Homotopy

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Contents

1 Homotopy groups **1**
I Homotopies 1
II The fundamental groupoïd 5
III Fibrations 8
IV Mapping spaces 13

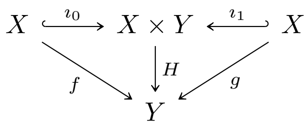
Chapter 1

Homotopy groups

We will denote by **Top** the categroy of topological spaces. The set of continuous maps between two topologicals psaces X and Y will be denoted $\mathbf{Top}(X,Y)$. We also set $I := [0,1]$.

I Homotopies

Definition (*Homotopy*):
For f, $g \in \mathbf{Top}(X,Y)$, a *homotopy* from f to g is a map $H : X \times I \longrightarrow Y$ such that the following diagram commutes.



We denote this by $f \overset{H}{\simeq} g$. We say that f and g are homotopic if ther exists a homotopy, then we denote this by $f \simeq g$.

Lemma:
the homotopy relation, $\simeq$, is an equivalence relation on $\mathbf{Top}(X,Y)$, compatible with composition; i.e. for $f \simeq f'$ and $g \simeq g'$ such that

$$X \underset{f'}{\overset{f}{\rightrightarrows}} Y \underset{g'}{\overset{g}{\rightrightarrows}} Z$$

we have $g \circ f \simeq g' \circ f'$.

proof: Let us proove alle the axioms of an equivalence relation.

- $f \simeq f$ with the constant homotopy defined by $H = f \times \mathrm{id}_I$.
- If $f \overset{H}{\simeq} g$, we denote $\bar{H}$ by $\bar{H}(\bullet, t) = H(\bullet, 1-t)$, then $g \overset{\bar{H}}{\simeq} f$.
- If $f \overset{H}{\simeq} h \overset{H'}{\simeq} h$, we have $f \overset{H*H'}{\simeq} h$ where $*$ denotes the concatenation of two paths induced to homotopies.

Let us now focus on the compatibility with composition.
If $f \overset{F}{\simeq} f$ and $g \overset{G}{\simeq} g'$, then $g \circ f \overset{H}{\simeq} g' \circ f \overset{K}{\simeq} g' \circ f'$ with

- $H = G \circ (f \times \mathrm{id}_I)$.
- $K = (g \times \mathrm{id}_I) \circ F$. □

Definition (*Homotopy category*):
We define **hoTop** to be the *homotopy category of spaces* which objects are **Top**'s objects and maps, continuous maps quotiented by the equivalence relation $\simeq$.

- $\mathrm{Ob}\,\mathbf{hoTop} = \mathrm{Ob}\,\mathbf{Top}$.
- $\mathbf{hoTop}(X,Y) = {}^{\mathbf{Top}(X,Y)}\!/_{\simeq} =: [X,Y]$

We denote by $[f]$ the quivalence class of a continuous map in **hoTop**.

remark: We have an obvious functor $H : \mathbf{Top} \longrightarrow \mathbf{hoTop}$.

Definition (*Homotopy equivalence*)**:**
A map $f \in \mathbf{Top}(X, Y)$ is called an *homotopy equivalence* if $[f] \in [X, Y]$ is an isomorphism, i.e. $\exists g \in \mathbf{Top}(Y, X)$ such that

- $fg \circ \mathrm{id}_Y$.
- $gf \circ \mathrm{id}_X$.

X, $Y \in \mathbf{Top}$ are said to be *homotopically equivalent* if there exists an homotopy equivalence between both. It is also said that X and Y have the same *homotopy type*. We note $X \simeq Y$

$X \in \mathbf{Top}$ is *contractible* if $X \simeq \{*\}$.

example: $[X, \mathbb{R}^n]$ has a single element since $\mathbb{R}^n$ using the convex homoptopy, hence $\mathbb{R}^n$ is contractible.

Definition (*Variants*)**:**

(1) $\mathbf{Top}^2$ is the category of *topological pairs.*

- Ob: (X, A), $\quad X \in \mathbf{Top}$, $A \subset X$.
- Hom: $f \in \mathbf{Top}(X, Y)$, $\quad f(A) \subset B$.
- Homotopies: $\mathbf{Top}^2((X, A) \times I, Y) = \mathbf{Top}^2((X \times I, A \times I), Y)$.

$\mathbf{hoTop}^2$ is defined in the same way.

(2) The *pointed spaces category*, $\mathbf{Top}_*$, is the full subcategory of $\mathbf{Top}^2$ where objects are pairs (X, x) with $x \in X$. Homotopies and $\mathbf{hoTop}_*$ are automatically induced by this definition.

(3) $\mathbf{Top}^2_*$ is the category of *pointed pairs.*

- Ob: (X, A, a), $\quad X \in \mathbf{Top}$, $A \subset X$, $a \in A$.
- Hom: $f \in \mathbf{Top}(X, Y)$, $\quad f(A) \subset B$, $\quad f(a) = b$.
- Homotopies: $\mathbf{Top}^2_*((X, A, a) \times I, Y)$.

$\mathbf{hoTop}^2_*$ is defined in the same way.

remark: These theories are limited, we shall give a small list of their downsides.

(1) Homotopy theorists do not work with **hoTop** in general; it is far too complicated:

- Hard to check if $f : X \longrightarrow Y$ is a homoptopy equivalence.
- Hard to produce maps.

Indeed, f is a homotopy equivalence *iff* $[f]$ is an isomorphism, which means we have to find a "homotopic inverse" g and show two homotopies to identities.
Solution: localise $\mathbf{Top} \rightsquigarrow \mathbf{Top}[w^{-1}]$ by inverting a class of morphisms.

(2) Neither **hoTop** nor $\mathbf{Top}[w^{-1}]$ are (co-)complete. The construction of objects in **hoTop** is very complicated. We could work in **Top**, but how would we ensure that the construction is meaningfull in **hoTop**?
Solution: Work with *models* for **hoTop** and $\mathbf{Top}[w^{-1}]$:

- Model categories (Quillem).
- $(\infty, 1)$-categories (Joyal-Lurie).

(3) We want sequence in an optimal setting: *triangulated categories.* **hoTop** or $\mathbf{Top}[w^{-1}]$ are not (in general) triangulated, only "pretriangulated".
Solution: work with *spectra.*

Definition:
For $(X, x) \in \mathbf{Top}_*$, we set for $n \in \mathbb{N}$

$$\pi_n(X, x) \coloneqq [(I^n, \partial I^n), (X, x)]$$

For $n > 0$, we consider the binary operation $+_i$ on $\pi_n(X, x)$, defined in $\mathbf{Top}^2((I^n, \partial I^n), (X, x))$ by

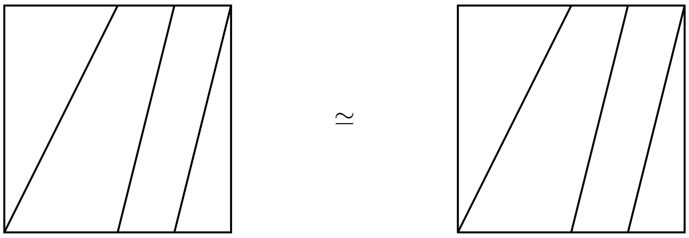
$$(f +_i g)(t_1, \ldots, t_n) \coloneqq \begin{cases} f(t_1, \ldots, t_{i-1}, 2t_i, t_{i+1}, \ldots, t_n) & \text{if } t_i \leqslant \frac{1}{2}. \\ g(t_1, \ldots, t_{i-1}, 2t_i - 1, t_{i+1}, \ldots, t_n) & \text{if } t_i > \frac{1}{2}. \end{cases}$$

inducing the operation in $\pi_n(X, x)$.

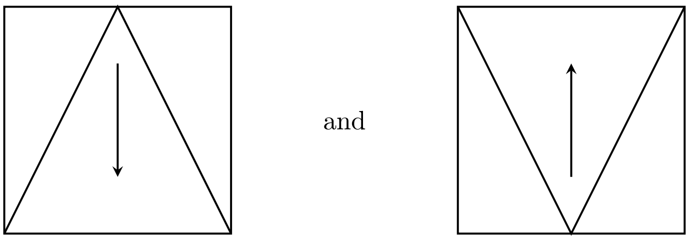
Lemma:
For $n > 0$ and $i \in [\![1, n]\!]$, $+_i$ endows $\pi_n(X, x)$ with a natural group structure. Moreover, if $n > 1$, and i, $j \in [\![1, n]\!]$, $+_i$ and $+_j$ coincide and are commutative.

proof: To show the group structure, it suffices to do it for $n = 1$, it is clear that it induces the result for any $n > 0$.

- The associativity is given by the following convex homotopy in $(I, \partial I) \times I$:

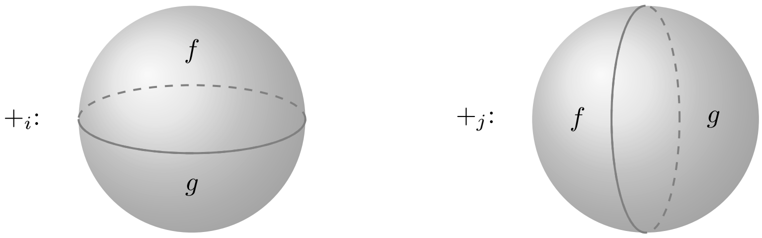


- The inverse is given by $f \mapsto \bar{f}$ where $\bar{f}(t) = f(1 - t)$. The homotopies of both concatenations are the following.



- For the neutral, we shall take the constant path c_x. □

Let us now proove that $+_i$ and $+_j$ coincide.
For $n > 1$, consider that $I^n / \partial I^n \cong \mathbb{S}^n$. We now have:



We could obtain the result easily from this drawing by considering a good rotation, but we prefer to show another property:

$$f +_i g \simeq (f +_j c_x) +_i (c_x +_j g) \stackrel{?}{=} (f +_i c_x) +_j (c_x +_i g) \simeq f +_i g$$

We just have to show the egality in $\mathrm{Vect}(e_i, e_j) \cap I^n \cong I^2$. It is given by this drawing.

c_x	g
f	c_x

For the commutativity, we have

$$f +_1 g \simeq f +_2 g \simeq (c_x +_1 f) +_2 (g +_1 c_x) \stackrel{?}{=} (c_x +_2 g) +_1 (f +_2 c_x) \simeq g +_1 f$$

given by the following drawing.

g	c_x
c_x	f

notation: For $n = 1$, we shall denote $*$ the operation $+_1$, and for $n > 1$ we shall use $+$.

Definition (*Homotopy groups*):
Th previous lemma ensures the following definitions; for $(X,x) \in \mathbf{Top}_*$,

- $\pi_0(X,x)$ is a pointed set.
- $(\pi_1(X,x), *)$ is a group called the *fundamental group*.
- $(\pi_n(X,x), +)$ for $n > 1$ is an abelian group.

We call $\pi_n(X,x)$ the *n-th homotopy group* of (X,x).

remark: The terminology can be misleading for $n \leqslant 1$.

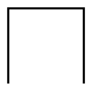
By definition, we have the functors

$$\Pi_n : \mathbf{Top}_* \longrightarrow \mathbf{hoTop}_* \xrightarrow{\pi_n} \begin{cases} \mathbf{Set}_* & \text{if } n = 0 \\ \mathbf{Grp} & \text{if } n = 1 \\ \mathbf{Ab} & \text{if } n > 1 \end{cases}$$

On question can emerge: do these form a "homology theory"? The answer is no, the excision axiom is not satisfied, but every other axiom is. Let us introduce a complementary notion.

Definition:
For $n > 0$, let $J^{n-1} \subset \partial I^n$ be defined as

$$J^{n-1} = \begin{cases} \{1\} & \text{for } n = 1 \\ \partial I^{n-1} \times I \cup I^{n-1} \times \{1\} & \text{for } n > 1 \end{cases} \qquad \text{e.g.} \qquad J^1 =$$



For $(X,A,a) \in \mathbf{Top}_*^2$ and $n > 0$, we set

$$\pi_n(X,A,x) \coloneqq \big[(I^n, \partial I^n, J^{n-1}), (X,A,a)\big]$$

For $n > 1$ and $i < n$, we can define $+_i$ the same way as before. They are still commutative and coinciding. The only drawback is for $i = n$, the concatenation does not work since J^{n-1} is missing a bottom, e.g.

Definition (*Relative homotopy groups*):
Th previous lemma ensures the following definitions; for $(X,x) \in \mathbf{Top}_*$,

- $\pi_1(X,A,a)$ is a pointed set.
- $(\pi_2(X,A,a), *)$ is a group.
- $(\pi_n(X,A,a), +)$ for $n > 2$ is an abelian group.

These objects are called the *relative homotopy groups* of (X,A,a).

We then have the functors

$$\Pi_n : \mathbf{Top}_*^2 \longrightarrow \mathbf{hoTop}_*^2 \xrightarrow{\pi_n} \begin{cases} \mathbf{Set}_* & \text{if } n = 1 \\ \mathbf{Grp} & \text{if } n = 2 \\ \mathbf{Ab} & \text{if } n > 2 \end{cases}$$

remark: For $n > 0$, we have $\pi_n(X, \{x\}, x) = \pi_n(X,x)$.

Definition-Proposition (*Connecting homomorphism*):
Let $(X,A,a) \in \mathbf{Top}_*^2$, we set $s : \begin{array}{ccc} I^{n-1} & \longrightarrow & I^n \\ t & \longmapsto & (t,0) \end{array}$, the section to the fibre over 0 of the projection on the n-th variable.

For $n > 0$, we set $\delta_n : \begin{array}{ccc} \pi_n(X, A, a) & \longrightarrow & \pi_{n-1}(A, a) \\ [f] & \longmapsto & [f \circ s] \end{array}$. It is a well defined group homomorphism for $n > 1$ and defines a natural transformation of functors $\Pi_n \overset{\delta_n}{\Longrightarrow} \Pi_n \circ U$ where U is the forgetfull functor.

$$\begin{array}{ccccc} & & \Pi_n & & \\ \mathbf{Top}^2_* & \nearrow & \Downarrow \delta_n & \searrow & \left\{ \begin{array}{l} \mathbf{Set}_* \\ \mathbf{Grp} \\ \mathbf{Ab} \end{array} \right. \\ & \underset{U}{\searrow} & \mathbf{Top}_* & \underset{\Pi_{n-1}}{\nearrow} & \end{array}$$

We call δ_n the *connecting homomorphism.*

proof: It all lies in the fact that $\delta_n = s^*$, and that s links I^{n-1} to I^n in a very pleasant way. □

Theorem:
For $(X, A, a) \in \mathbf{Top}^2_*$, we have a long exact sequence, natural in (X, A, a):

$$\begin{array}{l} \cdots \longrightarrow \pi_n(A, a) \longrightarrow \pi_n(X, a) \longrightarrow \pi_n(X, A, a) \xrightarrow{\delta_n} \\ \pi_{n-1}(A, a) \longrightarrow \pi_{n-1}(X, a) \longrightarrow \pi_{n-1}(X, A, a) \longrightarrow \\ \cdots\cdots \\ \pi_1(A, a) \longrightarrow \pi_1(X, a) \longrightarrow \pi_1(X, A, a) \xrightarrow{\delta_0} \\ \pi_0(A, a) \longrightarrow \pi_0(X, a) \longrightarrow 0 \end{array}$$

remark: We mean for a pointed set sequence to be exact if $\alpha(S) = \beta^{-1}(\{u\})$, where

$$(S, s) \xrightarrow{\alpha} (T, t) \xrightarrow{\beta} (U, U)$$

proof: The exactness in $\pi_n(X, a)$ is obvious for all $n > 0$, let us show the exactness on both sides of the long exact sequence.
To Fill! □

On question arises, how does $\pi_n(X, x)$ depends on x? We need to introduce the *fundamental groupoïd.*

II The fundamental groupoïd

Definition (*Groupoïd*)**:**
A *groupoïd* is a small category where alla morphisms are isomorphisms.

Definition (*Fundamental groupoïd*)**:**
Let $X \in \mathbf{Top}$, the *fundamental groupoïd* of X, Π_X is a small category with:

- Ob: the underlying set of X.
- Hom: for $a, b \in X$, $\Pi_X(a, b) := \Omega(X, a, b)/\simeq_3$.
  Where $\Omega(X, a, b) = \mathbf{Top}^3((I, \{0\}, \{1\}), (X, \{a\}, \{b\}))$ with composition given by

$$\begin{array}{ccc} \Pi_X(b, c) \times \Pi_X(a, b) & \longrightarrow & \Pi_X(a, c) \\ ([\beta], [\alpha]) & \longmapsto & [\alpha * \beta] \end{array}$$

remark: Checking the axioms of a category for Π_X is the same as showing that $\pi_1(X, x)$ is a group since it is a groupoïd. It is obvious since $[\alpha]^{-1} = [\bar{\alpha}]$.

Definition:
If G is a groupoïd, let us define $\Pi_0(G) := {}^{\mathrm{Ob}\, G}\!/\!_{\sim}$, where $a \sim b \Leftrightarrow G(a,b) \neq \varnothing$ (connexion).
We then set $\Pi_1(G,a) := \mathrm{Aut}_G(a)$.

Of course, we have $\begin{cases} \Pi_0(\Pi_X) = \pi_0(X,a) & \text{as sets.} \\ \Pi_A(\Pi_X, a) = \pi_1(X,a) & \text{as groups.} \end{cases}$

Proposition (*Functoriality of the fundamental groupoïd*)**:**
(1) A continuous map $f : X \longrightarrow Y$ gives a functor $f_* : \Pi_X \longrightarrow \Pi_Y$.

(2) A homotopy $f \overset{H}{\simeq} g$ gives a natural transformation $\Pi_X \underset{g_*}{\overset{f_*}{\Downarrow H_*}} \Pi_Y$.

proof: The result is obvious, we just need to define for $\alpha \in \Pi_X(a,b)$, $H_{*a} := [H(a, \bullet)]$. We then have the following commutative diagram:

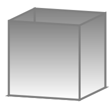
$$\begin{array}{ccc} f(a) & \xrightarrow{f_*\alpha} & f(b) \\ {\scriptstyle H_{*a}}\downarrow & & \downarrow{\scriptstyle H_{*b}} \\ g(a) & \xrightarrow[g_*\alpha]{} & g(b) \end{array}$$

Definition (*Local system of objects*)**:**
Let $X \in \mathbf{Top}$, a *local system if objects* in a category $\mathscr{C}$ is a functor:

$$\Pi_X \longrightarrow \mathscr{C}$$

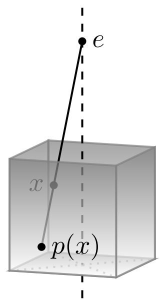
Lemma:
Let $n \geqslant 0$ and $K^{n-1} := \partial I^{n-1} \times I \cup I^{n-1} \times \{0\} \subset I^n$, then K^{n-1} is a retract of I^n.

The set K^{n-1} is the boundary of I^n with the top face removed, e.g.

$K^2 =$ 

proof: A retraction of X onto a subste $A \subset X$ is a homotopy H such that $\forall a \in A$, $H(a, \bullet) = a$ and $H(X, 1) \subset A$. The retraction we want here is the cubic equivalent of the radial projection for a cylinder.

We set $e := (1/2, \ldots, 1/2, 2) \in \mathbb{R}^n$, and for $x \in I^n$, we define $p(x)$ to be the unique point of the intersection $(e,x) \cap K^{n-1}$. Here is a drawing for $n = 3$:



We then just have to set $H(x, \bullet) = [x, p(x)]$, it is a well defined retraction. $\square$

Proposition:
For $X \in \mathbf{Top}$, $n \in \mathbb{N}$, we have a local system:

$$\begin{array}{cccc} \Pi_n : & \Pi_X & \longrightarrow & \mathscr{C} \\ & a & & \pi_n(X,a) \\ & {\scriptstyle [\alpha]}\downarrow & \longmapsto & \downarrow{\scriptstyle \tau_\alpha : [f] \mapsto [\alpha * f * \bar{\alpha}]} \\ & b & & \pi_n(X,a) \end{array}$$

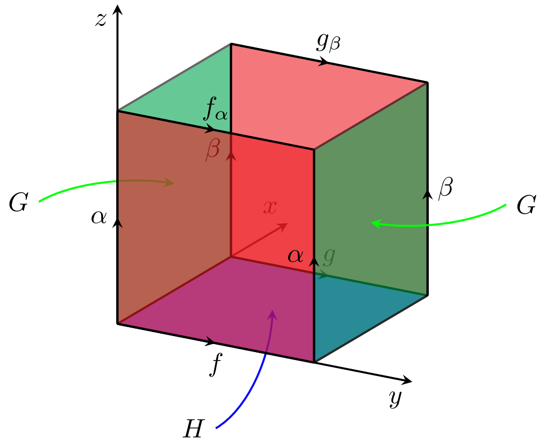
where $\mathscr{C} = \begin{cases} \textbf{Set} \\ \textbf{Grp} \\ \textbf{Ab} \end{cases}$ depending on n.

proof: We have to show it is well defined, let us use the previous lemma.

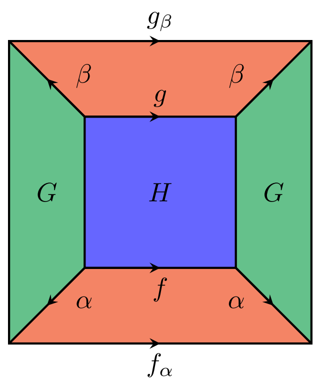
We denote $f_\alpha = \alpha * f * \bar{\alpha}$. If $f \overset{H}{\simeq} g$ and $\alpha \overset{G}{\simeq} \beta$ (relative to endpoints) we have to ensure that $f_\alpha \simeq g_\beta$. For that, we will extend a "usefull" map defined on K^{n+1} to I^{n+2} via the retraction. Let us set $\phi : K^{n+1} \longrightarrow X$ defined by:

- $\phi(x, t, 0) = H(x, t)$.
- $\phi(x, 0, t) = G(x, t) = \phi_(x, 1, t)$.
- $\phi(0, x, t) = f_{\alpha_t}(x)$, $\phi(1, x, t) = f_{\beta_t}(x)$; where $\alpha_t = a \circ [0, t]$. □

ϕ is continuous, here is a "drawing" of it for $n = 1$:



Thus, we just have to set $\Phi = \phi \circ r(\bullet, 1)$, which extends Φ to I^{n+2}, and take $F(x, t) = \Phi(x, t, 1)$ and $f_\alpha \overset{F}{\simeq} g_\beta$. Here is another "drawing" of F for $n = 1$.



Corollary:

From the previous proposition, we can deduce:

(*i*) Any path $\alpha : a \longrightarrow b$ induces an isomorphism:

$$\pi_n(X, a) \xrightarrow{\tau_\alpha} \pi_n(X, b)$$

(*ii*) $\pi_1(X, a)$ acts on $\pi_n(X, a)$.

(*iii*) If $A \subset X$, $\pi_1(A, a)$ acts on $\pi_n(X, A, a)$, and on the whole long exact sequence induced by the pointed pair (X, A, a).

example: $\pi_1(X, a)$ acts on itself by conjugation.

remark: We often use homotopies to lift maps on bigger source sets, but if some sets verify some lifting properties, is it the same for their homotopies?

III Fibrations

Definition (*HLP*):
A map $p : E \longrightarrow B$ has the *homotopy lifing property* (HLP) with respect to $X \in \textbf{Top}$, if, given a commutative square

$$\begin{array}{ccc} X & \xrightarrow{H_0} & E \\ {\scriptstyle \imath_0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[h]{} & B \end{array} \qquad \text{it exists a lifting} \qquad \begin{array}{ccc} X & \xrightarrow{H_0} & E \\ {\scriptstyle \imath_0}\big\downarrow & \overset{H}{\nearrow} & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[h]{} & B \end{array}$$

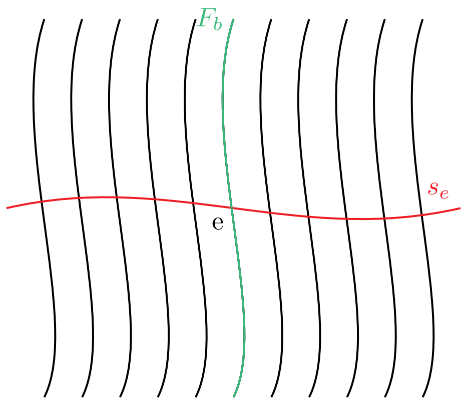
We now define two types of fibrations:

- We call p a Hurewicz-fibration if it has the HLP with respect to all $X \in \textbf{Top}$.
- We call p a Serre-fibration if it has the HLP with respect to I^n for all $n \in \mathbb{N}$.

<u>example</u>: If $E = B \times F \xrightarrow{p} B$ is a projection, then p is a Hurewicz-fibration.
With $H : \begin{array}{ccc} X \times I & \longrightarrow & B \times F \\ (x,t) & \longmapsto & (h(x,t), \mathrm{pr}_F \circ H_0(x)) \end{array}$, we have: $\begin{array}{ccc} X & \xrightarrow{H_0} & B \times F \\ {\scriptstyle \imath_0}\big\downarrow & \overset{H}{\nearrow} & \big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[h]{} & B \end{array}$, but it has no great interest, we would prefer to have the same property for non-trivial fibre bundles.

Definition (*Fibre and section*):
Given $p : (E,e) \longrightarrow (B,b) \in \textbf{Top}_*$, we call $F = F_b := p^{-1}(\{b\})$ the *fibre* of p (over b). A *section* (towards e) is a continuous map $s = s_e : (B,b) \longrightarrow (E,e)$ such that $p \circ s = \mathrm{id}_{(B,b)}$.

For a fibre bundle, it gives:

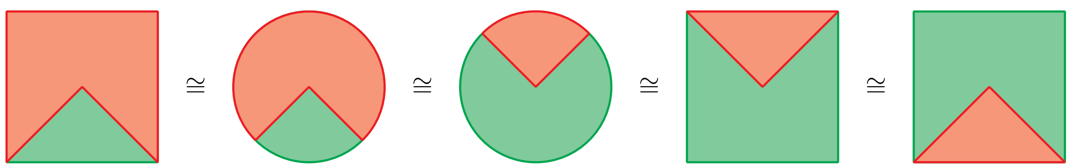


Proposition:
Let $p : (E,e) \longrightarrow (B,b)$ be a Serre-fibration and F the fibre, then

$$p_* : \pi_n(E,F,e) \longrightarrow \pi_n(B,\{b\},b) \simeq \pi_n(B,b)$$

is surjective for all $n \in \mathbb{N}$ and an isomorphism for all $n > 0$.

proof: For the surjectivity, let us first "show" that the pair $(I^{n-1} \times \{0\}, I^n)$ is homeomorphic to the pair (J^{n-1}, I^n):

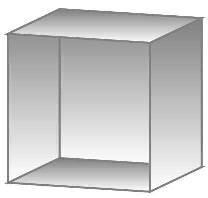


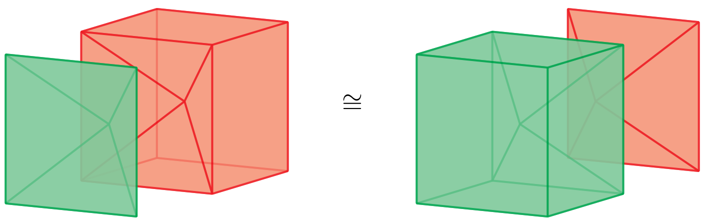
Let us now proove the surjectivity. Let $[f]_B \in \pi_n(B,b)$, we have:

$$\begin{array}{ccc} J^{n-1} & \xrightarrow{c_e} & (E,e) \\ \imath \big\downarrow & \overset{\exists \tilde{f}}{\nearrow} & \big\downarrow p \\ I^n & \xrightarrow[f]{} & (B,b) \end{array}$$

due to the Serre-fibration property transposed to (J^{n-1}, I^n) through the homeomorphism. Of course we have $p_*([\tilde{f}]_E) = [p \circ \tilde{f}]_B = [f]_B$.

For the injectivity, we define $L^n = I^n \times \{0\} \cup I^n \times \{1\} \cup J^{n-1} \times I$ (just K^n rotated). The pair (I^{n+1}, L^n) is homeomorphic to the pair $(I^{n+1}, I^n \times \{0\})$:

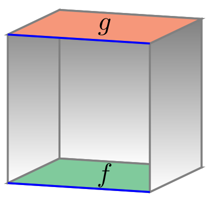
$L^2 =$ 



Thus we can use the HLP for the inclusion (I^{n+1}, L^n).
Let us then show the injectivity. Suppose $p_*([f]_E) = p_*([g]_E)$, it means that $[p\circ f]_B = [p \circ g]_B$, soit $p \circ f \overset{h}{\simeq} p \circ g$. We now define $H_0 : L^n \longrightarrow E$ by:

- $H_0(\bullet, 0) = f$.
- $H_0(\bullet, 1) = g$.
- $H_{0|J^{n-1}\times I} = e$

Of course, we have $h \circ \imath = p \circ H_0$ which follows frome the good definition of H_0, and the fact that $p(F) = \{b\}$. Here is a drawing of H_0 where the grey equals to e and the blue is in F:

H_0 : 

We can now properly lift h:

$$\begin{array}{ccc} L^n & \xrightarrow{H_0} & (E,e) \\ \imath \big\downarrow & \overset{\exists H}{\nearrow} & \big\downarrow p \\ I^{n+1} & \xrightarrow[h]{} & (B,b) \end{array}$$

and we have $p(H(x,0,\bullet)) = h(x,0,\bullet) = b$, so $p(H(x,0,\bullet)) \in F$, thus H is a relative homotopy and $f \overset{H}{\simeq} g$, so $[f]_E = [g]_E$. □

Corollary:
Let $p : (E, e) \longrightarrow (B, b)$ be a Serre-fibration, F the fibre of p and $\imath : (F, e) \longrightarrow (E, e)$ the inclusion, then the fibration sequence; $0 \longrightarrow (F, e) \longrightarrow (E, e) \longrightarrow (B, b) \longrightarrow 0$ induces a long exact sequence:

$$\begin{array}{l} \cdots \longrightarrow \pi_n(F,e) \longrightarrow \pi_n(E,e) \longrightarrow \pi_n(B,b) \\ \hookrightarrow \pi_{n-1}(F,e) \longrightarrow \pi_{n-1}(E,e) \longrightarrow \pi_{n-1}(B,b) \\ \cdots\cdots \\ \hookrightarrow \pi_1(F,e) \longrightarrow \pi_1(E,e) \longrightarrow \pi_1(B,b) \\ \hookrightarrow \pi_0(F,e) \longrightarrow \pi_0(E,e) \longrightarrow \pi_0(B,b) \longrightarrow 0 \end{array}$$

proof: We just need to replace (E, F, e) by (B, b) thanks to the previous proposition in the long exact sequence associated to the pointed pair (E, F, e). We must still be carefull at the exactness in $\pi_0(E, e)$. Let us suppose that $[e']_E \in \pi_0(E, e)$ verifies $p_*([e']_E) = [b]_B$. We have an arc $\gamma : p(e') \longrightarrow b$, which we can lift in the following diagramm:

$$\begin{array}{ccc} \{0\} & \xrightarrow{\;e\;} & (E,e) \\ {\scriptstyle \imath_0}\downarrow & \overset{\exists\tilde\gamma}{\nearrow} & \downarrow{\scriptstyle p} \\ I & \xrightarrow[\;\gamma\;]{} & (B,b) \end{array}$$

But $p \circ \tilde\gamma(1) = \gamma(1) = b$ so $f := \tilde\gamma(1) \in F$, hence $[f]_E = [e']_E$, thus $\imath_*([f]_F) = [e']_E$.
The reciprocity of the inclusion is trivial. □

The fibration property uses the same terminology as the fibre products, which are locally defined. The following proposition shall give us a very useful caracterisation of Serre-fibrations.

Proposition:
To be a Serre-fibration is a local property, i.e. if $p : E \longrightarrow B$ is given and $B = \bigcup_{\lambda\in\Lambda} U_\lambda$ has a covering of open sets such that for each $\lambda \in \Lambda$, $p^{-1}(U_\lambda) \xrightarrow{\;p\;} U_\lambda$ is a Serre-fibration, then p is a Serre-fibration.

Before attempting the proof, we shall demonstrate a very useful lemma.

Lemma (*Lebesgue number*)**:**
For X a compact metric space and $(U_\lambda)_{\lambda\in\Lambda}$ an open covering of X, there exists $\delta > 0$ such that:

$$\forall x \in X, \exists \lambda \in \Lambda, \quad \mathbb{D}(x, \delta) \subset U_\lambda.$$

proof: Since X is compact, we cans extract a finite cover of X given by $(U_1, \ldots, U_n)$. If there exists i such that $U_i = X$, the proof is obviously over, so we suppose that $\forall i \leqslant n$, $U_i \subsetneq X$.

We set $F_i = U_i^c \neq \varnothing$ and $f(x) := \frac{1}{n}\sum_{i=1}^{n} d(x, F_i)$ which is well defined and continuous since $F_i \neq \varnothing$. For it is continuous on a compact, it attains its minimum δ in x_0 and since (U_i) covers X, there exists i_0 such that $x_0 \notin F_{i_0}$, hence $\delta > 0$.

Now if there exists $x \in X$ such that $\forall i$, $\mathbb{D}(x, \delta) \not\subset U_i$, then $\forall i$, $d(x, F_i) < \delta$, hence $f(x) < \delta$ which is absurd, thus $\forall x \in X$, $\exists i \leqslant n$, $\mathbb{D}(x, \delta) \subset U_i$. □

Let us now proove the previous proposition.

proof: Let $n \in \mathbb{N}$, we consider the lifting problem of the following commutative square:

$$\begin{array}{ccc} I^n & \xrightarrow{\;H_0\;} & E \\ {\scriptstyle \imath_0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ I^{n+1} & \xrightarrow[\;h\;]{} & B \end{array}$$

We have an open covering of I^{n+1} given by the $h^{-1}(U_i)$. By LEBESGUE lemma, with $d = \|\bullet\|_1$, there exists $N \in \mathbb{N}$ such that we can cover I^{n+1} with small cubes defined as $W_\alpha = [\frac{\alpha_0 - 1}{N}, \frac{\alpha_0 - 1}{N}] \times \cdots \times [\frac{\alpha_n - 1}{N}, \frac{\alpha_n - 1}{N}]$ with $\alpha \in [\![1, N]\!]^{n+1}$. And $h(W_\alpha) \subset U_i$ for some i.
Of course if we have $W'_\alpha = W_{\mathrm{pr}_{<n}(\alpha)} = \imath_0^{-1}(W_\alpha) \subset H_0^{-1}(p^{-1}(U_i))$.

The Serre-fibration property on every U_i induces a lifing for every α (since $(W_\alpha, W'_\alpha) \cong (I^{n+1}, I^n \times \{0\})$):

$$\begin{array}{ccc} W'_\alpha & \xrightarrow{H_{0|W'_\alpha}} & p^{-1}(U_i) \\ {\scriptstyle \imath_{0|W_\alpha}} \downarrow & \exists H_\alpha \nearrow & \downarrow {\scriptstyle p} \\ W_\alpha & \xrightarrow[h]{} & U_i \end{array}$$

Now we just need to glue the results one by one with the lexicographic order on $[\![1, N]\!]^{n+1}$ to obtain H. $\square$

<u>example</u>: Fibre bundles are Serre-fibrations, since they are locally trivial, hence locally homoemorphic to a projection. Homogeneous spaces give good examples of fibre bundles, hence Serre-fibrations.

PROPOSITION:
Let G be a Hausdorff topological group and H a closed subgroup of G. Let G/H be the space of orbits for the right action of H on G.

Suppose the quotient map $G \xrightarrow{p} G/H$ has a local section at $[e]_H$, then if $K \leqslant H$ is closed, the quotient map $G/K \xrightarrow{q} G/H$ is a fibre bundle with fibre H/K.

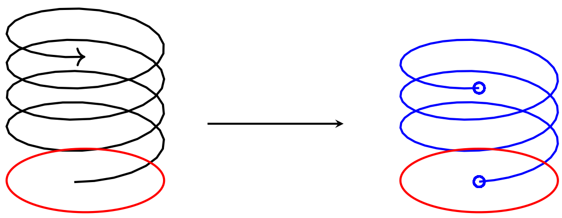
<u>example</u>: The most common example is the fibration $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ which gives us for every $k \in \mathbb{N}^*$, a fibration

$$0 \longrightarrow \mathbb{Z}/k\mathbb{Z} \longrightarrow \mathbb{R}/k\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

It is the embedding of $\mathbb{S}^1$ by itself of degree k:

$$\mathbb{S}^1 \xrightarrow{k\cdot} \mathbb{S}^1$$

Here is a drawing for $k = 3$.



proof: We recall that with Hausdorff topological groups, a description of the neighbourhood of any point (in particular the neutral) gives us a description of the whole group. By hypothesis, we have a local section $s : U \longrightarrow G$ with $[e]_H \in U \subset G/H$.

We set $\phi : \begin{array}{ccc} U \times H/K & \longrightarrow & G/K \\ ([g]_H, [h]_K) & \longmapsto & [s([g]_H) \cdot h]_K \end{array}$, it is oviously continuous well defined since we multiply on the left.
We have a commutative triangle:

$$\begin{array}{ccc} U \times H/K & \xrightarrow{\phi} & G/K \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle q} & \\ & U & \end{array}$$

Indeed, $q \circ \phi([g]_H, [h]_K) = q([s([g]_h) \cdot h]_K) = p(s([g]_h) \cdot h) = p \circ s([g]_H) = [g]_H$.
On top of that, ϕ is a homeomorphism with inverse given by $\psi([g]_K) = ([g]_H, [s([g]_H)^{-1} \cdot g]_K)$.

Now that we have a local trivialisation of the bundle at $[e]_H$, we deduce a local trivialisation at $[x]_H$ through the left translation by x: $p^{-1}U \mapsto xp^{-1}(U)$. $\square$

<u>example</u>: For $n \in \mathbb{N}$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we set $G(n) := \begin{cases} O_n(\mathbb{R}) & \text{if } \mathbb{F} = \mathbb{R}. \\ U_n(\mathbb{C}) & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$
If $k \leqslant n$, $G(k)$ can be seen as a closed subgroup of $G(n)$ through the inclusion $\begin{array}{ccc} G(k) & \longrightarrow & G(n) \\ A & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & I_{n-k} \end{pmatrix} \end{array}$.

We then have $p : G(n) \longrightarrow G(n)/G(k)$ admitting a local section at $[I_n]_k$.

To proove this, we will use the action of G_n on the following example.

<u>example</u> (Stiefel manifold): Let $1 \leqslant l \leqslant n$, we set $V_{k,l} = \left\{ (v_1, \dots, v_l) \in (\mathbb{F}^n)^l \middle| \langle v_i, v_j \rangle = \delta_i^j \right\}$.
Let $(e_1, \dots, e_n)$ be the canonical basis of $\mathbb{F}^n$ and $x_0 \coloneqq (e_{n-l+1}, \dots, e_n) \in V_{n,l}$, $G(n) \curvearrowright V_{n,l}$ transitively and $G(n)_{x_0} = G(n-l)$, it gives us a continuous bijection $\begin{array}{rcl} G(n)/G(n-l) & \longrightarrow & V_{n,l} \\ [A]_{n-l} & \longmapsto & A \cdot x_0 \end{array}$.

Since ${}^{G(n)}\!/\!_{G(k)}$ is compact and V_{n-l} Hausdorff, it is a homeomorphism, thuand we have the following commutative square:

$$\begin{array}{ccc} G(n) & \xrightarrow{p} & {}^{G(n)}\!/\!_{G(n-l)} \\ \downarrow & & \downarrow \\ V_{n,n} & \xrightarrow[\mathrm{pr}_{>n-l}]{} & V_{n,l} \end{array}$$

Therefore, finding a local section of p at $[I_n]_{n-l}$ is the same as fining a local section of $\mathrm{pr}_{>n-l}$ at x_0.

To do so, we take $U \coloneqq \{(v_1, \dots, v_l) \in V_{n,l} | (e_1, \dots e_{n-l}, v_1, \dots, v_{n-l}) \text{ is a basis}\}$ which is obviously an open set (through det) and contains x_0. We then define

$$\begin{array}{rrcl} s: & U & \longrightarrow & V_{n,n} \\ & (v_1, \dots, v_l) & \longmapsto & \underleftarrow{\text{G-S}}\,(e_1, \dots, e_n, v_1, \dots, v_l) \end{array}$$

it is continuous, since Gram-Schmidt can be obtained from a Gram-matrix, and we have $\mathrm{pr}_{>n-l} \circ s = \mathrm{id}_U$.

We now have a fibre bundle:

$$\begin{array}{ccccc} G(n-l) & \longrightarrow & G(n) & \longrightarrow & {}^{G(n)}\!/\!_{G(n-l)} \\ \downarrow & & \downarrow & & \downarrow \\ V_{n-l,n-l} & \longrightarrow & V_{n,n} & \longrightarrow & V_{n,l} \end{array}$$

The previous proposition gives many variations, e.g. for $1 \leqslant k \leqslant l \leqslant n$, we have:

$$\begin{array}{ccccc} {}^{G(n-k)}\!/\!_{G(n-l)} & \longrightarrow & {}^{G(n)}\!/\!_{G(n-l)} & \longrightarrow & {}^{G(n)}\!/\!_{G(n-k)} \\ \downarrow & & \downarrow & & \downarrow \\ V_{n-l,l-k} & \longrightarrow & V_{n,l} & \longrightarrow & V_{n,k} \end{array}$$

<u>example</u> (Grassman manifold): Let $1 \leqslant l \leqslant n$, we set $G_{n,l}$ to be the set of subspaces of $\mathbb{F}^n$ of dimension l. We have an obvious surjection:

$$\begin{array}{rcl} V_{n,l} & \longrightarrow & G_{n,l} \\ (v_1, \dots, v_l) & \longmapsto & \mathrm{Vect}(v_1, \dots, v_l) \end{array}$$

It gives $G_{n,l}$ the projection topology. If $X_0 = \mathrm{Vect}(x_0)$, we have $G(n)_{X_0} = G(n-l) \times G(l)$. It induces a homeomorphism

$$ {}^{G(n)}\!/\!_{G(n-l) \times G(l)} \longrightarrow G_{n,l}$$

With the previous proposition, we get more variations from the inclusions:

$$G(n-l) \hookrightarrow G(n-l) \times G(l) \hookrightarrow G(n)$$

Which give:

$$\begin{array}{ccccc} G(l) & \longrightarrow & {}^{G(n)}\!/\!_{G(n-l)} & \longrightarrow & {}^{G(n)}\!/\!_{G(n-l) \times G(l)} \\ \downarrow & & \downarrow & & \downarrow \\ V_{l,l} & \longrightarrow & V_{n,l} & \longrightarrow & G_{n,l} \end{array}$$

remark: The same works for $\mathbb{H}$ the unitary division ring of the quaternions, and we obtain fibre bundles:

$$V^{\mathbb{H}}_{n-l,l-k} \longrightarrow V^{\mathbb{H}}_{n,l} \longrightarrow V^{\mathbb{H}}_{n,k} \qquad \text{and} \qquad V^{\mathbb{H}}_{l,l} \longrightarrow V^{\mathbb{H}}_{n,l} \longrightarrow G^{\mathbb{H}}_{n,l}$$

This is used to solve the problem: can $\mathbb{R}^{2^n}$ be endowed with a unitary division ring structure?

Definition (*Hopf fibre bundle*):
We define the Hopf fibre bundle as:

$$\left\{\begin{array}{ccccc} \mathbb{S}^0 & \longrightarrow & \mathbb{S}^n & \longrightarrow & \mathbb{R}P^n \\ \mathbb{S}^1 & \longrightarrow & \mathbb{S}^{2n+1} & \longrightarrow & \mathbb{C}P^n \\ \mathbb{S}^3 & \longrightarrow & \mathbb{S}^{4n+3} & \longrightarrow & \mathbb{H}P^n \end{array}\right. \quad \text{using} \quad V^{\mathbb{F}}_{1,1} \longrightarrow V^{\mathbb{F}}_{n+1,1} \longrightarrow G^{\mathbb{F}}_{n+1,1} \quad \text{for} \quad \mathbb{F} = \left\{\begin{array}{c} \mathbb{R} \\ \mathbb{C} \\ \mathbb{H} \end{array}\right.$$

For $n = 1$, using $\left\{\begin{array}{l} \mathbb{R}P^1 \cong S^1 \\ \mathbb{C}P^1 \cong S^2 \\ \mathbb{H}P^1 \cong S^4 \end{array}\right.$, we have $\begin{array}{ccccc} \mathbb{S}^0 & \longrightarrow & \mathbb{S}^1 & \xrightarrow{2} & \mathbb{S}^1 \\ \mathbb{S}^1 & \longrightarrow & \mathbb{S}^3 & \xrightarrow{\eta} & \mathbb{S}^2 \\ \mathbb{S}^3 & \longrightarrow & \mathbb{S}^7 & \xrightarrow{\nu} & \mathbb{S}^4 \end{array}$, ckeck on YouTube.

remark: The octonions are not associative (nor commutative) so $\mathbb{O}P^n$ cannot be defined for $n \geqslant 3$, but it exists for $n = 1, 2$. We get one more example of fibre bundle:

$$\mathbb{S}^7 \longrightarrow \mathbb{S}^{15} \xrightarrow{\sigma} \mathbb{S}^8$$

We would like to replace maps by fibration, there is a problem: **Top** is not cartesian-closed. We must use a category with a more fitting structure for this.

IV Mapping spaces

Definition (*Cartesian-closed*):
We say that a category $\mathscr{C}$ is *cartesian-closed* if it is locally-small and follows the axioms:

(*i*) There exists $*$ a terminal object.

(*ii*) It admits finite cartesian product.

(*iii*) For each X, $Y \in \operatorname{Ob}\mathscr{C}$, $\hom_{\mathscr{C}}(X, Y) \in \operatorname{Ob}\mathbb{C}$.

(*iv*) There is an isomorphism, natural in the 3 variables:

$$\star : \hom_{\mathscr{C}}(X \times Y, Z) \longrightarrow \hom_{\mathscr{C}}(X, \hom_{\mathscr{C}}(Y, Z))$$

i.e. $(\bullet \times \bullet) \vdash \hom_{\mathscr{C}}(\bullet, \bullet)$.

(*v*) Some other axioms that will not be stated here.

notation: If $\mathscr{C}$ is cartesian-closed, $\hom_{\mathscr{C}}(X, Y)$ is denoted Y^X, and the adjunction gives us $Z^{X\times Y} \simeq (Z^Y)^X$.

Definition:
For X, $Y \in$ **Top**, we can endow **Top**(X, Y) with the *compact-open topology*, given by the following open-basis:

$$\mathcal{O}(K, U) = \mathbf{Top}^2((X, K), (Y, U)) \qquad \text{for} \quad \left\{\begin{array}{ll} K & \text{compact set of } X. \\ U & \text{open set of } Y. \end{array}\right.$$

We denote by $\operatorname{map}(X, Y)$ the set **Top**(X, Y) with this topology.

Lemma:
The mapping spaces have the following properties:

(*i*) If Y is Hausdorff, $\operatorname{map}(X, Y)$ is too.

(*ii*) $\begin{array}{cccc} \psi : & \operatorname{map}(X, Y \times Z) & \longrightarrow & \operatorname{map}(X, Y) \times \operatorname{map}(X, Z) \\ & f & \longmapsto & (\mathrm{pr}_1 \circ f, \mathrm{pr}_2 \circ f) \end{array}$ is a homeomorphism.

(*iii*) We have a continuous map $\begin{array}{cccc} \phi : & \operatorname{map}(X \times Y, Z) & \longrightarrow & \operatorname{map}(X, \operatorname{map}(Y, Z)) \\ & f & \longmapsto & \hat{f} : \begin{array}{ccc} X & \longrightarrow & \operatorname{map}(X, Y) \\ x & \longmapsto & f(x, \bullet) \end{array} \end{array}$.

(*iv*) If Y is Hausdorff and locally-compact and $g \in \operatorname{map}(X, \operatorname{map}(Y, Z))$, $\begin{array}{cccc} \check{g} : & X \times Y & \longrightarrow & Z \\ & (x, y) & \longmapsto & g(x)(y) \end{array}$ is continuous.

(*v*) If X is Hausdorff and Y Hausdorff and locally-compact, ϕ is an homeomorphism of inverse $g \mapsto \check{g}$.

proof: Let us proove all the assertions point by point.

(i) If Y is Hausdorff, for $f \neq g \in \mathbf{Top}(X, Y)$, $\exists x \in X$ such that $f(x) \neq g(x)$.
Since Y is Hausdorff, we have separating open sets of Y, U_f and U_g such that $U_f \cap U_g = \varnothing$ and $f(x) \in U_f$, $g(x) \in U_g$.
$\{x\}$ is compact since it is finite, thus $f \in \mathcal{O}(\{x\}, U_f)$ and $g \in \mathcal{O}(\{x\}, U_g)$ which are of empty intersection, hence $\mathrm{map}(X, Y)$ is Hausdorff.

(ii) Let $\mathcal{O}(K_X, U_Y)$ be a standard open set of $\mathrm{map}(X, Y)$, we set $V \coloneqq \mathcal{O}(K_X, U_Y) \times \mathrm{map}(X, Z)$.

$$\begin{aligned}\psi^{-1}(V) &= \left\{ f : X \longrightarrow Y \times Z \ \middle| \mathrm{pr}_1(f(K_X)) \subset U_Y \right\} \\ &= \mathcal{O}(K_X, \mathrm{pr}_1^{-1}(U_Y))\end{aligned}$$

which is open.
Symetrically, $\psi^{-1}(\mathrm{map}(X, Y) \times \mathcal{O}(K_X, U_Z)) = \mathcal{O}(K_X, \mathrm{pr}_2^{-1}(U_Z))$, hence, ψ is continuous.

(iii) It is obvious that the $\mathcal{O}(K_X, \mathcal{O}(K_Y, U_Z))$ form a basis of $\mathrm{map}(X, \mathrm{map}(Y, Z))$, thus we compute:

$$\begin{aligned}\phi^{-1}(\mathcal{O}(K_X, \mathcal{O}(K_Y, U_Z))) &= \left\{ f : X \times Y \longrightarrow Z \ \middle| \hat{f}(K_X) \subset \mathcal{O}(K_Y, U_Z) \right\} \\ &= \left\{ f : X \times Y \longrightarrow Z \ \middle| \hat{f}(K_X)(K_Y) \subset U_Z \right\} \\ &= \left\{ f : X \times Y \longrightarrow Z \ \middle| f(K_X, K_Y) \subset U_Z \right\} \\ &= \mathcal{O}(K_X \times K_Y, U_Z)\end{aligned}$$

A product of compact is compact for the product topology (Tychonoff's theorem) hence, ϕ is continuous.

(iv) If Y is Hausdorff and locally-compact, every point has a local base of relatively compact neighbourhoods. Let U_Z be an open set of Z, for $x \in X$, since $g(x)$ is continuous, if $g(x)(y) \in U_Z$, we have U_y relatively compact such that $g(x)(\bar{U}_y) \subset U_Z$. We have then $g(x) \in \mathcal{O}(\bar{U}_Y, U_Z)$, thus:

$$\check{g}^{-1}(U_Z) = \bigcup_{y \in Y} g^{-1}\mathcal{O}(\bar{U}_y, U_Z) \times U_y$$

which is open since g is continuous, hence $\check{g}$ is continuous.

(v) If X is Hausdorff and Y Hausdorff and locally-compact, we can obviously define ϕ^{-1} as stated, but we have to ensure that it is continuous. To fill! □

remark: **Top** with internal homomorphisms $\mathrm{map}(X, Y)$ is not cartesian-closed. It works with compactly-generated weak-Hausdorff spaces (CGWH). This category is cartesian-closed and mapping spaces are denoted Y^X.

We have a functor $k : \mathbf{Top} \longrightarrow \mathbf{CGWH}$ and a natural transformation $k \Longrightarrow \mathrm{id}$.

If X is Hausdorff and locally-compact, the morphism $k(X) \longrightarrow X$ induced by the natural transformation is an homeomorphism, but limits and colimits do not agree in general.

Definition:
Let $f \in \mathbf{Top}(X, Y)$, we define $F(f) \coloneqq \left\{(x, \omega) \in X \times Y^I \middle| \omega(0) = f(x)\right\}$. It is the pullback:

$$\begin{array}{ccc} F(f) & \longrightarrow & Y^I \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle 0^*} \\ X & \xrightarrow[f]{} & Y \end{array}$$

We set $\begin{array}{cccc} i : & X & \longrightarrow & F(f) \\ & x & \longmapsto & (x, c_{f(x)}) \end{array}$ and $\begin{array}{cccc} \epsilon_1 : & F(f) & \longrightarrow & Y \\ & (x, \omega) & \longmapsto & \omega(1) \end{array}$. We have $\epsilon_1 \circ i = f$.

Lemma:
i is a homotopy equivalence and ϵ_1 is a Serre-fibration (Hurewicz for CGWH).

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