Séminaire M2

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Introduction à la géométrie torique

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Introduction

This seminary is an introduction to affine toric varieties which form a large family of examples one often studies first to test algebraic geometry theorems. In this seminary we first introduce tori and affine toric varieties. After giving the definitions we show different point of views on affine toric varieties and then link them all so that we can have four characterizations of affine toric varieties. In the second part we introduce polyhedral cones and use duality and convexity to finally show that we can think of affine toric varieties as coming from cones.
I have studied this seminary with the book [1] and in the second part I have also used [2].

Je remercie Carl Tipler pour sa disponibilité et le temps qu'il m'a accordé pour me proposer ce sujet de séminaire.

I Introduction to affine toric varieties

I.A Tori

Definition 1:

A torus T is an affine variety isomorphic to $(\mathbb{C}^*)^n$ endowed with a group stucture inherited from the isomorphism.

Remark 2:

A torus is thus irreducible.

Character and one-parameter subgroups We remind you that a character of a group T is a group homomorphism $\chi : T \to \mathbb{C}^*$.
The characters of $(\mathbb{C}^*)^n$ are given by:

$$\text{for} \quad m = (a_1, \dots, a_n) \in \mathbb{Z}^n \quad \text{define} \quad \chi^m(t_1, \dots, t_n) = t_1^{a_1} \dots t_n^{a_n}.$$

Thus they form a group isomorphic to $\mathbb{Z}^n$ and the following remark stands.

Remark 3:

For a torus T, its characters form a free abelian group M of rank equal to the dimension of T

We also remind one-parameter subgroup of a torus T is a group homomorphism $\lambda : \mathbb{C}^* \to T$.
The one-parameter subgroups of $(\mathbb{C}^*)^n$ are given by:

$$\text{for} \quad u = (b_1, \dots, b_n) \in \mathbb{Z}^n \quad \text{define} \quad \lambda^u(t) = \left(t^{b_1}, \dots, t^{b_n}\right).$$

Thus they form a group isomorphic to $\mathbb{Z}^n$ and the following remark stands.

Remark 4:

For a torus T, its one-parameter subgroups form a free abelian group N of rank equal to the dimension of T

One can then define a bilinear pairing $\langle \cdot, \cdot \rangle : \left\{ \begin{array}{ccccc} M & \times & N & \to & \mathbb{Z} \\ m & , & u & \mapsto & k \end{array} \right.$ where $\chi^m, \lambda^u \mapsto \left(\chi^m \circ \lambda^u : t \mapsto t^k\right)$

Example 5:

If $T = (\mathbb{C}^*)^n$ and $m = (a_1, \dots, a_n) \in \mathbb{Z}^n$ and $u = (b_1, \dots, b_n) \in \mathbb{Z}^n$ then $\langle\, m, u \,\rangle = \sum_{i=1}^n a_i b_i$.

Moreover one obtains a canonical isomorphism $N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq T$; $u \otimes t = \lambda^u(t)$. Hence in the litterature we often refer to a torus by T_N.

Lemma 6:

Let T_1 and T_2 be two tori and let $f : T_1 \to T_2$ be a group homomorphism. Then the image $f(T_1)$ is a torus and is closed in T_2.

Proof. We can suppose $f : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^s$. Thus component wise $f_i : (\mathbb{C}^*)^n \to \mathbb{C}^*$ is a group homomorphism (a character) so is of the form $f_i(t_1, \dots, t_n) = t_1^{a_1} \dots t_n^{a_n}$ and its image is either $\{1\}$ or $\mathbb{C}^*$. So the image of f is $(\mathbb{C}^*)^d$ for some $d \leq s$ and thus is a torus. ∎

I.B Affine toric variety: definition

We can now define the object we want to study.

Definition 7:

An affine toric variety is an irreducible affine variety V containing a torus T_N which is a Zariski open subset and such that the action of T_N on itself extends to an action $T_N \times V \to V$ on V given by a morphism.

Example 8:

Let C be the curve $C = \mathbf{V}(x^3 - y^2) \subseteq \mathbb{C}^2$. This is an affine toric variety with torus:

$$C \backslash \{0\} = C \cap (\mathbb{C}^*)^2 = \{(t^2, t^3) \quad / \quad t \in \mathbb{C}^*\} \simeq \mathbb{C}^*$$

I.C Equivalent points of view

Construction of affine toric varieties Let T_N be a torus and let M be its character group. Let $\mathcal{A} = \{m_1, \dots, m_s\} \subset M$ be a finite subset. Consider the map

$$\phi_{\mathcal{A}} : \left\{ \begin{array}{ccc} T_N & \to & \mathbb{C}^s \\ t & \mapsto & (\chi^{m_1}(t), \dots, \chi^{m_s}(t)) \end{array} \right.$$

It can be regarded as a map between tori (with value in $(\mathbb{C}^*)^s$), hence its image $T = \phi_{\mathcal{A}}(T_N)$ is a torus by the preceeding lemma and is closed in $(\mathbb{C}^*)^s$.

Define $Y_{\mathcal{A}}$ the Zariski closure of the image. Then $Y_{\mathcal{A}}$ is an affine toric variety whose torus has character lattice $\mathbb{Z}\mathcal{A}$ (where lattice means free abelian group of finite rank: for example N and M are lattices of a torus T_N).

It is thus clear $T \subseteq Y_{\mathcal{A}} \cap (\mathbb{C}^*)^s$. The converse is true since $(\mathbb{C}^*)^s$ is an affine variety.
Thus $T = Y_{\mathcal{A}} \cap (\mathbb{C}^*)^s$. But since $(\mathbb{C}^*)^s$ is open in $\mathbb{C}^s$ with the usual topology, it is still open with the Zariski topology and so T is Zariski open in $Y_{\mathcal{A}}$. Since T is a torus, it is irreducible. So is $Y_{\mathcal{A}}$.
Consider now the action of T. The action of $t \in T$ on $\mathbb{C}^s$ sends varieties to varieties and then $T = t \cdot T \subseteq t \cdot Y_{\mathcal{A}}$ is a variety containing T. Hence $Y_{\mathcal{A}} \subseteq t \cdot Y_{\mathcal{A}}$ by definition of Zariski closure. The same is true with t^{-1} so the inclusion is an equality and finally $Y_{\mathcal{A}}$ is an affine toric variety.

Since $T = \phi_{\mathcal{A}}(T_N)$, the map factorizes through T:

$$\phi_{\mathcal{A}} = T_N \twoheadrightarrow T \hookrightarrow (\mathbb{C}^*)^s .$$

It induces a factorisation on characters lattices (where M' is the character lattice of T):

$$\widehat{\phi_{\mathcal{A}}} = \mathbb{Z}^s \twoheadrightarrow M' \hookrightarrow M.$$

defined by $\widehat{\phi_{\mathcal{A}}}(e_i) = m_i$ where $(e_1, \dots, e_s)$ is the standard basis of $\mathbb{Z}^s$; and its image is thus $\mathbb{Z}\mathcal{A}$. So factorization gives $M' = \mathbb{Z}\mathcal{A}$.

Toric ideals The previous map $\widehat{\phi_{\mathcal{A}}}$ provides an exact sequence:

$$0 \to L \to \mathbb{Z}^s \to M.$$

where L is the kernel of $\widehat{\phi_{\mathcal{A}}}$:

$$L = \left\{ \ell = (\ell_1, \dots, \ell_s) \quad / \quad \sum_{i=1}^{s} \ell_i m_i = 0 \right\}.$$

Let $\ell \in L$ then $\ell = \ell_+ - \ell_-$ where

$$\ell_+ = \sum_{\ell_i > 0} \ell_i e_i \qquad \text{and} \qquad \ell_- = -\sum_{\ell_i < 0} \ell_i e_i.$$

Consequently to the exact sequence, the map:

$$x \mapsto x^{\ell_+} - x^{\ell_-} = \prod_{\ell_i > 0} x^{\ell_i} - \prod_{\ell_i < 0} x^{-\ell_i}$$

vanishes on the image of $\phi_{\mathcal{A}}$ and thus on $Y_{\mathcal{A}}$.

Proposition 9:

The ideal of the affine toric variety $Y_{\mathcal{A}}$ is:

$$\mathbf{I}(Y_{\mathcal{A}}) = \langle\, x^{\ell_+} - x^{\ell_-} \quad / \quad \ell \in L \rangle = \langle x^\alpha - x^\beta \quad / \quad \alpha - \beta \in L \ , \ \alpha, \beta \in \mathbb{N}^s \rangle.$$

In the proof of this proposition, we will use monomial orders on mutlivariate polynomials so let's give two usual examples of orders on multivariate polynomials. Let $x_1^{a_1} \dots x_s^{a_s}$ and $x_1^{b_1} \dots x_s^{b_s}$ be two monomials in $\mathbb{C}[x_1, \dots, x_s]$ (thus $(a_1, \dots, a_s, b_1, \dots b_s) \in \mathbb{N}^{2s}$); here are classical orders one can find in the litterature, for example when studying Gröbner basis:

- *Lexicographic order* $\prec_{lex}$*:* $x_1^{a_1} \dots x_s^{a_s} \prec_{lex} x_1^{b_1} \dots x_s^{b_s}$ if there exists $j \in [\![1, s]\!]$ such that $a_1 = b_1, \dots, a_{j-1} = b_{j-1}$ and $a_j < b_j$.
  For example in $\mathbb{C}[x_1, x_2, x_3]$ we have $x_3^7 \prec_{lex} x_2^{100} \prec_{lex} x_1$ and $x_1 x_2 x_3^4 \prec_{lex} x_1 x_2^2 x_3^3$;
- *Graded lexicographic order* $\prec_{grlex}$*:* $x_1^{a_1} \dots x_s^{a_s} \prec_{grlex} x_1^{b_1} \dots x_s^{b_s}$ if, denoting $\alpha = \sum_{i=1}^s a_i$ and $\beta = \sum_{i=1}^s b_i$, we have that $\alpha < \beta$ or $\alpha = \beta$ and $x_1^{a_1} \dots x_s^{a_s} \prec_{lex} x_1^{b_1} \dots x_s^{b_s}$.
  For example in $\mathbb{C}[x_1, x_2, x_3]$ we have $x_1 \prec_{grlex} x_3^7 \prec_{grlex} x_2^{100}$ and $x_1 x_2 x_3^4 \prec_{grlex} x_1 x_2^2 x_3^3$;

Proof. Let I_L denote this ideal (we assume the second equality). It is immediate that $I_L \subseteq \mathbf{I}(Y_{\mathcal{A}})$. For the converse pick a monomial order $>$ on $\mathbb{C}[x_1, \dots, x_s]$. We assume $T_N \simeq (\mathbb{C}^*)^n$and $M = \mathbb{Z}^n$.
Suppose the inequality is strict, then take $f \in \mathbf{I}(Y_{\mathscr{A}}) \setminus I_L$ with minimal leading monomial $x^\alpha = \prod_{i=1}^s x_i^{a_i}$.

Since $(t_1, \dots, t_n) \mapsto f(t^{m_1}, \dots, t^{m_s})$ is identically zero, there exists a monomial $x^\beta = \prod_{i=1}^s x_i^{b_i} < x^\alpha$ such that

$$\prod_{i=1}^{s} (t^{m_i})^{a_i} = \prod_{i=1}^{s} (t^{m_i})^{b_i}.$$

Meaning

$$\sum_{i=1}^{s} a_i m_i = \sum_{i=1}^{s} b_i m_i,$$

so that $\alpha - \beta = \sum_{i=1}^s (a_i - b_i) e_i$ belongs to L. Then $x^\alpha - x^\beta \in I_L$. Hence $f - x^\alpha + x^\beta$ also lies in $\mathbf{I}(Y_{\mathscr{A}}) \setminus I_L$ but since f minimizes, it must be 0 and thus $f = x^\alpha - x^\beta$ belongs to I_L: this is ABSURD. ∎

Definition 10:

Let $L \subseteq \mathbb{Z}^s$ be a sublattice. We call the ideal $I_L = \langle x^\alpha - x^\beta \quad / \quad \alpha - \beta \in L \ , \ \alpha, \beta \in \mathbb{N}^s \rangle$ a lattice ideal.
Moreover if it prime, it is called a toric ideal.

Example 11:

Since toric varieties are irreducible, the ideals of the previous proposition are toric ideals

Example 12:

$\langle x^3 - y^2 \rangle \subset \mathbb{C}[x, y]$ is a toric ideal.

It follows from the definition that a toric ideal is prime and generated by binomials (where we call the maps $x \mapsto x^\alpha - x^\beta$ binomials).

Let's show that the converse is also true.
Let I be prime and generated by binomials $x^{\alpha_i} - x^{\beta_i}$. Then observe that $\mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is nonempty (it contains $(1, ..., 1)$) and is a subgroup of $(\mathbb{C}^*)^s$ (easy to check). Since $\mathbf{V}(I) \subseteq \mathbb{C}^s$ is irreducible, it follows that $\mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is an irreducible subvariety of $(\mathbb{C}^*)^s$ that is also a subgroup. By Proposition 1.1.1, we see that $T = \mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is a torus.

Projecting on the i^{th} coordinate of $(\mathbb{C}^*)^s$ gives a character $T \hookrightarrow (\mathbb{C}^*)^s \to \mathbb{C}^*$, which by our usual convention we write as $\chi^{m_i} : T \to \mathbb{C}^*$ for $m_i \in M$. It follows easily that $\mathbf{V}(I) = Y_\mathcal{A}$ for $\mathcal{A} = \{m_1, \ldots, m_s\}$, and since I is prime, we have $I = \mathbf{I}(Y_\mathcal{A})$ by the Nullstellensatz. Then I is toric by previous proposition.

Affine semigroups Let S be a semigroup, we say S is an affine semigroup if in addition the operation is commutative (hence it will be denoted by $+$ at of now), S is finitely generated and S can be embedded into a latice.
The definition of $\mathbb{N}\mathcal{A}$ is thus clear for $\mathcal{A}$ a finite subset of a semigroup S. Moreover we assume there that all the semigroups are of this form and notice a semigroup can therefore be embedded in a lattice.
Define then the semigroup algebra:

$$\mathbb{C}[S] = \left\{ \sum_{m \in S} c_m \chi^m \quad / \quad c_m \in \mathbb{C}, c_m = 0 \text{ for all but finitely many } m \right\}$$

endowed with the multiplication $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$.

Example 13:

$\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[\chi^{e_1}, \ldots, \chi^{e_n}]$ where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{Z}^n$.

Proposition 14:

If S is an affine semigroup then $\operatorname{Spec}(\mathbb{C}[S])$ is an affine toric variety whose torus has character lattice $\mathbb{Z}S$.
Moreover if $S = \mathbb{N}\mathcal{A}$ for a finite set $\mathcal{A}$ then $\operatorname{Spec}(\mathbb{C}[\mathrm{S}]) = Y_\mathcal{A}$.

Proof. Since S is an affine semigroup there exists a finite set $\mathcal{A} = \{m_1, \ldots, m_s\}$ such that $S = \mathbb{N}\mathcal{A} \subseteq M$ where M is a lattice. It follows that $\mathbb{C}[S] \subseteq \mathbb{C}[M]$ moreover, since $\mathcal{A} = \{m_1, \ldots, m_s\}$ then $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}]$. It is thus finitely generated. And since $\mathbb{C}[M]$ can be seen as the coordinate ring of a torus it is an integral domain. So is $\mathbb{C}[S]$.
The universal property of $\mathbb{C}$-algebras gives a morphism of $\mathbb{C}$-algebras:

$$\pi : \begin{array}{ccc} \mathbb{C}[X_1, \ldots, X_s] & \to & \mathbb{C}[M] \\ X_i & \mapsto & \chi^{m_i} \end{array} .$$

Using the link between affine varieties and their coordinate ring, one can remark π corresponds to $\phi_\mathcal{A} : T_N \to \mathbb{C}^s$ defined earlier. In addition, it can be proved that $\ker(\pi) = \mathbf{I}(Y_\mathcal{A})$. And hence, since $\operatorname{im}(\pi) = \mathbb{C}[S]$ it follows that:

$$\mathbb{C}[Y_\mathcal{A}] \overset{\text{def}}{=} \mathbb{C}[X_1, \ldots, X_s]/\mathbf{I}(Y_\mathcal{A}) \simeq \mathbb{C}[X_1, \ldots, X_s]/\ker(\pi) = \mathbb{C}[S].$$

Thus $\operatorname{Spec}(\mathbb{C}[S]) = Y_\mathcal{A}$ and since $\mathbb{N}\mathcal{A} = S$, the torus has character lattice $\mathbb{Z}\mathcal{A}$. ∎

Equivalence First we require a lemma conserning the action of the tous.

Let T_N be a torus and let's study its action on the semigroup algebra $\mathbb{C}[M]$:
T_N acts on itself by multiplication;
this induces an action on $\mathbb{C}[M]$ defined by:

$$\begin{array}{ccccc} T_N & \times & \mathbb{C}[M] & \longrightarrow & \mathbb{C}[M] \\ t & , & f & \longmapsto & t \cdot f \end{array} : \begin{array}{ccc} T_N & \to & \mathbb{C} \\ p & \mapsto & f(t^{-1} \cdot p) \end{array}$$

Lemma 15: Decomposition lemma

Let $A \subseteq \mathbb{C}[M]$ be a subspace stable under the action of T_N. Then

$$A = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m.$$

Proof. Let $A' := \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m$.

- It is thus clear that $A' \subseteq A$;
- Take $f \in A \setminus \{0\} \subseteq \mathbb{C}[M]$. Then write $f = \sum_{m \in \mathcal{B}} c_m \chi^m$, for a finite set $\mathcal{B} \subseteq M$. Then f lie in $B \cap A$, where $B = \text{Span}(\chi^m \quad / \quad m \in \mathcal{B}) \subseteq \mathbb{C}[M]$. This equation stands *a priori* in $\mathbb{C}[M]$, but we will prove it also stands in A.

  Since χ^m is a group homomorphism and the action on T_N is given by the multiplication gives $t \cdot \chi^m = \chi^m(t^{-1})\chi^m \in B$. It follows that B is stable under the action. Since both A and B are stable under the action, so is $A \cap B$. A standard result on tori gives that since $A \cap B$ is finite-dimensional, it is spanned by simultaneous eigenvectors of T_N; which are characters. So $A \cap B$ is spanned by characters. Then the above expression for $f \in B \cap A$ which stood in $\mathbb{C}[M]$ implies that $\chi^m \in A$ for $m \in \mathcal{B}$ so stands in A and therefore $f \in A'$.

■

Remind the aim of this first part was to give different point of view on the affines toric varieties. Then is a theorem linking general affine toric variety, the "particular" case $Y_{\mathcal{A}}$, toric ideals and affine semigroups.

Theorem 16:

Let V be an affine variety. The following are equivalent:

1. V is an affine toric variety
2. $V = Y_{\mathcal{A}}$ for a finite set $\mathcal{A}$ in a lattice
3. V is an affine variety defined by a toric ideal
4. $V = \text{Spec}(\mathbb{C}[S])$ for an affine semigroup S

Proof. Suppose 2. then 4. stands for $S = \mathbb{N}\mathcal{A}$ by proposition **14**. Conversly suppose 4. since S is a semigroup, it is of the form $\mathbb{N}\mathcal{A}$ for some $\mathcal{A}$ and then by theorem **14**. Thus 2. $\iff$ 4.
Since the ideal of $Y_{\mathcal{A}}$ is a toric ideal, it is prime and thus $\mathbf{V}(\mathbf{I}(Y_{\mathcal{A}})) = Y_{\mathcal{A}}$ so 2. $\implies$ 3.. The converse holds considering the lattice. So 2. $\iff$ 3.
It has been proved that 2. $\implies$ 1.

We prove now that 1. $\implies$ 4.
Let V be an affine toric variety containing the torus T_N with character lattice M. Since the coordinate ring of T_N is the semigroup algebra $\mathbb{C}[M]$, the inclusion $T_N \subseteq V$ induces an injective map $\mathbb{C}[V] \hookrightarrow \mathbb{C}[M]$. of coordinate rings, where injectivity comes from T_N being Zariski dense in V and leading to see $\mathbb{C}[V]$ as a subalgebra of $\mathbb{C}[M]$.

Remind the action of T_N on V is given by a morphism $T_N \times V \to V$. Hence for $t \in T_N$ and $f \in \mathbb{C}[V]$, we have that $t \cdot f$ is a morphism on V and consequently $\mathbb{C}[V] \subseteq \mathbb{C}[M]$ is stable under the action of T_N. By preceeding lemma, we get

$$\mathbb{C}[V] = \bigoplus_{\chi^m \in \mathbb{C}[V]} \mathbb{C} \cdot \chi^m.$$

And we have $\mathbb{C}[V] = \mathbb{C}[S]$ where S is the semigroup $S := \{m \in M \quad / \quad \chi^m \in \mathbb{C}[V]\}$.

Finally, since $\mathbb{C}[V]$ is finitely generated, we can find $f_1, \dots, f_s \in \mathbb{C}[V]$ such that $\mathbb{C}[V] = \mathbb{C}[f_1, \dots, f_s]$. And we can write each f_i as a linear combinaison of characters as in the proof of the preceeding lemma which leads to a finite generating set of S. Finally S is an affine semigroup. ■

II Cones

II.A Definitions and first properties

Cones and dual cones Let M and N be two real vector spaces dual to each other.

Definition 17:

A convex polyhedral cone in N is a set of the form

$$\sigma = \text{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u u \quad / \quad \lambda_u \geq 0 \right\}$$

where $S \subseteq N$ is a finite set and is said to generate σ.
Moreover $P = \text{Conv}(S) \subseteq N$ is a polytope, it is called the convex hull of S.

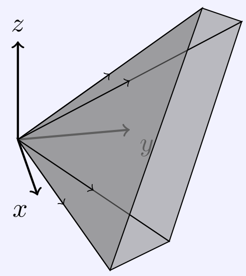
A convex polyhedral cone convex and is a cone. From now on we refer to σ simply as a polyhedral cone. Note that the term polyhedral comes from the finiteness of S which makes $P = \text{Conv}(S)$ a polytope.

Example 18:

The polyhedral cone generated by the vectors:

$$2e_1 + e_2 \ , \ e_2 - e_1 \ , \ 2e_2 + e_3 \ , \ -e_1 + e_2 - 2e_3$$

in $\mathbb{R}^3$ is represented by the shaded area on the right.



Let $P \subseteq N$ be a polytope, then $C(P) = \{\lambda \cdot (u, 1) \in N \times \mathbb{R} \quad / \quad u \in P, \lambda \geq 0\}$ is a polyhedral cone in $N \times \mathbb{R}$. It is thus conspicuous that if $P = \text{Conv}(S)$, then $C(P) = \text{Cone}(S \times \{1\})$.

Example 19:

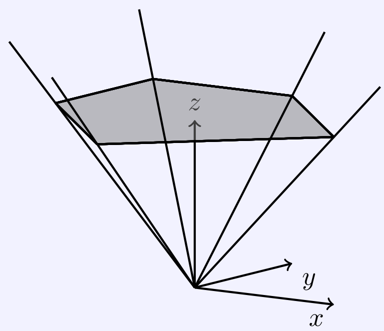
If

$$S = \{e_1, e_2, -e_1 + e_2, -e_1, -e_2\} \subset N \simeq \mathbb{R}^2$$

then

$$P = \text{Conv}(S)$$

is represented by the shaded area and the cone $C(P) \subset N \times \mathbb{R} \simeq \mathbb{R}^3$ is delimited by the lines joining $P \times \{1\} \subset \mathbb{R}^3$ to 0.



Definition 20:

If σ is a polyhedral cone, then we define $\dim\sigma$ as the dimension of the smallest subspace $W = \text{span}(\sigma)$ of N containing σ.

Since we've fixed M and N dual to each other, we can define dual cones:

Definition 21:

Let $\sigma \subseteq N$ be a polyhedral cone. The dual cone of σ is

$$\sigma^\vee = \{m \in M \quad / \quad \langle m, u\rangle \geq 0, \forall u \in \sigma\}.$$

If σ is a polyhedral cone in N, we can check easily that $\sigma^\vee$ is a polyhedral cone in M and $(\sigma^\vee)^\vee = \sigma$. Since we have introduced duality, what seems natural to do now is to deal with objects associated to dual ones: hyperplans.

Hyperplanes / Half-spaces For $m \in M\backslash\{0\}$ (a linear form on N) define the corresponding hyperplane by

$$H_m \overset{\text{def}}{=} \ker(m) = \{u \in N \quad / \quad \langle m, u\rangle = 0\} \subset N.$$

and define the closed half-space by

$$H_m^+ = \{u \in N \quad / \quad \langle m, u\rangle \geq 0\} \subseteq N.$$

If $\sigma \subseteq N$ is a polyhedral cone, and $\sigma \subset H_m^+$, then H_m is said to be a supporting hyperplane.
The followings are equivalent:

- H_m is a supporting hyperplane of σ
- $m \in \sigma^\vee\backslash\{0\}$

Indeed, if $m \in \sigma^\vee\backslash\{0\}$ then $\forall u \in \sigma,\ \langle m, u\rangle \geq 0$ by definition. So H_m is a supporting hyperplane.
Conversely, if H_m is a supporting hyperplane, then $H_m^+ = \{u \in N \quad / \quad \langle m, u\rangle \geq 0\} \supset \sigma$ so in particular for all $u \in \sigma, \langle m, u\rangle \geq 0$ which leads to $m \in \sigma^\vee$. Thus in particular $\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+$.

It follows that if $\sigma^\vee$ is generated by $m_1, \ldots, m_s$.

Definition 22:

A face of a cone of the polyhedral cone σ is $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$.
If τ is a face such that $\tau \neq \sigma$ then τ is said to be a proper faces, written $\tau \prec \sigma$.

Remark if σ is a polyhedral cone then since $0 \in \sigma^\vee$ we have that σ is a face of itself and more genereally, a face is a polyhedral cone.
Remark also that the intersection of two faces is still a face since $\sigma^\vee$ is a cone. In particular, if τ is a face of σ then each face γ of τ is a face of σ because $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$ and $\gamma = H_k \cap \tau$ for some $k \in \tau^\vee$. If $m = 0$ then the previous remarks make it trivial so suppose $m \neq 0$. Suppose there exists $u \in \sigma$ such that for any positive p we have:

$$\langle pm + k, u\rangle = p\langle m, u\rangle + \langle k, u\rangle < 0$$

Since $m \in \sigma^\vee$ and $u \in \sigma$ we have that $\langle m, u\rangle \geq 0$. So the previous is equivalent to $\langle m, u\rangle = 0$ and $\langle k, u\rangle < 0$. But since $k \in \tau^\vee$ then $u \in \sigma\backslash\tau$. Thus $u \in \sigma$ and $u \notin H_m$ so $\langle m, u\rangle > 0$ and then for large positive p we have $\langle pm+k, u\rangle = p\langle m, u\rangle + \langle k, u\rangle \geq 0$. So there is no such $u \in \sigma$ and then for large positive p we have $pm + k \in \sigma^\vee$.

Lemma 23:

Let τ be a face of a polyhedral cone σ. If $v, w \in \sigma$ and $v + w \in \tau$, then $v, w \in \tau$.

Proof. Since τ is a face, we get $m \in \sigma^\vee$ such that $\tau = H_m \cap \sigma$. So since $v, w \in \sigma$ we have $\langle m, v\rangle \geq 0$ and so for w, and since $v + w \in \tau$ we have $\langle m, v\rangle + \langle m, w\rangle = \langle v + w, m\rangle = 0$. Hence using non-negativity we have proved the lemma. ■

A facet τ of σ is a face of codimension 1 in the sens that $\dim \tau = \dim \sigma - 1$. An edge of σ is a face of dimension 1.

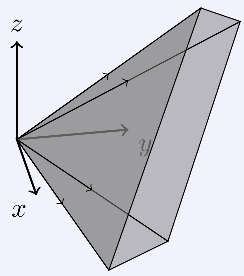
If $\sigma \subseteq N_{\mathbb{R}}$ is a polyhedral cone such that $\sigma = H^+_{m_1} \cap \cdots \cap H^+_{m_s}$ where $m_i \in \sigma^\vee$ for $1 \leq i \leq s$, then $\sigma^\vee = \text{Cone}(m_1, \ldots, m_s)$. Besides if $\dim \sigma = n (\leq s)$, then we can assume that the facets of σ are $\tau_i = H_{m_i} \cap \sigma$ and a proper face is the intersection of the facets which cointain it.
Since we are working there in $\mathbb{R}^n$, we can identify a space to its dual and then the vectors $m_1, \ldots, m_s$ in the previous formula of σ are "facet normal" (i.e. perpendicular to their corresponding facet).

Example 24:

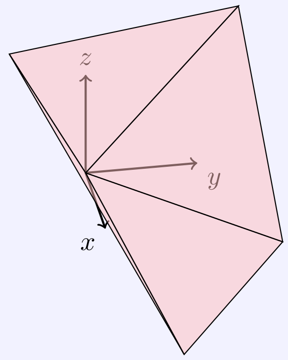
Refering to the example 18:

- the facets are the shaded areas;
- the edges are the rays determined by the generating vectors;
- the facet normals are

$e_1 - 2e_2 + 4e_3$
$e_1 + e_2 - 2e_3$
$e_1 + e_2$
$-2e_1 + 4e_2 + 3e_3$



Thus $\sigma^\vee = \text{Cone}(e_1 - 2e_2 + 4e_3\ ,\ e_1 + e_2 - 2e_3\ ,\ e_1 + e_2\ ,\ -2e_1 + 4e_2 + 3e_3)$. And it is represented by the pink shaded area.



Rational Polyhedral Cones Let N and M be dual lattices with associated vector spaces $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$.

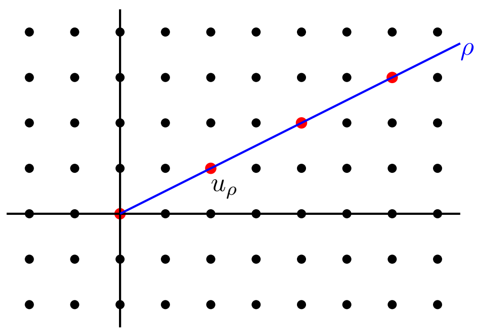
Definition 25:

A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is rational if $\sigma = \text{Cone}(S)$ for some finite set $S \subseteq N$.

In the example 18 if we take the lattice $N = \mathbb{Z}^3$, then the cone is rational. We can remark that a face of this cone is also rational. More generally if σ is a rational polyhedral cone then so are its faces.

If we suppose $\{0\}$ is a face of rational cone σ (we say σ is strongly convex), then σ has a canonical generating set constructed as follows:
Let ρ be an edge of σ then ρ is a ray (from 0). Since ρ is rational $\rho \cap N$ is a semigroup generated by a unique element $u_\rho \in \rho \cap N$ called the ray generator of ρ. Here is an exemple with $N = \mathbb{Z}^2$. The dots are the lattice $N = \mathbb{Z}^2$ and the red ones are $\rho \cap N$.

And it follows σ is generated by the ray generators of its edges.



The example 18 is a strongly convex polyhedral cone and the given generators are the ray generators.

II.B Link with affine toric varieties

Semigroup of lattice points Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational polyhedral cone, the lattice points

$$S_\sigma = \sigma^\vee \cap M \subseteq M$$

form a semigroup.

Notice that since N and M are lattices dual to each other, there exists n such that $N \simeq \mathbb{Z}^n$ and $M \simeq \mathbb{Z}^n$.

Proposition 26: Gordan's Lemma

$S_\sigma = \sigma^\vee \cap M$ is finitely generated and hence is an affine semigroup.

Proof. Since M and N are dual to each other, $\sigma^\vee$ can be proved to be a rational polyhedral cone. Thus there exists a finite set $T \subseteq M$ such that $\sigma^\vee = \text{Cone}(T)$. Then

$$K = \left\{ \sum_{m \in T} \delta_m m \quad / \quad 0 \leq \delta_m < 1 \right\} \subset \sigma^\vee$$

is a bounded area of $M_{\mathbb{R}}$, so since M is discrete $K \cap M$ is finite. So that $T \cup (K \cap M) \subseteq S_\sigma$ is finite.

Let's prove $T \cup (K \cap M)$ generates S_σ as a semigroup.
So let $v \in S_\sigma$. Write $v = \sum_{m \in T} \lambda_m m$ where $\lambda_m \geq 0$. Then $\lambda_m = \lfloor \lambda_m \rfloor + \delta_m$ with $\lfloor \lambda_m \rfloor \in \mathbb{N}$ and $0 \leq \delta_m < 1$, so that

$$v = \sum_{m \in T} \lfloor \lambda_m \rfloor m + \sum_{m \in T} \delta_m m.$$

The first sum is in M since $T \subset M$. So since $v \in M$ we have that

$$M \ni v - \sum_{m \in T} \lfloor \lambda_m \rfloor m = \sum_{m \in T} \delta_m m \in K.$$

Thus the second sum is in $K \cap M$, denote it u, it follows that

$$v = \sum_{m \in T} \lfloor \lambda_m \rfloor m + u.$$

is a nonnegative integer combination of elements of $T \cup (K \cap M)$. ■

Affine toric varieties associated to rational polyhedral cones Since affine semigroups give affine toric varieties, we get the following.

Theorem 27:

Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a rational polyhedral cone with semigroup $S_\sigma = \sigma^\vee \cap M$. Then

$$U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma]) = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$

is an affine toric variety. Furthermore,

$$\dim U_\sigma = n \iff \text{the torus of } U_\sigma \text{ is } T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \iff \sigma \text{ is strongly convex.}$$

Remark that the first equivalence stands since the dimension of an affine toric variety is the dimension of its torus, which is the rank of its character lattice as we discussed in the first part. Consequently, what is left to prove is $\dim U_\sigma = n \iff \sigma$ is strongly convex.

Proof. By Gordan's Lemma and proposition 14, U_σ is an affine toric variety whose torus has character lattice $\mathbb{Z}S_\sigma \subseteq M$.

Let's prove that $M/\mathbb{Z}S_\sigma$ is torsion free so that $\mathbb{Z}S_\sigma = M \iff \text{rank}(\mathbb{Z}S_\sigma) = n$. But since $\mathbb{Z}S_\sigma$ is the character lattice of U_σ and denoting T its torus then the link between the group of one-parameter subgroups and the torus gives $\mathbb{Z}S_\sigma = M \iff$ the torus of U_σ is T_N.

Suppose there exist $k > 1$ and $m \in M$ such that $km \in \mathbb{Z}S_\sigma$. Then write $km = m_1 - m_2$ for $m_1, m_2 \in S_\sigma = \sigma^\vee \cap M$. Since $\sigma^\vee$ is convex, we have

$$m + m_2 = \frac{1}{k}m_1 + \left(1 - \frac{1}{k}\right) m_2 \in \sigma^\vee.$$

It follows that $M \ni m = (m + m_2) - m_2 \in \sigma^\vee - S_\sigma$ and thus $m \in \mathbb{Z}S_\sigma$, so that $M/\mathbb{Z}S_\sigma$ is torsion-free. Hence, as we said earlier:

$$\text{the torus of } U_\sigma \text{ is } T_N \iff \mathbb{Z}S_\sigma = M \iff \operatorname{rank} \mathbb{Z}S_\sigma = n.$$

Since one can remark σ is strongly convex if and only if $\dim \sigma^\vee = n$, what is left to prove is:

$$\dim U_\sigma = n \iff \operatorname{rank} \mathbb{Z}S_\sigma = n \iff \dim \sigma^\vee = n. \tag{1}$$

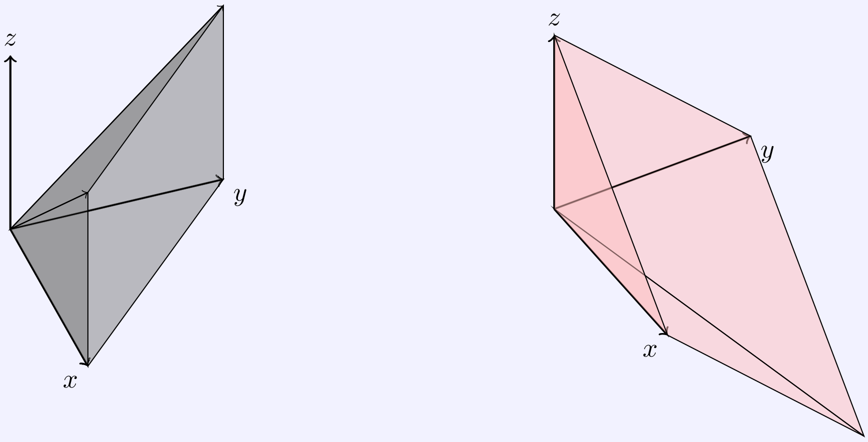
But, since from Gordan's lemma S_σ is finitely generated, considering a finite generating set of S_σ, one can remark that $\operatorname{rank}(\mathbb{Z}S_\sigma) = \dim(\operatorname{Span}(S_\sigma))$.

On the other hand, it follows from the proof of the Gordan's lemma that a finite generating set of S_σ is also a finite generating set of $\sigma^\vee$, but since $S_\sigma \subset \sigma^\vee$, we finally have that $\sigma^\vee = \operatorname{Cone}(S_\sigma)$ and by definition of $\operatorname{Span}(S_\sigma)$ and definition 20 we get $\dim(\operatorname{Cone}(S_\sigma)) = \dim(\sigma^\vee) = \dim(\operatorname{Span}(S_\sigma))$.

And thus is proved (1). ■

Example 28:

Let $\sigma = \operatorname{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subseteq N_\mathbb{R}$ with $N = \mathbb{Z}^3$. This is the cone pictured in grey below. Its dual cone is $\sigma^\vee = \operatorname{Cone}(e_1, e_2, e_3, e_1 + e_2 + e_3) \subseteq M_\mathbb{R}$ pictured in pink below.



The lattice points S_σ consists in the $\mathbb{N}$-linear combinations of $e_1, e_2, e_3, e_1 + e_2 - e_3$.

However if we consider the variety $V = \mathbf{V}(xy - zw) \subset \mathbb{C}^4$, then we have seen that it is an affine toric variety. Besides its torus is $(\mathbb{C}^*)^3$ via the map $(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$. Thus the affine semigroup S generated by $e_1, e_2, e_3, e_1 + e_2 - e_3$ gives the character lattice $\mathbb{Z}S$ of the torus. So from proposition 14 and theorem 16, we have that S determines the affine toric variety V.

It follows from the previous theorem that U_σ is the affine toric variety $\mathbf{V}(xy - zw)$.

References

[1] D. Cox, J. Little, and H. Schenck. *Toric Varieties.* American Mathematical Society, 2011.

[2] W. Fulton. *Introduction to toric varieties.* Annals of mathematics studies ; no. 131. Princeton University Press, Princeton, N.J, 1993.