neq \omega'_v | \omega_{N(v)} = \alpha_{N(v)}, \omega'_{N(v)} = \alpha'_{N(v)}).$$

If $\mathbb{P}_{\mathbf{p}}(\text{"There exists an infinite open path"}) = 0$, then $\mu = \mu'$.

Proof. Let for all $v \in V$, $\mathcal{J}_v$ be the σ-algebra generated by the random variables ω_w and ω'_w for all $w \in V \setminus \{v\}$ and let $O \in V$. Since μ and μ' are Markov fields, for $v \neq O$, we easily have

$$(\mu \otimes \mu')(\text{"There is a path of disagreement from } O \text{ to } v | \mathcal{J}_v) \leq \mathbf{p}_v.$$

We have the following :

$$\begin{aligned}
&(\mu \otimes \mu')(\text{"}(\omega, \omega') \text{ has an infinite path of disagreement containing } O\text{"}) \\
&\qquad = (\mu \otimes \mu')(\text{"}(\omega, \omega') \text{ has an infinite path } \Pi \text{, not containing } O \text{, such that for each } v \text{ on } \Pi, \\
&\qquad\qquad \text{there exists a path of disagreement from } O \text{ to } v\text{"}) \\
&\qquad \leq \mathbb{P}_{\mathbf{p}}(\text{"}(\omega, \omega') \text{ has an infinite path } \Pi \text{, not containing } O \text{, such that each } v \text{ on } \Pi \text{ is open}) \\
&\qquad\qquad \leq \mathbb{P}_{\mathbf{p}}(\text{"There is an infinite open path'}) = 0.
\end{aligned}$$

The first equality is due to the fact that the two events are the same. The second equality is due to the same reason. The inequality is true thanks to what we developed before.

Since the above holds for any $O \in V$ and V is countable, we have

$$(\mu \otimes \mu')(\text{"}(\omega, \omega') \text{ has an infinite path of disagreement}) = 0$$

and with theorem 2.7, we have the conclusion. □

Using corollary 2.8, we can find a link between the critical probability and the critical activity thanks to theorem 2.9.

Theorem 2.9 ([BS94]) — Let $G = (V, E)$ be a countably infinite, locally finite, connected graph. Let $\mathbf{a} \in (R_+^*)^V$ and $\mathbf{p} = (\frac{\mathbf{a}_v}{1+\mathbf{a}_v})_{v \in V}$.

1. If $\mathbb{P}_{G,\mathbf{p}}(\text{"There exists an infinite open path"}) = 0$, then the hard-core model on G with activity $\mathbf{a}$ has a unique Gibbs measure.
2. The critical activity of G satisifies $a_G^c \geq \frac{p_G^c}{1-p_G^c}$.

Proof. Let $v \in V$, we will calculate $\mathbf{p}_v$. Let us define $f : \{0,1\}^{N(v)} \times \{0,1\}^{N(v)}$ by

$$\forall \alpha, \alpha' \in \{0,1\}^{N(v)}, f(\alpha, \alpha') = (\mu \otimes \mu')(\omega_v \neq \omega'_v | \omega_{N(v)} = \alpha_{N(v)}, \omega'_{N(v)} = \alpha'_{N(v)}).$$

In case there are $k, l \in N(v)$, $\alpha_k = \alpha'_l = 1$, then $f(\alpha, \alpha') = 0$. Hence, since by the definition we may assume that $\alpha \neq \alpha'$, there is two cases : either $\alpha = 0 \neq \alpha'$ or $\alpha' \neq 0 = \alpha$. By symmetry it suffices to take the first, in which case it is easily seen that

$$f(\alpha, \alpha') = \mu(\omega_i = 1 | \omega_{N(v)} = 0) = \frac{a_v}{1 + a_v}.$$

Hence, by Theorem 2.8, we have the first result.

The second part is a consequence of the first part and of the definition of the critical activity. $\square$

Theorem 2.9 allows to extend the results about percolation's critical probability in order to obtain results about hard-core model's critical activity.

Moreover, we can prove the following.

Theorem 2.10 — Let $G = (V, E)$ be a finite connected graph and let $\mathbf{a} \in (\mathbb{R}_+^*)^V$. For ω and ω two independent realisations under $\mu_{G,\mathbf{a}}$. Then for all v and w vertices we have :

$$\begin{aligned} 2\,\mathrm{Cov}_{\mu_{G,\mathbf{a}}}(\omega_v, \omega_w) &= (\mu_{G,\mathbf{a}} \otimes \mu_{G,\mathbf{a}})(\text{"}(\omega, \omega') \text{ has a path of disagreement with even length from } i \text{ to } j\text{"}) \\ &= (\mu_{G,\mathbf{a}} \otimes \mu_{G,\mathbf{a}})(\text{"}(\omega, \omega') \text{ has a path of disagreement with odd length from } i \text{ to } j\text{"}). \end{aligned}$$

In particular, if G is bipartite, then

$$2\,\mathrm{Cov}_{\mu_{G,\mathbf{a}}}(\omega_v, \omega_w) = \begin{cases} \frac{1}{2}(\mu_{G,\mathbf{a}} \otimes \mu_{G,\mathbf{a}})(\text{"}(\omega, \omega') \text{ has a path of disagreement from } i \text{ to } j\text{"}) & \text{, if } \{v, w\} \subset \mathcal{O} \text{ or } \{v, w\} \subset \mathcal{E} \\ -\frac{1}{2}(\mu_{G,\mathbf{a}} \otimes \mu_{G,\mathbf{a}})(\text{"}(\omega, \omega') \text{ has a path of disagreement from } i \text{ to } j\text{"}) & \text{, otherwise.} \end{cases}$$

Proof. We have easily

$$\mathrm{Cov}_{\mu_{G,\mathbf{a}}}(\omega_v, \omega_w) = \frac{1}{2}\langle(\omega_v - \omega'_v)(\omega_w - \omega'_w)\rangle_{\mu_{G,\mathbf{a}} \otimes \mu_{G,\mathbf{a}}}.$$

For $v \in V$, let us define $C(v) = \{l \in V : (\omega, \omega')$ has a path of disagreement from v to $l\}$ and let us define the transformation $T : \Omega_G \times \Omega_G \to \Omega_G \times \Omega_G$. For $v, w \in V$, define the transformation $T_{v,w} : \Omega_G \times \Omega_G \to \Omega_G \times \Omega_G$ by :

- if $w \in C(v)$, then $T_{v,w} = \mathrm{id}_{\Omega_G \times \Omega_G}$,
- if $w \notin C(v)$, then for all $(\omega, \omega') \in \Omega_G \times \Omega_G$, $T_{v,w}(\omega, \omega') = (\omega'_{C(v)} \omega_{C(v)^c}, \omega_{C(v)} \omega'_{C(v)^c})$.

$T_{v,w}$ is bijective and since $\mu_{G,a}$ is a Markov field, $T_{v,w}$ is measure preserving. Let us define $f_{i,j} : \Omega_G \times \Omega_G \to \mathbb{R}$ by $f(\omega, \omega') = (\omega_v - \omega'_v)(\omega_w - \omega'_w)$, we then have

$$\mathrm{Cov}_{\mu_{G,\mathbf{a}}}(\omega_v, \omega_w) = \frac{1}{2} \langle f(\omega, \omega') \rangle_{\mu_{G,\mathbf{a}} \otimes \mu_{G,\mathbf{a}}} = \frac{1}{4} \langle f(\omega, \omega') + f(T(\omega, \omega')) \rangle_{\mu_{G,\mathbf{a}} \otimes \mu_{G,\mathbf{a}}}.$$

By definition, we have

- if $w \in C(v)$, $f(\omega, \omega') = f(T(\omega, \omega'))$,
- if $w \notin C(v)$, $f(\omega, \omega') + f(T(\omega, \omega')) = 0$.

Hence

$$\mathrm{Cov}_{\mu_{G,\mathbf{a}}}(\omega_v, \omega_w) = \frac{1}{2} \langle f(\omega, \omega') I_{v,w}(\omega, \omega') \rangle_{\mu_{G,\mathbf{a}} \otimes \mu_{G,\mathbf{a}}},$$

where $I_{v,w}(\omega, \omega')$ is equal to 1 if there is a path of disagreement from v to w for (ω, ω') and 0 otherwise. If there is a path of disagreement Π for (ω, ω') from v to w, Since they have no two adjacent 1s, the values of ω and ω' altern between 0 and 1 on Π. That is why,

- if Π is of even length, $\omega_v = \omega_v = 1 - \omega'_v = 1 - \omega'_w$, then $f(\omega, \omega') = 1$,
- if Π is of odd length, $\omega_v = \omega'_w = 1 - \omega'_v = 1 - \omega_w$, then $f(\omega, \omega') = -1$.

That gives the first result.

The second part is a direct consequence of the first one. □

Remark. We only used in this proof the fact that $\mu_{G,\mathbf{a}}$ is a Markov field. Hence, this result is also true for any finite graph equipped with a Markov field measure.

3 Results in the bipartite case

We previously defined the hard-core model in the most general case, we will now restrict it to the case of finite or countably finite, locally finite, connected, *bipartite* graphs.

In subsection 3.1, we prove results useful to conclude for us to get the existence of the hard-core model in the infinite bipartite graph case.

In subsection 3.2, we prove the existence of the hard-core model in the infinite bipartite graph case and two criteria for uniqueness.

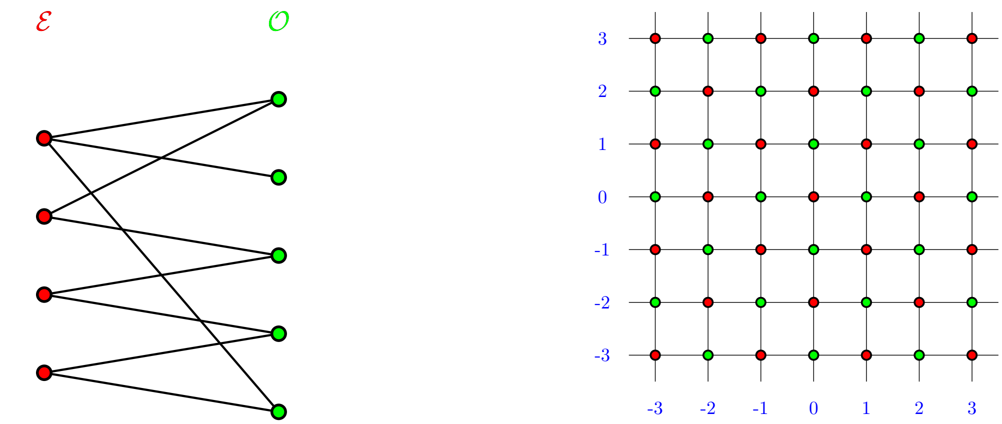
3.1 Preliminary results

We recall the definition of a bipartite graph in definition 3.1.

Definition 3.1 — We say that a graph $G = (V, E)$ is *bipartite* if and only if there exists $\mathcal{E} \subset V$ and $\mathcal{O} = V \setminus \mathcal{E}$ such that :

$$\begin{cases} \forall e, e' \in \mathcal{E}, e \not\sim e'; \\ \forall o, o' \in \mathcal{O}, o \not\sim o'. \end{cases}$$

The vertices of $\mathcal{E}$ (resp. $\mathcal{O}$) are called the *even* (resp. *odd*) vertices.



(a) A finite bipartite graph.

(b) $\mathbb{Z}^2$ is a bipartite graph.

Figure 10: Examples of bipartite graphs.

For instance, for all $d \geq 1$, $\mathbb{Z}^d$ is a bipartite graph where we say that a vertex v is *even* (resp. *odd*) if and only if $\sum_{k=1}^{d} v_k$ is even (resp. odd).

With this special property and because $\{0, 1\}$ has a total order, we can define in the following proposition-definition 3.2 a partial order on Ω_G.

Proposition-Definition 3.2 — Let $G = (V, E)$ be a countably infinite or finite, locally finite, connected, bipartite graph. We define the binary relation $\leq$ on the elements of Ω_G by :

$$\forall \omega, \omega' \in \Omega_G, \omega \leq \omega' \iff \begin{cases} \forall e \in \mathcal{E}, \omega_e \leq \omega'_e \\ \forall o \in \mathcal{O}, \omega_o \geq \omega'_o \end{cases}.$$

$\leq$ is a partial order on the elements of Ω_G and for all $A \subset \Omega_G$, we say that A is *increasing* if and only if

$$\forall \omega \in A, \omega' \in \Omega, \omega \leq \omega' \Rightarrow \omega' \in A.$$

Proof. In order to prove that $\leq$ is a partial order on Ω_G, we need to prove that $\leq$ is reflexive, anti-symmetric and transitive.

For all $\omega, \omega', \omega'' \in \Omega_G$,

- We easily have $\omega \leq \omega$ because for all $v \in V$, $\omega_v \leq \omega_v$ and $\omega_v \geq \omega_v$.
- We easily have that if $\omega \leq \omega'$ and $\omega' \leq \omega$ then $\omega = \omega'$ because the first condition gives that for all $v \in V$, $\omega_v \leq \omega'_v$ and $\omega'_v \leq \omega_v$.
- If $\omega \leq \omega'$ and $\omega' \leq \omega''$, then for all $e \in \mathcal{E}$ and all $o \in \mathcal{O}$, $\omega_e \leq \omega'_e \leq \omega''_e$ and $\omega_o \geq \omega'_o \geq \omega''_o$, then $\omega \leq \omega''$. □

Let us recall the definition of distributive lattice in definition 3.3, before giving one more property of the partial order in proposition 3.4.

Definition 3.3 — A partially ordered set $(L, \leq)$ is called a *lattice* if and only if each two-element subset $\{a, b\} \subset L$ has a join (*i.e.* least upper bound, denoted by $a \wedge b$) and dually a meet (*i.e.* greatest lower bound, denoted by $a \vee b$).

A lattice $(L, \leq)$ is called a *distributive lattice* if and only if the following additional identity holds for all x, y and z in L,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

(or equivalently $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$). The previous equalities are called the FKG condition.

We can now prove that $(\Omega_G, \leq)$ is a distributive lattice.

Proposition 3.4 ([BS94]) — With the previous notations, $(\Omega_G, \leq)$ is a lattice and for all $\omega, \omega' \in \Omega_G$,

$$\begin{array}{rcl} (\omega \wedge \omega') & = & (\min(\omega_v, \omega'_v))_{v \in \mathcal{E}} (\max(\omega_v, \omega'_v))_{v \in \mathcal{O}} \\ (\omega \vee \omega') & = & (\max(\omega_v, \omega'_v))_{v \in \mathcal{E}} (\min(\omega_v, \omega'_v))_{v \in \mathcal{O}} \end{array}.$$

Moreover, $(\Omega_G, \leq)$ is a distributive lattice.

Proof. For $\omega, \omega' \in \Omega_G$, let η be $(\min(\omega_v, \omega'_v))_{v \in \mathcal{E}} (\max(\omega_v, \omega'_v))_{v \in \mathcal{O}}$ and let $\alpha \in \Omega_G$ verify $\alpha \leq \omega$ and $\alpha \leq \omega'$. Because of its definition, we easily have that $\eta \leq \omega$ and $\eta \leq \omega'$. Moreover, for $e \in \mathcal{E}$ and for $o \in \mathcal{O}$, $\alpha_e \leq \omega_e$ and $\alpha_e \leq \omega'_e$, hence $\alpha_e \leq \eta_e$ and, for the same reason, $\alpha_o \geq \eta_o$. That's why, $\alpha \leq \eta$.

To sum up, we have for $\alpha \in \Omega_G$ verifying $\alpha \leq \omega$ and $\alpha \leq \omega'$,

$$\alpha \leq \eta \leq \omega \text{ and } \alpha \leq \eta \leq \omega',$$

that's why there is a join for $\{\omega, \omega'\}$ which is $(\min(\omega_v, \omega'_v))_{v \in \mathcal{E}} (\max(\omega_v, \omega'_v))_{v \in \mathcal{O}}$.

Similarly, we can prove that there is a meet for $\{\omega, \omega'\}$ which is $(\max(\omega_v, \omega'_v))_{v \in \mathcal{E}} (\min(\omega_v, \omega'_v))_{v \in \mathcal{O}}$.

The fact that $(\Omega_G, \leq)$ is a distributive is a consequence of the fact that for all $x, y, z \in \mathbb{R}$,

$$\min(x, \max(y, z)) = \max(\min(x, y), \min(x, z)) \text{ and } \max(x, \min(y, z)) = \min(\max(x, y), \max(x, z)). \qquad \square$$

The reason why we defined this partial order on the configurations is given in the proposition 3.5.

Proposition 3.5 — For all $G = (V, E)$ finite, locally finite, connected, bipartite graph equipped with the partial order $\leq$ defined in the proposition-proposition 3.2, for all $\mathbf{a} = (\mathbf{a}_v)_{v\in V}$, $\mu_{G,\mathbf{a}}$ satisfies the FKG condition with respect to $\leq$ *i.e.*

$$\forall \omega, \omega' \in \Omega_G, \mu_{G,\mathbf{a}}(\omega \wedge \omega')\mu_{G,\mathbf{a}}(\omega \vee \omega') \geq \mu_{G,\mathbf{a}}(\omega)\mu_{G,\mathbf{a}}(\omega').$$

Proof. The result is easy to prove because for all $\omega, \omega' \in \Omega_G$, $\prod_{v\in V} \mathbf{a}_v^{(\omega\wedge\omega')_v+(\omega\vee\omega')_v} = \prod_{v\in V} \mathbf{a}_v^{\omega_v+\omega'_v}$ which gives

$$\mu_{G,\mathbf{a}}(\omega \wedge \omega')\mu_{G,\mathbf{a}}(\omega \vee \omega') = \mu_{G,\mathbf{a}}(\omega)\mu_{G,\mathbf{a}}(\omega'). \qquad \square$$

The FKG condition gives us the FKG inequality which is really useful when it comes to statistical physics.

Theorem 3.6 (FKG inequality [FKG71]) — Let $(X, \leq)$ be a finite distributive lattice and μ a non-negative function on it verifying the FKG condition.

Then for all f and g two non-decreasing functions on X, we have

$$\left(\sum_{x\in X} f(x)g(x)\mu(x)\right)\left(\sum_{x\in X} \mu(x)\right) \geq \left(\sum_{x\in X} f(x)\mu(x)\right)\left(\sum_{x\in X} g(x)\mu(x)\right).$$

We will not prove theorem 3.6 here but the reader can find its original proof in [FKG71]. Moreover, they can find its proof in the chapter 3 of the book of Friedli and Velenik [FV17] based on the following theorem 3.7 whose proof is in the pages 129 and 130.

Theorem 3.7 ([FV17]) — Let $\mu = \bigotimes_{v\in V} \mu_v$ be a product measure on Ω_G. Let $f_1, f_2, f_3, f_4 : \Omega_G \to \mathbb{R}$ be non-negative functions on Ω_G such that

$$\forall \omega, \omega' \in \Omega_G, f_1(\omega)f_2(\omega') \leq f_3(\omega \wedge \omega')f_4(\omega \vee \omega').$$

Then

$$\langle f_1\rangle_\mu \langle f_2\rangle_\mu \leq \langle f_3\rangle_\mu \langle f_4\rangle_\mu.$$

One of the interesting consequences of theorem 3.7 is the lemma 3.10 based on proposition-definition 3.8.

Proposition-Definition 3.8 — For all $G = (V, E)$ finite, connected, bipartite graph, we define the partial order $\preceq$ on the probability measures on Ω_G by for all ν and ν' two probability measures on Ω_G, $\nu \preceq \nu'$ if and only if for all increasing subset A of Ω_G, $\nu(A) \leq \nu'(A)$.

Proof. In order to prove that $\preceq$ is a partial order on the probability measures on Ω_G, we need to prove that $\preceq$ is reflexive, anti-symmetric and transitive.

For all ν, ν', ν'' three probability measures on Ω_G,

- We easily have $\nu \leq \nu$ because for all $A \subset \Omega_G$, $\nu(A) \leq \nu(A)$.
- We easily have that if $\nu \leq \nu'$ and $\nu' \leq \nu$, then for all A increasing subset of Ω_G, $\nu(A) = \nu'(A)$. Moreover for all $\omega \in \Omega_G$, if we denote $A_\omega = \{\omega' \in \Omega_G : \omega \leq \omega'\}$, it is an increasing set and

$$\nu(\{\omega\}) = \nu(A_\omega) - \nu(A_\omega \setminus \{\omega\}) = \nu'(A_\omega) - \nu'(A_\omega \setminus \{\omega\}) = \nu'(\{\omega\}).$$

- If $\nu \leq \nu'$ and $\nu' \leq \nu''$, we easily have $\nu \leq \nu''$. □

In the same way, we can define the following partial order.

Proposition-Definition 3.9 — For all $G = (V, E)$ countably infinite, locally finite, connected, bipartite graph, we define the partial order $\preceq$ on the probability measures on Ω_G by for all ν and ν' two probability measures on Ω_G, $\nu \preceq \nu'$ if and only if for all f non-decreasing continuous functions of Ω_G, $\langle f \rangle_\nu \leq \langle f \rangle_{\nu'}$.

Remark. This definition gives in the case of a finite graph the same of the previous one that is why we can use the same notation.

An interesting consequence of the partial order is the useful lemma 3.10.

Lemma 3.10 ([BS94]) — Let $G = (V, E)$ be a finite connected bipartite graph, let $\mathbf{a} = (\mathbf{a}_v)_{v \in V}$, let $W \subset V$, let $\alpha, \alpha' \in F_G$ such that $\alpha \leq \alpha'$. Then

$$\mu_{G,\mathbf{a}}(\cdot\ |\{\omega \in \Omega_G : \omega_W = \alpha_W\}) \preceq \mu_{G,\mathbf{a}}(\cdot\ |\{\omega \in \Omega_G : \omega_W = \alpha'_W\}).$$

Proof. Thanks to the proposition 2.2, we know that $\mu_{G,\mathbf{a}}(\cdot) = \mathbb{P}_{G,\mathbf{p}}(\cdot\ |F_G)$ with $\mathbf{p}$ defined as $\mathbf{p} = (\mathbf{p}_v)_{v \in V} = (\frac{\mathbf{a}_v}{1+\mathbf{a}_v})_{v \in V}$, the probability measure on Ω_G $\mathbb{P}_{G,\mathbf{p}} = \bigotimes_{v \in V} \mu_v$ with for all $v \in V$, μ_v following the Bernoulli distribution with parameter $\mathbf{p}_v$.

Let A be an increasing subset of Ω_G, let $A_W^\alpha = \{\omega \in A : \omega_W = \alpha_W\}$, $A_W^{\alpha'} = \{\omega \in A : \omega_W = \alpha'_W\}$, $\Omega_W^\alpha = \{\omega \in \Omega_G : \omega_W = \alpha_W\}$ and $\Omega_W^{\alpha'} = \{\omega \in \Omega_G : \omega_W = \alpha'_W\}$.

Let f_1, f_2, f_3 and f_4 be respectively $\mathbb{1}_{A_W^\alpha}\mathbb{1}_{F_G}$, $\mathbb{1}_{\Omega_W^{\alpha'}}\mathbb{1}_{F_G}$, $\mathbb{1}_{\Omega_W^\alpha}\mathbb{1}_{F_G}$ and $\mathbb{1}_{A_W^{\alpha'}}\mathbb{1}_{F_G}$ four non-negative functions. For all $\omega, \omega' \in \Omega_G$, on the one hand, if $\omega \notin A_W^\alpha \cap F_G$ or $\omega \notin \Omega_W^{\alpha'} \cap F_G$, we have easily

$$f_1(\omega)f_2(\omega') \leq f_3(\omega \wedge \omega')f_4(\omega \vee \omega').$$

On the other hand, if $\omega \in A_W^\alpha \cap F_G$ and $\omega \in \Omega_W^{\alpha'} \cap F_G$,

- We have $(\omega \wedge \omega')_W = \alpha_W$ and for all $v \in V$ such that $(\omega \wedge \omega')_v = 1$:
  - If $v \in \mathcal{E}$, then $\omega_v = \omega'_v = 1$ and then for all w adjacent to v, $\omega_w = \omega'_w = 0$ then for all w adjacent to v, $(\omega_\wedge\omega')_w = 0$.
  - If $v \in \mathcal{O}$, then either $\omega_v = 1$ or $\omega'_v = 1$ and then either for all w adjacent to v, $\omega_w = 0$ or for all w adjacent to v, $\omega'_w = 0$ then for all w adjacent to v, $(\omega_\wedge\omega')_w = 0$.

  Hence, $\omega \wedge \omega' \in \Omega_W^\alpha \cap F_G$.

- We have $(\omega \vee \omega')_W = \alpha'_W$, $(\omega \vee \omega') \geq \omega$ and for all $v \in V$ such that $(\omega \wedge \omega')_v = 1$:
  - If $v \in \mathcal{E}$, then either $\omega_v = 1$ or $\omega'_v = 1$ and then either for all w adjacent to v, $\omega_w = 0$ or for all w adjacent to v, $\omega'_w = 0$ then for all w adjacent to v, $(\omega_\wedge\omega')_w = 0$.
  - If $v \in \mathcal{O}$, then $\omega_v = \omega'_v = 1$ and then for all w adjacent to v, $\omega_w = \omega'_w = 0$ then for all w adjacent to v, $(\omega_\wedge\omega')_w = 0$.

  Hence, $\omega \wedge \omega' \in A_W^{\alpha'} \cap F_G$.

Hence, we always have $f_1(\omega)f_2(\omega') \leq f_3(\omega \wedge \omega')f_4(\omega \vee \omega')$. Thanks to theorem 3.7, we can deduce that

$$\langle \mathbb{1}_{A_W^\alpha} \rangle_{\mu_{G,\mathbf{a}}} \langle \mathbb{1}_{\Omega_W^{\alpha'}} \rangle_{\mu_{G,\mathbf{a}}} = \langle \mathbb{1}_{A_W^\alpha}\mathbb{1}_F \rangle_{\mathbb{P}_{G,\mathbf{p}}} \langle \mathbb{1}_{\Omega_W^{\alpha'}}\mathbb{1}_F \rangle_{\mathbb{P}_{G,\mathbf{p}}} \leq \langle \mathbb{1}_{\Omega_W^\alpha}\mathbb{1}_F \rangle_{\mathbb{P}_{G,\mathbf{p}}} \langle \mathbb{1}_{A_W^{\alpha'}}\mathbb{1}_F \rangle_{\mathbb{P}_{G,\mathbf{p}}} = \langle \mathbb{1}_{\Omega_W^\alpha} \rangle_{\mu_{G,\mathbf{a}}} \langle \mathbb{1}_{A_W^{\alpha'}} \rangle_{\mu_{G,\mathbf{a}}},$$

that is why

$$\mu_{G,\mathbf{a}}(A|\Omega_W^\alpha) = \frac{\mu_{G,\mathbf{a}}(A_W^\alpha)}{\mu_{G,\mathbf{a}}(\Omega_W^\alpha)} \leq \frac{\mu_{G,\mathbf{a}}(A_W^{\alpha'})}{\mu_{G,\mathbf{a}}(\Omega_W^{\alpha'})} = \mu_{G,\mathbf{a}}(A|\Omega_W^{\alpha'}). \qquad \square$$

With the tools we developed in the finite case, we can now resume the proofs in the infinite case. We will firstly show the existence of at least one Gibbs measure for the hard-core model in the bipartite case and then give criteria for uniqueness in this case.

3.2 Existence of at least one Gibbs measure for the hard-core model and two criteria for uniqueness

The goal of this subsection is to prove the existence of at least one Gibbs measure in the bipartite case and to have a first criterion for the uniqueness of the Gibbs measure, thanks to a similar proof for the Ising model that can be found in [Lig85].

In order to prove the existence of at least one Gibbs measure, we will define two sequences of measures whose limit is a Gibbs measure.

In order not to have to repeat the exact same things several times, we will fix in the rest of the section $G = (V, E)$ a countably infinite, locally finite, connected, bipartite graph whose set of even (resp. odd) vertices is $\mathcal{E}$ (resp. $\mathcal{O}$) and $\mathbf{a} = (\mathbf{a}_v)_{v \in V}$ a sequence of positive real numbers.

Before proceeding with proof, we need the following measures.

Definition 3.11 — Let $\alpha \in F_G$ and $\Lambda \Subset V$.

- We denote $\mu^{\alpha}_{\Lambda,\mathbf{a}}(\cdot) = \mu_{\overline{\Lambda},\mathbf{a}}(\cdot \mid \{\omega \in F_{\overline{\Lambda}} : \omega_{\partial\Lambda} = \alpha_{\partial\Lambda}\})$.
- We denote $\nu^{\alpha}_{\Lambda,\mathbf{a}}$ the probability measure associated with the random variable $X_{\overline{\Lambda}}\alpha_{\overline{\Lambda}^c}$ where $X \sim \mu^{\alpha}_{\Lambda,\mathbf{a}}$.

In both of the case, we say that α is the *boundary condition*.

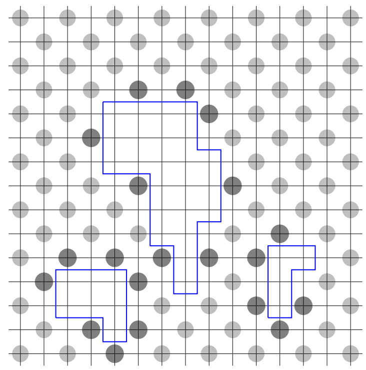


Figure 11: Illustration of how we construct the sequences of measures: the sites in the blue border correspond to the subgraph Λ and the grey balls are the boundary conditions, the darkest ones having a direct influence on how the n-th probability measures are defined.

With definition 3.11, we can prove the following proposition 3.12.

Proposition 3.12 — Let $\Lambda \Subset V$ and take $v \in V$ be an even vertex. Let ω and ω' be two independent realisations respectively under $\mu^{\mathfrak{e}}_{\Lambda,\mathbf{a}}$ and under $\mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}}$. We then have

$$\mu^{\mathfrak{e}}_{\Lambda,\mathbf{a}}(\omega_v = 1) - \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}}(\omega'_v = 1) = (\mu^{\mathfrak{e}}_{\Lambda,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}})((\omega,\omega') \text{ has a path of disagreement from } v \text{ to } \partial\Lambda).$$

Proof. Let Ω be $\{0,1\}^{\overline{\Lambda}}$ and let $C(v) = \{w \in V : \text{there is path of disagreement for } (\omega,\omega') \text{ from } v \text{ to } w\}$. Let $T_v : \Omega \times \Omega \to \Omega \times \Omega$ be the transformation such that

- if $C(v) \cap \partial\Lambda \neq \emptyset$ *i.e.* there is a path of disagreement from v to $\partial\Lambda$, $T_v = \mathrm{id}_{\Omega\times\Omega}$,
- otherwise, for $(\alpha,\alpha') \in \Omega \times \Omega$, $T(\alpha,\alpha') = (\alpha'_{C(i)}\alpha_{C(i)^c}, \alpha_{C(i)}\alpha'_{C(i)^c})$.

Then T is bijective and preserves $\mu^{\mathfrak{e}}_{\Lambda,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}}$. Hence

$$\begin{aligned}\mu^{\mathfrak{e}}_{\Lambda,\mathbf{a}}(\omega_v = 1) &= (\mu^{\mathfrak{e}}_{\Lambda,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}})(\omega_v = 1)\\ &= (\mu^{\mathfrak{e}}_{\Lambda,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}})(C(i) \cap \partial\Lambda = \emptyset, \omega_v = 1) + (\mu^{\mathfrak{e}}_{\Lambda,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}})(C(i) \cap \partial\Lambda \neq \emptyset, \omega_v = 1)\end{aligned}$$

Similarly

$$\mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}}(\omega'_v = 1) = (\mu^{\mathfrak{c}}_{\Lambda,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}})(C(i) \cap \partial\Lambda = \emptyset, \omega'_v = 1) + (\mu^{\mathfrak{c}}_{\Lambda,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}})(C(i) \cap \partial\Lambda \neq \emptyset, \omega'_v = 1)$$

It is easy to see that T transform the first event of the first term of the first sum into the event of the first term of the second sum. That is why the two terms are equal. Because of the boundary conditions, because ω and ω' are feasible, because there is a path of disagreement for (ω, ω') from v to $\partial\Lambda$, $\omega'_v = 1 - \omega_v = 0$. Hence we have $\mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}}(\omega'_v = 1) = (\mu^{\mathfrak{c}}_{\Lambda,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}})(C(i) \cap \partial\Lambda = \emptyset, \omega'_v = 1)$. Then we have the result $\square$

Thanks to the definition 3.11, we will be able to construct sequences of measures whose exterior condition is the same and such that the subgraph of definition is growing andmore and more covering the whole graph. Meanwhile, we can prove the following.

Proposition 3.13 — For all $\Lambda_1 \subset \Lambda_2 \Subset V$, for all $\alpha \in F_G$,

- $\nu^{\alpha}_{\Lambda_2,\mathbf{a}}(\cdot \mid \{\omega \in F_G : \omega_{\Lambda_2 \setminus \Lambda_1} = \alpha_{\Lambda_2 \setminus \Lambda_1}\}) = \nu^{\alpha}_{\Lambda_1,\mathbf{a}}(\cdot)$.
- $\nu^{\alpha}_{\Lambda_2,\mathbf{a}} = \sum_{\substack{\gamma \in F_G \\ \gamma_{V\setminus\Lambda_2} = \alpha_{V\setminus\Lambda_2}}} \nu^{\alpha}_{\Lambda_2,\mathbf{a}}(\{\omega \in F_G : \omega_{\Lambda_2 \setminus \Lambda_1} = \gamma_{\Lambda_2 \setminus \Lambda_1}\}) \nu^{\gamma}_{\Lambda_1,\mathbf{a}}$.

Proof. The first result is easy to obtain thanks to the definition of $\nu^{\alpha}_{\Lambda_2,\mathbf{a}}$. The second result is a consequence of the first one and of the following equality

$$\nu^{\alpha}_{\Lambda_2,\mathbf{a}}(\cdot) = \sum_{\substack{\gamma \in F_G \\ \gamma_{V\setminus\Lambda_2} = \alpha_{V\setminus\Lambda_2}}} \nu^{\gamma}_{\Lambda_2,\mathbf{a}}(\{\omega \in F_G : \omega_{\Lambda_2 \setminus \Lambda_1} = \gamma_{\Lambda_2 \setminus \Lambda_1}\}) \nu^{\gamma}_{\Lambda_2,\mathbf{a}}(\cdot \mid \{\omega \in F_G : \omega_{\Lambda_2 \setminus \Lambda_1} = \gamma_{\Lambda_2 \setminus \Lambda_1}\}). \qquad \square$$

We can also prove the following.

Proposition 3.14 — For all $\alpha, \alpha' \in F_G$ such that $\alpha \leq \alpha'$, $\nu^{\alpha}_{\Lambda,\mathbf{a}} \preceq \nu^{\alpha'}_{\Lambda,\mathbf{a}}$.

Proof. It is a direct consequence of lemma 3.10. $\square$

We will now need two particular boundary conditions in order to prove the existence of at least one Gibbs measure. They are both illustrated in the Figure 12.

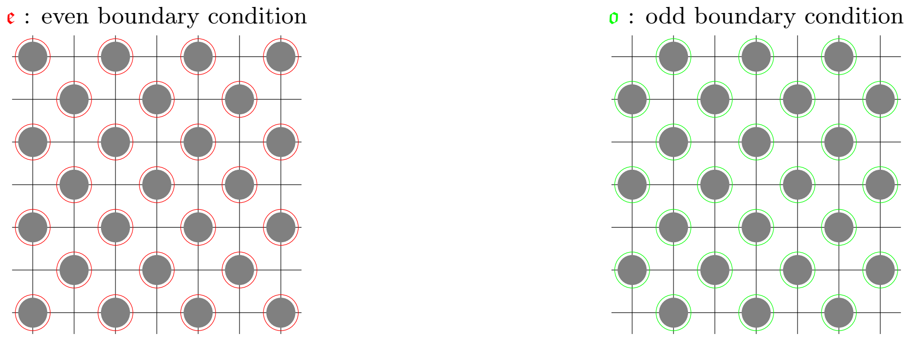


Figure 12: Illustration of the two boundary conditions.

Proposition-Definition 3.15 — We denote by $\mathfrak{e} = (\mathbb{1}_{\mathcal{E}}(v))_{v\in V}$ (resp. $\mathfrak{o} = (\mathbb{1}_{\mathcal{O}}(v))_{v\in V}$) the *even boundary condition* (resp. the *odd boundary condition*).

In particular, for all $\omega \in \Omega_G$,

$$\mathfrak{o} \leq \omega \leq \mathfrak{e}.$$

Moreover, we have for all $\Lambda \Subset V$, for all $\alpha \in F$,

$$\nu^{\mathfrak{o}}_{\Lambda,\mathbf{a}} \preceq \nu^{\alpha}_{\Lambda,\mathbf{a}} \preceq \nu^{\mathfrak{e}}_{\Lambda,\mathbf{a}}.$$

Proof. The order between the configurations is a consequence of

$$\forall \omega \in \Omega_G, \begin{cases} \forall e \in \mathcal{E}, 0 \leq \omega_e \leq 1 \\ \forall o \in \mathcal{O}, 1 \geq \omega_o \geq 0 \end{cases}.$$

The order between the measures is a consequence of the previous point and proposition 3.14. □

Remark. We can see that $\mathfrak{o}$ and $\mathfrak{e}$ play in the hard-core model the same role as $+$ and $-$ in the Ising model. What explains that we took those two paticular boundary conditions is that when the activity is big enough, the hard-core mesure concentrates itself only on the two configurations $\mathfrak{e}$ and $\mathfrak{o}$.

In order to have a limit measure, we can prove the monotonicity of the sequences of measures in proposition 3.16.

Proposition 3.16 — For all $\Lambda_1 \subset \Lambda_2 \Subset V$, $\nu^{\mathfrak{o}}_{\Lambda_1,\mathbf{a}} \preceq \nu^{\mathfrak{o}}_{\Lambda_2,\mathbf{a}}$ and $\nu^{\mathfrak{e}}_{\Lambda_2,\mathbf{a}} \preceq \nu^{\mathfrak{e}}_{\Lambda_1,\mathbf{a}}$ *i.e.* $(\nu^{\mathfrak{o}}_{\Lambda,\mathbf{a}})_\Lambda$ is increasing and $(\nu^{\mathbf{e}}_{\Lambda,\mathbf{a}})_\Lambda$ is decreasing.

Proof. The two statements have the same proof, let us prove the first inequality. We already know, by proposition 3.14, for $\alpha \in F$, $\nu^{\alpha}_{\Lambda_1,\mathbf{a}} \preceq \nu^{\mathfrak{e}}_{\Lambda_1,\mathbf{a}}$. That's why,

$$\nu^{\mathfrak{e}}_{\Lambda_2,\mathbf{a}} = \sum_{\substack{\gamma \in F_G \\ \gamma_{V\setminus\Lambda_2} = \mathfrak{e}_{V\setminus\Lambda_2}}} \nu^{\mathfrak{e}}_{\Lambda_2,\mathbf{a}}(\{\omega \in F_G : \omega_{\Lambda_2\setminus\Lambda_1} = \gamma_{\Lambda_2\setminus\Lambda_1}\})\nu^{\gamma}_{\Lambda_1,\mathbf{a}} \preceq \nu^{\mathfrak{e}}_{\Lambda_1,\mathbf{a}},$$

we can now conclude. □

Remark. We find here once again a similar behaviour in the hard-core model as in the Ising-Lenz model.

Thanks to the fact that there is a monotonicity for the measures, we can define a weak limit for the measures. We want them to be a hard-core measures for G with activity $\mathbf{a}$.

Theorem 3.17 — Let us denote $\mu^{\mathfrak{e}}_{G,\mathbf{a}} = \lim_{\Lambda\uparrow G} \nu^{\mathfrak{e}}_{\Lambda,\mathbf{a}}$ and $\mu^{\mathfrak{o}}_{G,\mathbf{a}} = \lim_{\Lambda\uparrow G} \nu^{\mathfrak{o}}_{\Lambda,\mathbf{a}}$. Then $\mu^{\mathfrak{e}}_{G,\mathbf{a}}$ and $\mu^{\mathfrak{o}}_{G,\mathbf{a}}$ are hard-core measures for G with activity $\mathbf{a}$.

Proof. For all $B \Subset V$, $\eta \in F_G$, $\alpha = \{0,1\}^{\Lambda}$, for all Λ be a set verifying $\overline{B} \subset \Lambda \Subset V$, we easily see that $\nu^{\mathfrak{e}}_{\Lambda,\mathbf{a}}$ and $\nu^{\mathfrak{o}}_{\Lambda,\mathbf{a}}$ verify that for all $\omega \in \Omega_G$, for $\mathfrak{i} \in \{\mathfrak{e},\mathfrak{o}\}$,

$$\mu^{\mathfrak{i}}_{\Lambda,\mathbf{a}}(\{\omega\}) = \frac{1}{Z^{\eta}_{B,\mathbf{a}}} \mathbb{1}_{F_G}(\alpha_B \eta_{B^c}) \exp(-H^{\eta}_{B,\mathbf{a}}(\alpha)).$$

Because it is true for all Λ verifying $\overline{B} \subset \Lambda \Subset V$, because $\mu^{\mathfrak{e}}_{G,\mathbf{a}}$ and $\mu^{\mathfrak{o}}_{G,\mathbf{a}}$ are weak limits for the $\nu^{\mathfrak{e}}_{\Lambda,\mathbf{a}}$ and $\nu^{\mathfrak{o}}_{\Lambda,\mathbf{a}}$ when $\Lambda \uparrow V$, the two weak limits verify definition 1.7, that is why they are hard-core measures for G with activity $\mathbf{a}$. □

Now we prove the existence of at least one Gibbs measure thanks to two different methods.

Before proceeding with a proof of a uniqueness criterion for the hard-core model, we need the following lemma.

Lemma 3.18 — Let us denote $\mathcal{G}_{G,\mathbf{a}}$ the class of all hard-core measure for G with activity $\mathbf{a}$ and for all $\Lambda \Subset V$, let us denote $\mathcal{G}_{G,\mathbf{a}}(\Lambda)$ the closed (for the topology of weak convergence) convex hull of $\{\nu^{\alpha}_{\Lambda,\mathbf{a}}, \alpha \in F_G\}$.

1. $\Lambda_1 \subset \Lambda_2 \Subset V$ implies $\mathcal{G}_{G,\mathbf{a}}(\Lambda_1) \supset \mathcal{G}_{G,\mathbf{a}}(\Lambda_2)$.
2. If $\nu \in \mathcal{G}$, then for finite $\Lambda \subset V$ and $\zeta \in F_G$, $\nu(\cdot\ |\{\omega \in \Omega_G : \omega_{T^c} = \zeta_{T^c}\}) = \nu^{\zeta}_{T,\mathbf{a}}(\cdot)$.
3. $\mathcal{G}_{G,\mathbf{a}} = \bigcap_{\Lambda \Subset V} \mathcal{G}_{G,\mathbf{a}}(T)$.
4. $\mathcal{G}_{G,\mathbf{a}}$ is nonempty, convex and compact.

Proof. 1. Let $\Lambda_1 \subset \Lambda_2 \Subset V$, let $\zeta \in F_G$. thanks to proposition 3.13 we can write $\nu^{\zeta}_{\Lambda_2,\mathbf{a}}$ as a convex combinaison of elements of $\mathcal{G}_{G,\mathbf{a}}(\Lambda_1)$, that is why $\mathcal{G}_{G,\mathbf{a}}(\Lambda_1) \supset \mathcal{G}_{G,\mathbf{a}}(\Lambda_2)$.

2. By definition 1.7, it is really clear.

3. By the second point, we easily have $\mathcal{G}_{G,\mathbf{a}} \subset \bigcap_{\Lambda \Subset V} \mathcal{G}_{G,\mathbf{a}}(T)$ because it is true for all $\Lambda \Subset V$ and the converse is true by definition 1.7.

4. Because Ω_G is compact (as a product of compact spaces), because the net $(\mathcal{G}_{G,\mathbf{a}}(\Lambda))_{\Lambda \Subset V}$ verify the first point, because $\mathcal{G}_{G,\mathbf{a}} = \bigcap_{\Lambda \Subset V} \mathcal{G}_{G,\mathbf{a}}(T)$, all the $\mathcal{G}_{G,\mathbf{a}}(\Lambda)$ are compact, convex, nonempty and then we have the conclusion.

This concludes the proof. □

Thanks to this lemma, we can prove the following theorem.

Theorem 3.19 — With the notations of theorem 3.17, for all ν hard-core measure for G with activity $\mathbf{a}$, we then have $\nu^{\mathfrak{o}}_{\Lambda,\mathbf{a}} \preceq \nu \preceq \nu^{\mathfrak{e}}_{\Lambda,\mathbf{a}}$.

In particular, there is a unique hard-core measure for G with activity $\mathbf{a}$ if and only if $\nu^{\mathfrak{e}}_{\Lambda,\mathbf{a}} = \nu^{\mathfrak{o}}_{\Lambda,\mathbf{a}}$.

Proof. We keep the notations of lemma 3.18, for all $\Lambda \Subset V$, for all $\nu \in \mathcal{G}_{G,\mathbf{a}}(T)$, by definition of $\mathcal{G}_{\mathbf{a}}(T)$ and by proposition-definition 3.15, we have $\nu^{\mathfrak{o}}_{\Lambda,\mathbf{a}} \preceq \nu \preceq \nu^{\mathfrak{e}}_{\Lambda,\mathbf{a}}$, then for all $\nu \in \mathcal{G}_{G,\mathbf{a}}$, $\nu^{\mathfrak{o}}_{\Lambda,\mathbf{a}} \preceq \nu \preceq \nu^{\mathfrak{e}}_{\Lambda,\mathbf{a}}$ by lemma 3.18, that is why

$$\forall \nu \in \mathcal{G}_{G,\mathbf{a}}, \mu^{\mathfrak{o}}_{G,\mathbf{a}} \preceq \nu \preceq \mu^{\mathfrak{e}}_{G,\mathbf{a}}.$$

□

It is already knew that the set of hard-core measures was convex, we now know that it behaves like a segment and we know the two measures on its border.

Even though this result is really powerful to prove uniqueness, the fact that we need to compare two measures is not ideal. However we can translate the problem in the domain of percolation in order to get an other way to prove uniqueness.

Theorem 3.20 ([BS94]) — The hard-core model on G with activity $\mathbf{a}$ has a unique Gibbs measure if and only if

$$(\mu^{\mathfrak{e}}_{G,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{G,\mathbf{a}})(\text{"There is an infinite path of disagreement for } (\omega,\omega')\text{"}) = 0.$$

Proof. Thanks to theorem 2.7, if $(\mu^{\mathfrak{e}}_{G,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{G,\mathbf{a}})(\text{"There is an infinite path of disagreement for } (\omega,\omega')\text{"}) = 0$, $\mu^{\mathfrak{e}}_{G,\mathbf{a}} = \mu^{\mathfrak{o}}_{G,\mathbf{a}}$, hence there is a unique hard-core measure.

Conversely, let us suppose the hard-core model has a unique Gibbs measure *i.e.* $\mu^{\mathfrak{e}}_{G,\mathbf{a}} = \mu^{\mathfrak{o}}_{G,\mathbf{a}}$. We say that a path Π is perfect for (ω,ω') if and only if every even vertex v on Π verify $\omega_v = 1 - \omega'_v = 1$ and every odd vertex v on Π verify $\omega_v = 1 - \omega'_v = 0$. Then a path Π of disagreement for (ω,ω') is either a perfect path for (ω,ω') or for (ω',ω). That is why $\Lambda \Subset V$, let $v \in \mathcal{E}$. Since $\mu^{\mathfrak{e}}_{G,\mathbf{a}} = \mu^{\mathfrak{o}}_{G,\mathbf{a}}$ and by symmetry, we have for (ω,ω') a realisation under $\mu^{\mathfrak{e}}_{G,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{G,\mathbf{a}}$.

$$\begin{aligned}(\mu^{\mathfrak{e}}_{G,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{G,\mathbf{a}})&(\text{"There is an infinite path of disagreement for } (\omega,\omega') \text{ containing } v\text{"}) \\ &= 2(\mu^{\mathfrak{e}}_{G,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{G,\mathbf{a}})(\text{"There is an infinite perfect path for } (\omega,\omega') \text{ containing } v\text{"}).\end{aligned}$$

Moreover, for $v \in \Lambda' \subset \Lambda$,

$$(\mu^{\mathfrak{c}}_{G,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{G,\mathbf{a}})(\text{"There is a perfect path from } v \text{ to } \partial\Lambda\text{"})$$
$$\leq (\mu^{\mathfrak{c}}_{\Lambda,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}})(\text{"There is a perfect path from } v \text{ to } \partial\Lambda\text{"})$$
$$\leq (\mu^{\mathfrak{c}}_{\Lambda,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}})(\text{"There is a perfect path from } v \text{ to } \partial\Lambda'\text{"}).$$

The first inequality is the consequence of proposition 3.13 and proposition 3.14. The second is trivial. Now we first let Λ grow to V and then we let Λ' grow to V. We now have

$$(\mu^{\mathfrak{c}}_{G,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{G,\mathbf{a}})(\text{"There is an infinite perfect path containing } v\text{"})$$
$$= \lim_{\Lambda \uparrow G} (\mu^{\mathfrak{c}}_{\Lambda,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}})(\text{"There is a perfect path from } v \text{ to } \partial\Lambda\text{"}).$$

Moreover as with (ω, ω') a realisation under $(\mu^{\mathfrak{c}}_{\Lambda,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}})$, $(\omega_{\partial\Lambda}, \omega'_{\partial\Lambda}) = (\mathfrak{c}_{\partial\Lambda}, \mathfrak{o}_{\partial\Lambda})$, then a path is perfect if and only if it is a path of disagreement.

Hence, with lemma 3.12,

$$(\mu^{\mathfrak{c}}_{G,\mathbf{a}} \otimes \mu^{\mathfrak{o}}_{G,\mathbf{a}})(\text{"There is an infinite path of disagreement for } (\omega, \omega') \text{ containing } v\text{"})$$
$$= 2(\mu^{\mathfrak{c}}_{\Lambda,\mathbf{a}}(\omega_v = 1) - \mu^{\mathfrak{o}}_{\Lambda,\mathbf{a}}(\omega'_v = 1)) = 0,$$

by assumption.

Since an infinite path must contain an even vertex, the proof is complete. $\square$

Thanks to the previous theorem, we can easily understand why there is no phase transition in $\mathbb{Z}$ even though the proof of Dobrushin in [Dob68] does not require such a theorem.

Conclusion

The hard-core model gives the behaviour of a more realistic gas than the lattice gas model derived of the Ising-Lenz model what explains the necessity to analyse it rigorously.

However the question of knowing if the critical activity separates the case of phase transition and ergodicity is still open. Moreover this model is an easiest version of reality, the following step is to translate it in a continuous graph (for instance $\mathbb{R}$) and see how to translate the properties of the first model.

In any case as the Ising model is a very stimulating model in the domain of statistical physics, the hard-core model which seems near from the Ising model needs new objects to be studied correctly as we can see for the analysis of [GK04].

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Annexes

A Algorithm for the simulation of the hard-core model

HCM.py

```python
"""Simulate the hard-core model on a subgraph of Z^2"""

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#IMPORTS

import matplotlib.pyplot as plt
import matplotlib.image as mpimg

import numpy as np

#FUNCTIONS

def is_free(G,i,j):
    """Says if the vertex (i,j) is surrounded by no particles in G
    Input : G : boolean list list
                a square matrix of booleans
            i : int
                the row index
            j : int
                the column index
    Output: boolean"""
    l = len(G) - 1
    if i == 0:
        if j == 0:
            return not(G[0][1] or G[1][0])
        if j == l:
            return not(G[0][l - 1] or G[1][l])
        return not(G[0][j - 1] or G[0][j + 1] or G[1][j])
    if i == l:
        if j == 0:
            return not(G[l][1] or G[l - 1][0])
        if j == l:
            return not(G[l][l - 1] or G[l - 1][l])
        return not(G[l][j - 1] or G[l][j + 1] or G[l - 1][j])
```

```python
    if j == 0:
        return not(G[i][1] or G[i - 1][0] or G[i + 1][0])
    if j == l:
        return not(G[i][l - 1] or G[i - 1][l] or G[i + 1][l])
    return not(G[i - 1][j] or G[i + 1][j] or G[i][j - 1] or G[i][j + 1])

if __name__ == "__main__":

    # Model's information
    L = 500            # Size of the square
    V = L ** 2         # Set of vertices
    a = 1.             # Activity
    p = a / (a + 1)    # Probability of drawing heads

    # Number of loops done, must be greater than the number of vertices
    n = 10 * V

    # Configuration of the beginning
    G = [[False for j in range(L)] for i in range(L)]

    # Evolution of the configuration
    for k in range(n):
        v = np.random.randint(0, V)    # Takes a random vertex
        v1, v2 = v // L, v % L         # Gets its coordinates
        b = np.random.binomial(1, p)   # Draw a coin
        if b and is_free(G, v1, v2):   # If heads and possible,
            G[v1][v2] = True           # Puts a particle
        else:                          # Else
            G[v1][v2] = False          # Puts no particles

    ## Displays the simulated configurated if uncommented
    #plt.imshow(G,cmap="binary")
    #plt.show()

    # Sets the file's name
    file_name = "HCM_a=" + str(a) + ".png"

    # Saves the image
    mpimg.imsave(file_name,G,cmap="binary")
```