# Introduction to toric geometry

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# Affine toric varieties

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*Definition* : Affine toric variety

An affine toric variety is an irreducible affine variety $V$ containing a torus $T_N$ which is a Zariski open subset and such that the action of $T_N$ on itself extends to an action $T_N \times V \to V$ on $V$ given by a morphism.

$\mathbf{V}(x^3 - y^2) \subseteq \mathbb{C}^2$ is an affine toric variety with torus:

$$\{(t^2, t^3) \mid t \in \mathbb{C}^*\} \simeq \mathbb{C}^*$$

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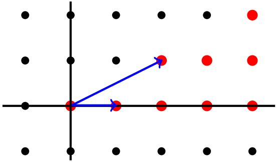
Let $\mathcal{A} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^2$ then $\mathbb{N}\mathcal{A}$ is the affine semigroup composed of the red dots.



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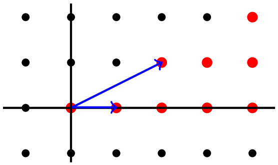

*Theorem* :

Let $V$ be an affine variety. The following are equivalent:

1. $V$ is an affine toric variety
2. $V = \mathrm{Spec}(\mathbb{C}[S])$ for an affine semigroup $S$

# Convex polyhedral cones

Fix $M$ and $N$ two dual vector spaces.

*Definition* : Convex polyhedral cone

$$\sigma = \mathrm{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u u \;\middle|\; \lambda_u \geq 0 \right\} \subseteq N$$

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*Definition* : Dual cone

The dual cone of convex polyhedral cone $\sigma$ is:

$$\sigma^\vee = \{ m \in M \mid \langle m, u \rangle \geq 0, \forall u \in \sigma \} \subseteq M.$$

$\dim(\sigma) := \dim(\mathrm{Span}(\sigma))$

# Example : $\sigma = \mathrm{Cone}(2e_1 + e_2\ ,\ e_2 - e_1\ ,\ 2e_2 + e_3\ ,\ -e_1 + e_2 - 2e_3)$

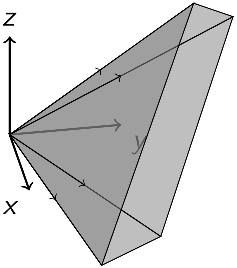
- $\dim(\sigma) = 3$
- the facets are the shaded areas;
- the facet normals are:

$e_1 - 2e_2 + 4e_3$

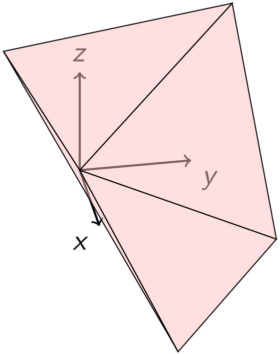
$e_1 + e_2 - 2e_3$

$e_1 + e_2$

$-2e_1 + 4e_2 + 3e_3$




Thus $\sigma^\vee = \mathrm{Cone}(e_1 - 2e_2 + 4e_3\ ,\ e_1 + e_2 - 2e_3\ ,\ e_1 + e_2\ ,\ -2e_1 + 4e_2 + 3e_3)$.



# Link with affine toric varieties

Let $N$ and $M$ be dual lattices and fix $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$.
A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is rational if $\sigma = \mathrm{Cone}(S)$ for some finite set $S \subseteq N$.

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*Proposition* : Gordan's Lemma

$S_\sigma = \sigma^\vee \cap M$ is finitely generated and hence is an affine semigroup.

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*Theorem* :

Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a rational polyhedral cone with semigroup $S_\sigma = \sigma^\vee \cap M$. Then

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]) = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$

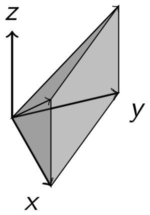
is an affine toric variety. Furthermore,

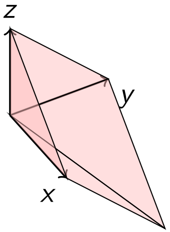
$$\dim U_\sigma = n \iff \text{the torus of } U_\sigma \text{ is } T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \iff \sigma \text{ is strongly convex.}$$

# Example

$\sigma = \mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subseteq N_{\mathbb{R}}$

$\sigma^{\vee} = \mathrm{Cone}(e_1, e_2, e_3, e_1 + e_2 - e_3) \subseteq M_{\mathbb{R}}$




$V = \mathbf{V}(xy - zw) \subset \mathbb{C}^4$ is an affine toric variety with torus $(\mathbb{C}^*)^3$

$S = \mathbb{N}\{e_1, e_2, e_3, e_1 + e_2 - e_3\}$ is an affine semigroup which generates the character lattice of the torus of $V$.

$$U_\sigma = \mathbf{V}(xy - zw)$$