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FLC Delone Sets and their hulls

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Abstract
We define and study FLC Delone sets in locally compact second countable (lcsc) groups. They generalize discrete and relatively dense subgroups. We study them espacially through their hulls and the continuous functions on their hulls. To do that, it will be convenient to consider a hull as an inverse limit of some approximants. Finally, we introduce the notion of periodization complexity and establish the periodization compexity of some FLC Delone sets in $\mathbb{R}$.

Key words: FLC Delone sets, lattices, inverse limits, continuous functions, periodization, Fibonacci word

Contents

1 Introduction

Tiling is a popular subject in geometry, giving fascinating and beautiful pictures. There exist periodic tilings (for instance the kitchen or bathroom tilings, or in the crystal structure) but also aperiodic ones, such as the Penrose tilings. Since the discovery of quasicrystals, aperiodic tilings have attracted more and more interest.

In this report we will almost not speak about tilings but only about point sets. Let us think about a tiling of $\mathbb{R}$, we can describe it as well by its tiles as by the boundaries of its tiles, namely as a point set whose points are not too close nor far from one another. We will not look further at the link between tilings and point sets, but the spaces we study here can be relevant to answer physical and chemical questions (See [14]).

2 Point sets

We introduce a few notions about point sets:

Definition 1 (Delone set)**.** *Let X be a metric space. Let $\Lambda \subset X$ be a subset. Λ is said to be:*

(i) discrete *if every subset of Λ is open in Λ with respect to the subset topology;*

(ii) locally discrete *if it is discrete and closed, or equivalently if its intersection with every pre-compact subset of X is finite;*

(iii) r-uniformly discrete *for some $r > 0$ if for every distinct points $x, y \in \Lambda$, $d(x, y) > r$;*

(iv) R-relatively dense *for some $R > 0$ if $X = \bigcup_{x \in \Lambda} \mathcal{B}(x, R)$;*

(v) (r, R)-Delone set *if it is both r-uniformly discrete and R-relatively dense.*

Remark 1: Note that being uniformly discrete or relatively dense depends on the metric on X and not just on the topology.

Remark 2: The notion of Delone sets in $\mathbb{R}^3$ would well describe a gas. Each atom cannot be too near from the other ones (so the point set is uniformly discrete) and the atoms use all the space they have with a minimum pressure (so the point set is relatively dense).

This definition is really generic, we will restrict ourselves to cases where X is a group, and to fix ideas, we will mostly think of the special case where $X = \mathbb{R}$. Before going further and establish some properties, we give some examples.

Example 1.

- $\left\{ \frac{1}{n+1} \mid n \in \mathbb{N} \right\}$ is a discrete set of $\mathbb{R}$ with the Euclidean metric, but it is non locally finite.
- $\mathbb{N} \cup \left\{ n + \frac{1}{n+1} \mid n \in \mathbb{N} \right\}$ is a locally finite set of $\mathbb{R}$ with the Euclidean metric, but it is non uniformly discrete (nor relatively dense).
- $\mathbb{Z}$ is a (r, R)-Delone set for any $0 < r < 1 < R$.
- If we put the discrete topology on $\mathbb{R}$, which is metrizable with $\delta(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{else} \end{cases}$, then every subset of $\mathbb{R}$ is a (r, R)-Delone set for any $0 < r < 1 < R$.

More precisely, we will consider a locally compact[1] and second countable[2] (lcsc) group G. In particular, G is a topological group, namely the group law and the inverse are continuous. The following theorem, due to Struble, establishes the existence of nice metrics on such a group.

Theorem 1 (Struble). *Let G be a locally compact group. Then G is second countable if and only if there exists a left-invariant proper compatible metric on G.*

We recall that a metric d is said to be *left-invariant* whenever:

$$\text{for all } x, y \text{ and } g \text{ in } G,\ d(g.x, g.y) = d(x, y).$$

[1] Each point has a basis of compact neighbourhoods.
[2] There exists a countable basis of open sets.

It is said to be *proper* if the balls are pre-compact.
It is said to be *compatible* if it leads to the same topology on G.

There is another equivalent property, but we will not use it here. The equivalence between the three properties is proved in [8].

Example 2. Since $\mathbb{R}^d$ is an lcsc group (with the addition and the canonical topology), there exists a left-invariant proper compactible metric on $\mathbb{R}^d$. Actually, there exist many such metrics. For instance, the Euclidean metric (corresponding to the norm L^2) and more generally the distances corresponding to the norms L^1, L^∞ and L^p are left-invariant proper compatible metrics on $\mathbb{R}^d$.

Moreover, as established in [5], any proper, continuous, left-invariant metric leads to the same set of Delone sets:

Proposition 1. *Let G be an lcsc group, $\Lambda \subset G$ be a subset and d be a proper left-invariant compatible metric on G. Then the following hold:*

(i) Λ is uniformly discrete in G if and only if the identity $e \in G$ is not an accumulation point of $\Lambda^{-1}\Lambda = \{x^{-1}y \mid x, y \in \Lambda\}$.

(ii) Λ is relatively dense in G if and only if there exists a compact subset $K \subset G$ such that $G = \Lambda K$.

In particular, the property of being a Delone set in G is independent of the choice of d.

Proof. *(i)* If Λ is not uniformly discrete, then there exist elements $x_n \neq y_n$ in Λ such that $d(x_n, y_n) < 1/n$. Hence, $z_n := y_n^{-1}x_n \in \Lambda^{-1}\Lambda$ is such that $d(z_n, e) = d(x_n, y_n) < 1/n$, which means that (z_n) converges to e, so e is an accumulation point of $\Lambda^{-1}\Lambda$.
Conversely, if e is an accumulation point of $\Lambda^{-1}\Lambda$, then there exists a sequence of $z_n = y_n^{-1}x_n \in \Lambda^{-1}\Lambda$, where $z_n \neq e$, which converges to e, hence $d(x_n, y_n) = d(z_n, e) \to 0$ and Λ is not uniformly discrete.

(ii) Assume that Γ is R-relatively dense and let $K = \overline{\mathcal{B}(e, R)} \subset G$.
Let $g \in G$. There exists $\gamma \in \Gamma$ such that $d(g, \gamma) < R$. Since d is left-invariant, $d(\gamma^{-1}g, e) = d(g, \gamma) < R$, hence $g = \gamma.\gamma^{-1}g \in \Gamma.K$. So $G = \Gamma.K$, and K is compact.
Conversely, if $G = \Gamma.K$ with K compact, then every $g \in G$ can be written as $g = \gamma.k$ with $\gamma \in \Gamma$ and $k \in K$, and then $d(g, \gamma) = d(k, e) \leqslant \max_{k \in K} d(k, e) =: R$ which is finite because d is continuous and K compact. Since R does not depend on g, Γ is R-relatively dense in G.

□

Definition 2 (Finite local complexity). *A subset Λ of an lcsc group G is said to have* finite local complexity *(FLC) if $\Lambda^{-1}\Lambda$ is locally finite.*

Remark 3: Any FLC set is uniformly discrete but the converse is false:

$$S = 2\mathbb{N} \cup \left\{2n - 1 + \frac{1}{n+1} \,\middle|\, n \in \mathbb{N}^*\right\} = \left\{0, 1 + \frac{1}{2}, 2, 3 + \frac{1}{3}, 4, 5 + \frac{1}{4}, 6, 7 + \frac{1}{5}, 8, ...\right\}$$

is r-uniformly discrete for any $r < 0.5$ but has not FLC because 1 is an accumulation point of the set of the distances between two successive points.

Remark 4: When a set $\Lambda \subset \mathbb{R}$ is locally finite, we can describe it with a sequence[3] $(d_n)_{n\in\mathbb{Z}}$ (or, if Λ is not relatively dense, $(d_n)_{0\leqslant n}$ or $(d_n)_{n\leqslant 0}$ or a finite sequence) and a base point $x_0 \in \Lambda$ through the 1-to-1 correspondence:

$$Q: \begin{array}{ccc} \mathbb{R} \times (\mathbb{R}_+^*)^{\mathbb{Z}} & \to & LF(\mathbb{R}) \\ (x_0, (d_n)_{n\in\mathbb{Z}}) & \mapsto & (x_n)_{n\in\mathbb{Z}} \end{array} \text{ such that: } \begin{cases} x_n = x_0 + \sum_{k=0}^{n-1} d_k & \text{if } n \geqslant 0 \\ x_n = x_0 - \sum_{k=1}^{-n} d_{-k} & \text{if } n < 0 \end{cases}$$

where $LF(\mathbb{R})$ denotes the set of locally finite point sets in $\mathbb{R}$.

Definition 3 (Symbolic coding of a locally finite point set in $\mathbb{R}$). *Let Λ be a locally finite point set in $\mathbb{R}$. We call* symbolic coding *of Λ every sequence $(d_n)_{n\in\mathbb{Z}}$ such that $\Lambda = \{Q(x_0, (d_n))\}$ for some x_0.*

Note that if $\Lambda \neq \alpha\mathbb{Z}$, then Λ has many (possibly infinitely many) different symbolic codings: indeed if $(d_n)_{n\in\mathbb{Z}}$ is a symbolic coding of Λ, then for all $k \in \mathbb{Z}$, $(d_{n+k})_{n\in\mathbb{Z}}$ is also a symbolic coding of Λ. Λ is determined up to translation by (one of) its symbolic coding (which can be fixed with the base point information).
A symbolic coding is very useful to describe one locally finite point set in $\mathbb{R}$. Particularly because of the following properties:

[3]This is because $\mathbb{R}$ is totally ordered.

Proposition 2. *Let Λ be a locally finite set in $\mathbb{R}$. Let us note d a symbolic coding of Λ.*

(i) Λ is uniformly discrete if and only if d has a non-zero infimum.

(ii) Λ is relatively dense if and only if d is a bi-infinite sequence which has a supremum.

(iii) Λ is an FLC Delone set if and only if d is a bi-infinite sequence which takes a finite number of values (and they are all non-zero).

Proof. *(i)* By definition, Λ is r-uniformly discrete in G if and only if every point of Λ is at distance at least r of any other point of Λ. In $\mathbb{R}$, this is equivalent to saying that every two successive points are at distance at least r, which means that every value of d is greater than $r > 0$.

(ii) By definition, Λ is R-relatively dense in G if and only if every point of G is at distance at most R of a point of Λ. In $\mathbb{R}$, this is equivalent to saying that Λ is not bounded and that every two successive points are at distance at most $D < 2R$, which means that d is bi-infinite and its values are all less than $2R$.

(iii) If Λ is an FLC Delone set, it is uniformly discrete and relatively dense so d is a bi-infinite bounded sequence with non-zero infimum. If d had an infinite number of values, then one of them would be an accumulation point and $\Lambda - \Lambda = \{x - y \mid x, y \in \Lambda\}$ would not be discrete, so Λ would not have FLC.
Conversely, if d is a bi-infinite sequence which takes a finite number of values and which are all non-zero then Λ is uniformly discrete and relatively dense, hence a Delone set. Moreover if $\Lambda - \Lambda$ had an accumulation point $z = x - y$, then there exists $z_n = x_n - y_n \to z$, which means that sums of values of d can be as near to z (but not equal to) as one wants, but this is not possible because d takes a finite number of values.

□

Example 3.

- $\mathbb{N} \cup \left\{n + \frac{1}{n+1} \,\middle|\, n \in \mathbb{N}\right\} = \left\{0, 1, \frac{3}{2}, 2, \frac{7}{3}, 3, \frac{13}{4}, 4, \frac{21}{5}, 5, ...\right\}$ is described by the symbolic coding $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{4}{5}, ...)$ and the base point 0.
- $\mathbb{Z}$ is described by the symbolic coding $(..., 1, 1, 1, ...)$ and the base point 0 (or any other integer).
- $\mathbb{Z}\backslash\{0\}$ is described by the symbolic coding $(..., 1, 2, 1, ...)$ and the base point -1 (if $2 = d_0$).
- $-2\mathbb{N} \cup \mathbb{N}$ is described by the bi-infinite word $(..., 2, 2, 1, 1, ...)$ and the base point 0 (if $d_{-1} = 2$ an $d_0 = 1$).

Remark 5: One can also see point sets in $\mathbb{R}$ as tilings of $\mathbb{R}$ (*i.e.* sets of closed intervals $(I_n)_{n\in\mathbb{Z}}$ with non empty interior such that $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} I_n$ and the $(\mathring{I}_n)$ are disjoint) through the 1-to-1 correspondence:

$$T: \begin{array}{ccc} \mathfrak{I} & \to & \mathcal{T}(\mathbb{R}) \\ (x_n)_{n\in\mathbb{Z}} & \mapsto & \{I_n = [x_n, x_{n+1}] \mid n \in \mathbb{Z}\} \end{array}$$

where $\mathcal{T}(\mathbb{R})$ denotes the set of tilings of $\mathbb{R}$ and $\mathfrak{I} \subset (\mathbb{R})^{\mathbb{Z}}$ denotes the set of strictly increasing bi-infinite sequences.

3 Groups and lattices

Let us first study the case where Λ is a uniform lattice of $X = G$, a topological lcsc group. To fix ideas, one can think of G as $\mathbb{R}$ and Λ as $\alpha\mathbb{Z}$). G acts over itself through the left multiplication (in $\mathbb{R}$, it is through the addition).

Definition 4 (Uniform lattice)**.** *Let G be an lcsc group and Γ a discrete subgroup of G.*
We call Γ a uniform lattice *whenever G/Γ is compact.*

We emphasize that Γ is not assumed to be a normal subgroup, so G/Γ is a set quotient, but not necessarily a group.

Proposition 3. *Let G be an lcsc group and Γ a subgroup of G. Then:*

(i) G/Γ is a topological space on which G acts continuously.

(ii) Γ *has FLC (so* Γ *is also uniformly discrete) if and only if* Γ *is discrete.*

(iii) G/Γ *is compact if and only if* Γ *is relatively dense.*

Proof. *(i)* G/Γ, with the quotient topology inherited from the toplogy of G, is a topological space. G is a topological group, which means that the multiplication by an element of G is continuous. So the action of G over itself is continuous, so the action of G over G/Γ is also continuous.

(ii) If Γ has FLC, then it is uniformly discrete, and hence discrete.
Conversely, if Γ is discrete then the identity element $e \in G$ is not an accumulation point of Γ. But $\Gamma = \Gamma^{-1}\Gamma$ since Γ is a group. Since $e \in G$ is not an accumulation point of $\Gamma^{-1}\Gamma$, Γ is uniformly discrete (Proposition 1). Then $\Gamma^{-1}\Gamma = \Gamma$ is uniformly discrete, hence discrete, which is the definition of Γ having FLC.

(iii) If Γ is relatively dense, we know that $G = \Gamma.K$ where K is a compact space (Proposition 1). So we have:

$$G/\Gamma = \{\Gamma.g \mid g \in G\} \subset \{\Gamma.\gamma.k \mid \gamma \in \Gamma,\, k \in K\} = \{\Gamma.k \mid k \in K\}$$

G/Γ is closed in the compact set $\{\Gamma.k \mid k \in K\}$ so G/Γ is compact.
Conversely, if G/Γ is compact, then $G = \Gamma.G/\Gamma$ and Γ is relatively dense.

□

Corollary 1. *Let* G *be an lcsc group and* Γ *a subgroup of* G*. The following statements are equivalent:*

(i) Γ *is a Delone set.*

(ii) Γ *is an FLC Delone set.*

(iii) Γ *is a uniform lattice.*

Proof. By definition, we have $(ii) \Rightarrow (i)$. By Proposition 2, we have $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$. It remains to prove $(iii) \Rightarrow (i)$. If Γ is a uniform lattice, then by definition, it is discrete and G/Γ is compact. So Γ has FLC and is relatively dense (Proposition 2). Γ is uniformly discrete and relatively dense, hence a Delone set.

□

4 Hull

We want to generalize the dynamical system $G \curvearrowright G/\Gamma$ when we do not consider an FLC Delone subgroup Γ anymore, but only an FLC Delone set Λ. In other words, we are looking for a compact space, giving some information of Λ, and on which G acts continuously. Throughout this section, G denotes an lcsc group.

4.1 Topologies and definition of the hull

The first space one could think of is the set of all translates of Λ, namely the orbit of Λ under the action of G. But in most cases, this will only be a copy of G, which does not give us any information on Λ. For example, if $G = \mathbb{R}$ and $\Lambda = \mathbb{Z}\backslash\{0\}$, then there is a 1-to-1 correspondence between the set of translates of Λ, denoted $\mathscr{T}_\Lambda$, and $\mathbb{R}$:

$$\Lambda + x \in \mathscr{T}_\Lambda \longleftrightarrow x \in \mathbb{R}$$

This will happen whenever Λ is such that there is no $t > 0$ verifying: for all x in Λ, $x + t$ and $x - t$ are also in Λ. To fix this problem, we will consider not only the orbit of Λ but also its closure (that of the orbit of Λ) for a certain topology on closed sets. We first define two topologies on closed sets.

Definition 5 (Local topology)**.** *The* local topology *is the unique topology on LF(G), the set of locally finite point sets in* G*, such that for any closed point set* Λ *in* G*, a neighbourhood basis of* Λ *is given by the sets:*

$$U_{K,V}(\Lambda) = \{L \in LF(G) \mid \exists \tau \in V : \tau.\Lambda \cap K = L \cap K\}$$

where K *is a compact subset of* G *and* V *an identity neighbourhood of* G*.*

With regard to the local topology, the following sequence of sets:

$$F_{2n} = \left\{-1 + \frac{1}{n}, 1\right\} \text{ and } F_{2n+1} = \left\{-1, 1 - \frac{1}{n}\right\}$$

does not converge for the local topology, though one could want to say that it converges to $F_\infty = \{-1, 1\}$. Moreover this example shows that $LF(\mathbb{R})$ is not compact with regard to the local topology.
We will consider another topology, in which the sequence $(F_n)_{n\in\mathbb{N}}$ does converge to F_∞.

Definition 6 (Chabauty-Fell topology). *A sequence (P_n) of closed subsets of G is said to converge to P with respect to the* Chabauty-Fell topology *whenever the following two properties hold:*

(i) If (n_k) is an unbounded sequence of natural numbers and $p_{n_k} \in P_{n_k}$ such that (p_{n_k}) converges to $p \in G$, then $p \in P$.

(ii) For every $p \in P$ there exist elements $p_n \in P_n$ such that (p_n) converges to p.

Remark 6: Local topology and Chabauty-Fell here are well-defined, see [6] for details.
The Chabauty-Fell topology is metrizable, see [8].

Definition 7 (Hull). *We denote by Ω_Λ (resp. $\Omega_\Lambda^{(loc)}$) the* hull *of Λ (resp. local hull of Λ), namely the topological space of the closure of all the translates of Λ (*i.e.* the closure of the orbit $\mathscr{T}_\Lambda$ of Λ under the action of G) with regard to the Chabauty-Fell topology (resp. local topology).*

In order to give a more explicit description of the elements of the hull, we define the notion of local translates.

Definition 8 (Local translates). *L is said to be a* local translate *of Λ whenever:*

$$\forall K \subset G \text{ compact},\ \exists g \in G,\ (g.\Lambda) \cap K = L \cap K.$$

Any translate L of Λ is also a local translate of Λ. Indeed it exists $g \in G$ such that $L = g.\Lambda$, so for all compact subset K of G, $(g.\Lambda) \cap K = L \cap K$ (always with the same g).

Proposition 4. *Let Λ be a locally finite point set in G. Then $T \in \Omega_\Lambda^{(loc)}$ if and only if T is a local translate of Λ.*

The proof of this statement is in [11].
We know that the Chabauty-Fell and local topologies are not equivalent since one is compact and not the other. So, generally, $\Omega_\Lambda \neq \Omega_\Lambda^{(loc)}$ but we will restrict our study to subsets which are FLC Delone sets, and for them $\Omega_\Lambda = \Omega_\Lambda^{(loc)}$, as the following theorem establishes.

Theorem 2. *Let G be an lcsc group. If Λ is an FLC Delone set, then the hull and the local hull are the same as sets and the Chabauty-Fell and local topologies are the same on them. Moreover, $\Omega_\Lambda = \Omega_\Lambda^{(loc)}$ is compact (with regard to these topologies).*

For a proof of this theorem, see [6]. Since we restrict ourselves to lcsc groups and FLC Delone subsets, the Chabauty-Fell and local topologies will lead to the same topology. We can then use both of them, chosing the most convenient for us, but we will only use the notation Ω_Λ.

The following statement shows that Ω_Λ is a good candidate for the space we were looking for to replace G/Λ when Λ is not a discrete subgroup anymore but only an FLC Delone set.

Proposition 5. *G acts continuously over Ω_Λ with regard to the Chabauty-Fell topology.*

Proof. Since G and Ω_Λ are metric spaces we can prove continuity with sequences. Let (g_n) and (x_n) be sequences of points of G and Ω_Λ respectively, such that (g_n) converges to $g \in G$ and (x_n) converges to $x \in \Omega_\Lambda$ (with regard to the Chabauty-Fell topology).
Let us show that $g_n.x_n \overset{C-F}{\to} g.x$.

(i) If (n_k) is an unbounded sequence of natural numbers and $p_{n_k} \in g_{n_k}.x_{n_k}$ such that (p_{n_k}) converges to $p \in G$, then $g_{n_k}^{-1}.p_{n_k} \in x_{n_k}$ and $(g_{n_k}^{-1}.p_{n_k})$ converges to $g^{-1}.p$ (because the multiplication in G is continuous). But $x_{n_k} \overset{C-F}{\to} x$, so $g^{-1}.p$ must be in x.

(ii) If $p \in g.x$, then $g^{-1}p \in x$ and (since $x_n \overset{C-F}{\to} x$) there exist elements $y_n \in x_n$ such that (y_n) converges to $g^{-1}p$. Then, $g_n.y_n \to g.g^{-1}p = p$ where $g_n.y_n \in x_n$.

We conclude that $g_n.x_n \overset{C-F}{\to} g.x$: the action is continuous.

□

We will study this space Ω_Λ. First we give some examples in order to picture a bit more what a hull is and then we give another description of the hull of point sets in $\mathbb{R}$, which will be more convenient to use in some cases.

4.2 First Examples

Here are some examples of hulls which can be easily pictured.

- The hull $\Omega_\mathbb{Z}$ of $\mathbb{Z}$ is the closure of all translates of $\mathbb{Z}$:

$$\Omega_\mathbb{Z} = \overline{\{\mathbb{Z} + x \mid x \in \mathbb{R}\}} = \overline{\{\mathbb{Z} + x \mid x \in [0,1]\}_{/\mathbb{Z}+0\sim\mathbb{Z}+1}}\,.$$

  But $\{\mathbb{Z} + x \mid x \in \mathbb{R}\}$ is a closed set with regard to the Chabauty-Fell topology, so the hull $\Omega_\mathbb{Z}$ of $\mathbb{Z}$ is exactly the set of all the translates of $\mathbb{Z}$. It's a circle. More generally, when Λ is a subgroup of G, then $\Omega_\Lambda = G/\Lambda$.

- The hull $\Omega_{\mathbb{Z}\setminus\{0\}}$ of $\mathbb{Z}\setminus\{0\}$ is the closure of all translates of $\mathbb{Z}\setminus\{0\}$:

$$\Omega_{\mathbb{Z}\setminus\{0\}} = \overline{\{\mathbb{Z}\setminus\{0\} + x \mid x \in \mathbb{R}\}}\,.$$

  But this time $\{\mathbb{Z}\setminus\{0\} + x \mid x \in \mathbb{R}\}$ is not a closed set. Indeed, we have for example

$$\mathbb{Z}\setminus\{n\} = \mathbb{Z}\setminus\{0\} + n \overset{C-F}{\to} \mathbb{Z}.$$

  More generally: $\forall x \in \mathbb{R},\ \mathbb{Z}\setminus\{0\} + n + x \overset{C-F}{\to} \mathbb{Z} + x.$

  And conversely, if $L \in \Omega_{\mathbb{Z}\setminus\{0\}}$, then L is either $\mathbb{Z}\setminus\{0\} + x$ or $\mathbb{Z} + x$ for some $x \in \mathbb{R}$. Indeed, $L \in \Omega_{\mathbb{Z}\setminus\{0\}}$ means that there exist elements $L_n \in \{\mathbb{Z}\setminus\{0\} + x \mid x \in \mathbb{R}\}$ such that $L_n \overset{C-F}{\to} L$. Say $L_n = \mathbb{Z}\setminus\{0\} + x_n$, with $x_n = k_n + y_n$ where $k_n \in \mathbb{Z}$ and $y_n \in [0,1]$. We can assume without loss of generality that $y_n \to y \in [0,1]$. If $|k_n| \to +\infty$, then $L = \mathbb{Z} + y$, else (k_n) must converge to $k \in \mathbb{Z}$ (because (L_n) converges) and $L = \mathbb{Z}\setminus\{0\} + k + y$.

  Topologically, it's a slinky whose two extremities accumulate on a same circle. On figure 1, we see the hull first as a "straight" slinky with two circles, but which are the same, and then as a slinky whose two extremities accumulates this time really on the same circle.

- The hull Ω_Λ of $\Lambda = -2\mathbb{N} \cup \mathbb{N}$ is the closure of all translates of Λ:

$$\Omega_\Lambda = \overline{\{-2\mathbb{N} \cup \mathbb{N} + x \mid x \in \mathbb{R}\}}\,.$$

  Once again, $\{-2\mathbb{N} \cup \mathbb{N} + x \mid x \in \mathbb{R}\}$ is not a closed set. Indeed, we have for example

$$-2\mathbb{N} \cup \mathbb{N} + 2n \overset{C-F}{\to} 2\mathbb{Z}.$$

  More generally: $\begin{cases} \forall x \in \mathbb{R}, & -2\mathbb{N} \cup \mathbb{N} + 2n + x \overset{C-F}{\to} 2\mathbb{Z} + x \\ \forall x \in \mathbb{R}, & -2\mathbb{N} \cup \mathbb{N} - n + x \overset{C-F}{\to} \mathbb{Z} + x \end{cases}.$

  And conversely, if $L \in \Omega_{\mathbb{Z}\setminus\{0\}}$, then L is either $-2\mathbb{N} \cup \mathbb{N} + x$ or $\mathbb{Z} + x$ or $2\mathbb{Z} + x$ for some $x \in \mathbb{R}$. Indeed, $L \in \Omega_\Lambda$ means that there exist elements $L_n \in \{-2\mathbb{N} \cup \mathbb{N} + x \mid x \in \mathbb{R}\}$ such that $L_n \overset{C-F}{\to} L$. Say $L_n = -2\mathbb{N} \cup \mathbb{N} + x_n$, with $x_n = k_n + y_n$ where $k_n \in 2\mathbb{Z}$ and $y_n \in [0,2]$. We can assume without loss of generality that $y_n \to y \in [0,2]$. If $k_n \to +\infty$, then $L = 2\mathbb{Z} + y$, else if $k_n \to -\infty$, then $L = \mathbb{Z} + y$, else (k_n) must converge to $k \in \mathbb{Z}$ (because (L_n) converges) and $L = -2\mathbb{N} \cup \mathbb{N} + y$.

  Topologically, it's a slinky whose "left" extremity accumulates on a circle of radius 1 and "right" extremity accumulates on a circle of radius 2.

The next subsection introduces the notion of lexicon, which is actually a more convenient way to describe the hull of an FLC Delone set in $\mathbb{R}$.

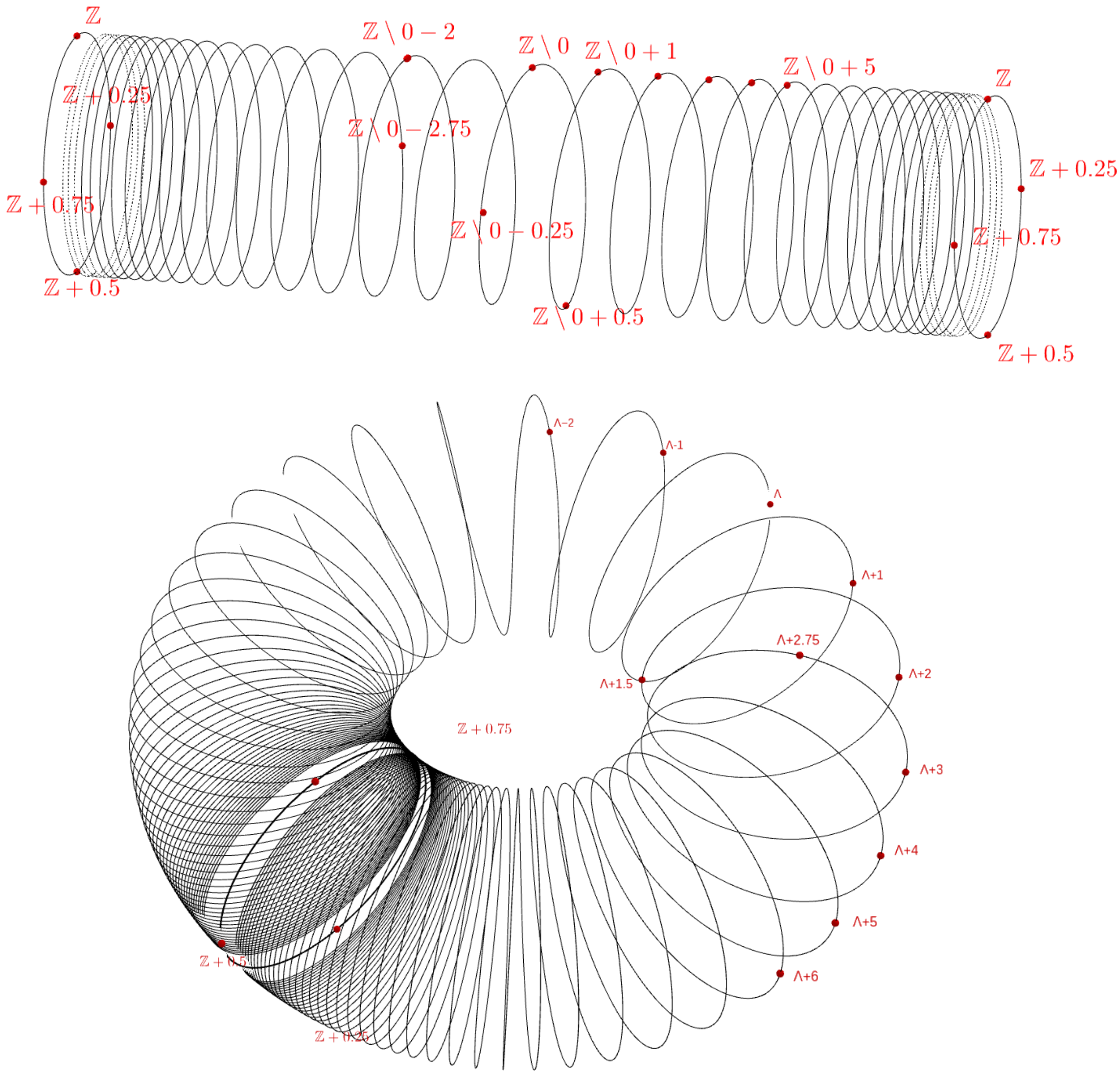


Figure 1: $\Omega_{\mathbb{Z}\setminus\{0\}}$

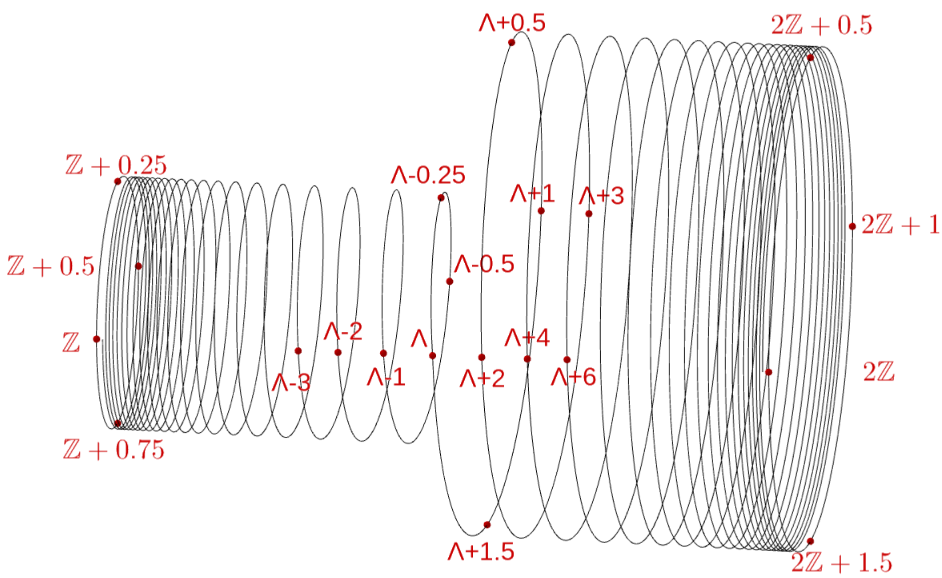


Figure 2: $\Omega_{-2\mathbb{N}\cup\mathbb{N}}$

4.3 Hull and lexicon

Definition 9 (Words and subwords)**.** *We call* word *a finite or infinite sequence of letters from a fixed alphabet. We say that v is a* subword *of a word w if v is a word and if there exists other words u_1, u_2 possibly finite, infinite or even empty, such that $w = u_1 v u_2$.*

Definition 10 (Lexicon)**.** *Let w be a word. We call the set of all finite subwords of w the* lexicon *of w, and we denote it by $\mathcal{L}_w$.*

Remark 7: A word w is finite if and only if its lexicon $\mathcal{L}_w$ is also finite.

Proposition 6. *Let Λ be an FLC Delone set in $\mathbb{R}$. Let v and w be two symbolic codings of Λ. Then $\mathcal{L}_v = \mathcal{L}_w$.*

Proof. If $v = (v_n)_{n\in\mathbb{Z}}$ and $w = (w_n)_{n\in\mathbb{Z}}$ are two symbolic codings of Λ, then there exists $k \in \mathbb{Z}$ such that for all $n \in \mathbb{N}$, $v_n = w_{n+k}$. Thus v and w have the same set of subwords.

□

We may then refer to them as *lexicon of* Λ, denoted by $\mathcal{L}_\Lambda$. Note that we can speak about lexicon of a point set Λ if and only if Λ is a subset of $\mathbb{R}$, since we need Λ to have a symbolic coding.

Proposition 7. *Let Λ and L be two FLC Delone sets in $\mathbb{R}$. Then $\mathcal{L}_L \subset \mathcal{L}_\Lambda$ if and only if L is a local translate of Λ.*

Proof. If L is a local translate of Λ, every "pattern" seen in L and be seen in Λ. In terms of symbolic codings this means that every subword of a symbolic coding of L is a subword of any symbolic coding of Λ.

□

Proposition 8. *Let Λ_1 and Λ_2 be two FLC Delone sets, and let us denote $\mathcal{L}_1$, $\mathcal{L}_2$, Ω_1 and Ω_2 their lexicons and their hulls respectively. Then:*

$$\mathcal{L}_1 = \mathcal{L}_2 \text{ if and only if } \Omega_1 = \Omega_2.$$

Proof. This is a corollary of the previous proposition and the fact that the local hull, hence the hull since Λ is an FLC Delone set, is the set of local translates.

□

4.4 Fibonacci hull

Fibonacci word

Definition 11 (Substitution maps)**.** *We will call* substitution map *a map from a finite set Σ, called* alphabet, *to Σ^*, the set of the finite words over the alphabet Σ.*

We can extend a substitution map σ defined on an alphabet Σ to a unique morphism on finite words on Σ. We can extend it the same way to infinite words on the alphabet Σ. We will note as well σ its extension to finite and infinite words.

Example 4. The following substitution can be used to define the Fibonacci word.

$$\begin{array}{cccc} \sigma : & \{1,\phi\} & \to & \{1,\phi\}^* \\ & 1 & \mapsto & \phi \\ & \phi & \mapsto & 1\phi \end{array}$$

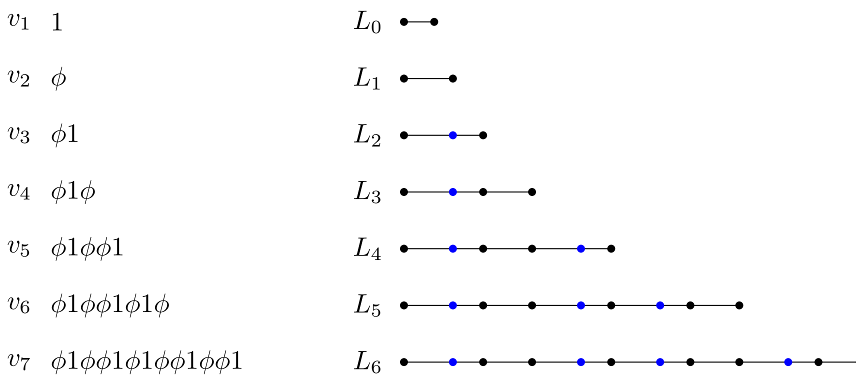
We define the infinite Fibonacci word w_F as follows[4]:

$$\begin{aligned} &v_1 = 1 \\ &v_{n+1} = \sigma(v_n) \\ &w_F = \lim v_n \quad \text{where the limit means that for all } n \in \mathbb{N},\ \forall k \in [\![0,n]\!], (w_F)_k = (v_n)_k{}^5. \end{aligned}$$

[4] There are other ways to define the Fibonacci word.

One can consider w_F as a symbolic coding[6] of a point set Λ_F in $\mathbb{R}$ with letter 1 standing for length 1 and letter ϕ standing for length $\varphi = \frac{1+\sqrt{5}}{2}$ (a uniformly discrete but not relatively dense point set). Actually, one could define w_F through this point set Λ_F. Let us define Λ_F:

$$L_0 = \{0, 1\}$$
$$L_{n+1} = \varphi.L_n \bigcup_{x \in L_n \,|\, x+\varphi \in L_n} \{\varphi.x + \varphi\} \quad \text{where } \varphi = \tfrac{1+\sqrt{5}}{2} \text{ is the golden ratio}$$
$$\Lambda_F = \bigcup_{n \in \mathbb{N}^*} L_n \quad \text{which is well defined because } \forall n \in \mathbb{N}^* L_n \subset L_{n+1}.$$



Black points in L_{n+1} are those that are also in φL_n and blue points are the others.

Figure 3: Construction of the Fibonacci word w_F

Proposition 9. *The Fibonacci word w_F is a fixpoint of the substitution map σ:*

$$\sigma(w_F) = w_F.$$

This can be proved by induction.
Remark 8: The Fibonacci word w_F could be *defined* as a fixpoint of the substitution map σ.

The Fibonacci word w_F is a famous word in combinatorics on words, with lots of useful properties, such as the fact that every subword of w_F appears infinitely many times in w_F (which can be proved by induction).

Fibonacci hull

We want to construct an FLC Delone point set *coming from* the Fibonacci word w_F. But if we consider a point set Λ whose (one of its) symbolic coding is w_F, then Λ is not relatively dense because w_F is not a bi-infinite word. We have to find another way to define Λ, still linked with w_F. But what exactly do we want Λ to be? We want it to be such that each of its local patterns is also a local pattern of w_F. More formally, if s is a symbolic coding of Λ, then we want Λ to be such that s is a bi-infinite word and $\mathcal{L}_s = \mathcal{L}_{w_F}$.
Remark 9: If s is an infinite word such that $\mathcal{L}_s \subset \mathcal{L}_{w_F}$ then $\mathcal{L}_s = \mathcal{L}_{w_F}$. This is because for all $n \in \mathbb{N}$, there exists an $N_n > n$ such that every subword of w_F of length n is also a subword of each subword of length N_n.

One way to define a such Λ is to construct *step by step* a sympbolic coding for Λ: first we chose a letter $s_0 \in \{1, \phi\} \subset \mathcal{L}_{w_F}$, then we chose two letters s_{-1}, $s_1 \in \{1, \phi\}$ such that the finite word $s_{-1}s_0s_1$ is in $\mathcal{L}_{w_F}$, then again we chose s_{-2}, $s_2 \in \{1, \phi\}$ such that $s_{-2}s_{-1}s_0s_1s_2 \in \mathcal{L}_{w_F}$, *etc.*
Remark 10: Since w_F is a fixpoint for the substitution σ (see proposition 3), $u \in \mathcal{L}_{w_F}$ if and only if $\sigma(u) \in \mathcal{L}_{w_F}$. Another really important property of w_F is that any finite subword $u \in \mathcal{L}_{w_F}$ of the Fibonacci word does appear

[5] Actually, this is an inverse limit. We define this notion in subsection 5.1.
[6] We have here identified words and sequences, which is natural since words are sequences of letters.

infinitely many times in w_F. This ensures that any finite subword $s_{-n}...s_0...s_n \in \mathcal{L}_{w_F}$ can always be extended into $s_{-n-1}s_{-n}...s_0...s_n s_{n+1} \in \mathcal{L}_{w_F}$.

Definition 12 (Fibonacci point set)**.** *We will call* Fibonacci point set *any FLC Delone set which can be obtained as described above.*

Of course, this construction requires infinitely many choices, which means that there are infinitely many Fibonacci point sets. But they will all have the same hull.

Proposition 10. *If Λ_1 and Λ_2 are txo Fibonacci point set, then $\Omega_{\Lambda_1} = \Omega_{\Lambda_2}$.*

Proof. This results from proposition 7: Since two Fibonacci words have the same lexicons, their hulls are also the same.

□

Remark 11: We could define more generally *substitution point sets* the same way as Fibonacci point sets but with another substitution map. A trivial other example would be $\Lambda = \mathbb{Z}$ obtained with the word $w = 1111....$, which is a fixpoint of the substitution map $\sigma : 1 \mapsto 11$.

The hull of a Fibonacci point set Λ is the set of all local translates of Λ, but as we have seen when constructing Λ, there are infinitely many choices to do, which means that there are infinitely many "types" of local translates (on the contrary, for $\mathbb{Z}\setminus\{0\}$, there were only two types of local translates: either $\mathbb{Z}\setminus\{0\} + x$ or $\mathbb{Z} + x$). If we want to understand the hull of a Fibonacci point set, we have no choice but to look at *approximants* of the hull.

5 Approximants

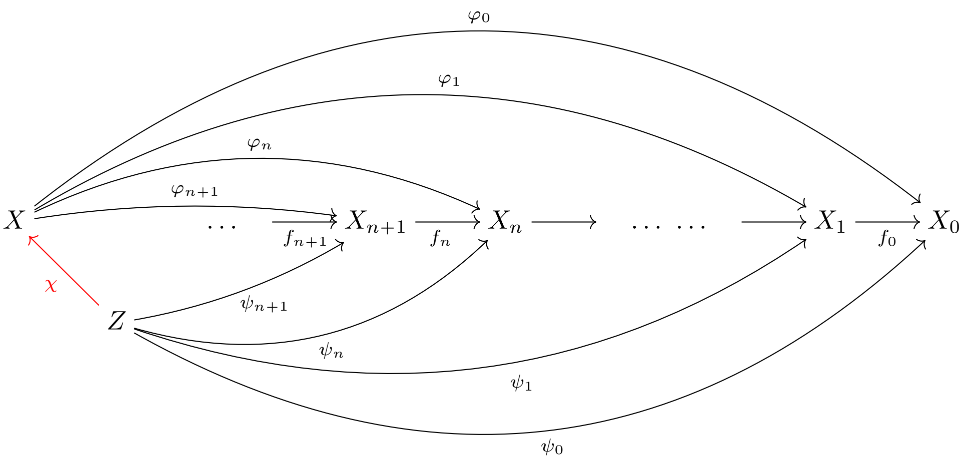
In order to understand the hull of a point set, we will study sequences of spaces which *approximate* the hull, namely that converge *in a certain way* to the hull. We first need to explain what this certain way is.

5.1 Inverse limit

Definition 13 (Inverse limit)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of spaces with continuous maps $f_n : X_{n+1} \to X_n$:*

$$\dots \xrightarrow{f_{n+1}} X_{n+1} \xrightarrow{f_n} X_n \longrightarrow \quad \dots\ \dots \quad \longrightarrow X_1 \xrightarrow{f_0} X_0$$

A space X is called an inverse limit *of $((X_n), (f_n))_{n\in\mathbb{N}}$, if there exists a collection of functions $\varphi_n : X \to X_n$ compatible with f_n, namely such that $f_n \circ \varphi_{n+1} = \varphi_n$, and such that for each other space Z with a collection of compatible functions $\psi_n : Z \to X_n$, there is a unique map $\chi : Z \to X$ so the following diagram commutes.*



Remark 12: X does depend on the maps f_n (not only on the X_n).

Theorem 3. *If there is an inverse limit X of the (X_n), then it is unique up to unique isomorphism making the whole diagram commute, and a model for X is a subset of the product $Y = \prod_{i\in\mathbb{N}} X_i$ with consistant information. We will note this model $\varprojlim X_n$: $\varprojlim X_n = \{(x_i)_{i\in\mathbb{N}} \mid \forall n \in \mathbb{N},\ f_n(x_{n+1}) = x_n\}$.*

For a proof of this theorem, see [10].
So we can think of X as the set of all sequences $(x_n)_{n\in N}$ such that for all $n \in \mathbb{N}, x_n \in X_n$ and $f_n(x_{n+1}) = x_n$. We may refer to it later on as *the* inverse limit of $(X_n)_{n\in\mathbb{N}}$.

Proposition 11. *If for all $n \in \mathbb{N}$, X_n is a compact space, then $X = \varprojlim X_n$ is also a compact space.*

Proof. This a corollary of the previous theorem and the Tychonoff theorem. □

5.2 Approximants of the hull

In all the constructions described in the following subsections, we approximate the hull by approximating its points (which are FLC Delone sets) with finite point sets. Anderson and Putnam ([14]) introduced such approximants for the first time in 1998. In 2002, Gähler ([14]) gave another construction. They were actually speaking of tilings, but we will here only speak of point sets. Karasik ([11]) has also introduced another contruction, motivated by the previous ones. In the mean time, lots of other matematicians worked on the subject ([14]) but we will not present their constructions here.

Geometric approximants

All the following notions and statements come from [11].

Definition 14 (Geometric approximation of the hull)**.** *Given an FLC Delone point set Λ of an lcsc group G and an open pre-compact subset K of G, we define the* geometric approximation*, associated with the compact K, of the hull Ω_Λ of Λ as*

$$\mathscr{G}_\Lambda^K = \{g.\Lambda \cap K \mid g \in G\}\,.$$

There exists a canonical map from the hull Ω_Λ of Λ to each geometrical approximant:

$$\begin{array}{rccc} \pi_K: & \Omega_\Lambda & \to & \mathscr{G}_\Lambda^K \\ & M & \mapsto & M \cap K \end{array}$$

Indeed, if $M \in \Omega_\Lambda$ is a translate of Λ then $M \cap K$ is in $\mathscr{G}_\Lambda^K$, else M is a local translate of Λ and there is a sequence $(M_n = g_n.\Lambda)$ of translates of Λ, which converges to M with regard to the local topology: for every identity neighbourhood V of G, there exists an $N \in \mathbb{N}$ such that for all $n \leqslant N$, $M_n \in U_{K,V}(M)$. In other words, there exists a sequence (e_n) in V such that $(g_n.\Lambda) \cap K = M_n \cap K = (e_n.M) \cap K$, so $M \cap K = (e_n^{-1}g_n.\Lambda) \cap K$ is in $\mathscr{G}_\Lambda^K$.
We will study geometrical approximants $\mathscr{G}_\Lambda^K$ with regard to the quotient topology associated with π_K.

Let us re-write this in the case of $\mathbb{R}$. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of open balls in $\mathbb{R}$, such that $B_n = \mathcal{B}(0, r_n)$ with $r_n < r_{n+1}$ for all $n \in \mathbb{N}$ and $r_n \underset{n\to+\infty}{\longrightarrow} +\infty$. We will denote

$$\mathscr{G}_\Lambda^n = \{(\Lambda + x) \cap B_n \mid x \in \mathbb{R}\}\,.$$

So, $\mathscr{G}_\Lambda^n$ corresponds to all the patterns of size $|B_n|$ one can see in Λ. We will consider the inverse limit of the $(\mathscr{G}_\Lambda^n)$ associated with the forgetful maps f_n:

$$\begin{array}{rccc} f_{\mathcal{G},n}: & \mathscr{G}_\Lambda^{n+1} & \to & \mathscr{G}_\Lambda^n \\ & p & \mapsto & p \cap B_n \end{array}$$

which are well defined and continuous with regard to the quotient topology.

Theorem 4. *The hull $\Omega_\Lambda = \Omega_\Lambda^{(loc)}$ of Λ is homeomorphic to the inverse limit $\varprojlim \mathscr{G}_\Lambda^n$ through the following map:*

$$\begin{array}{ccc} \Omega_\Lambda & \longrightarrow & \varprojlim \mathscr{G}_\Lambda^n \\ & & \\ p & \to & (p \cap B_n)_{n\in\mathbb{N}} \\ \bigcup_{n\in\mathbb{N}} p_n & \leftarrow & (p_n)_{n\in\mathbb{N}} \end{array}$$

For a proof of this theorem, see [11].

Theorem 5. *Let Λ be an FLC Delone set. Then $\mathscr{G}^B_\Lambda$ is homeomorphic to a finite graph.*

We give the main ideas of the proof:

1. Every point in $\mathscr{G}^B_\Lambda$ is either a *regular point*, meaning that it has a neighbourhood in $\mathscr{G}^B_\Lambda$ homeomorphic to an interval of $\mathbb{R}$, or a *special point*, meaning that it has a neighbourhood in $\mathscr{G}^B_\Lambda$ homeomorphic to a branch of intervals of $\mathbb{R}$ glued together (like a star whose center is the singular point).

2. There is a finite number of singular points.

For a complete proof, see [11].

Example 5. Let $B_n = \mathcal{B}(0, n)$ be open balls in $\mathbb{R}$. In figures 6 and 5, we show pictures of the geometric approximants of the hull Ω_Λ of $\Lambda = \mathbb{Z}\setminus\{0\}$ and $\Lambda = -2\mathbb{N} \cup \mathbb{N}$ respectively associated with the sequence (B_n).

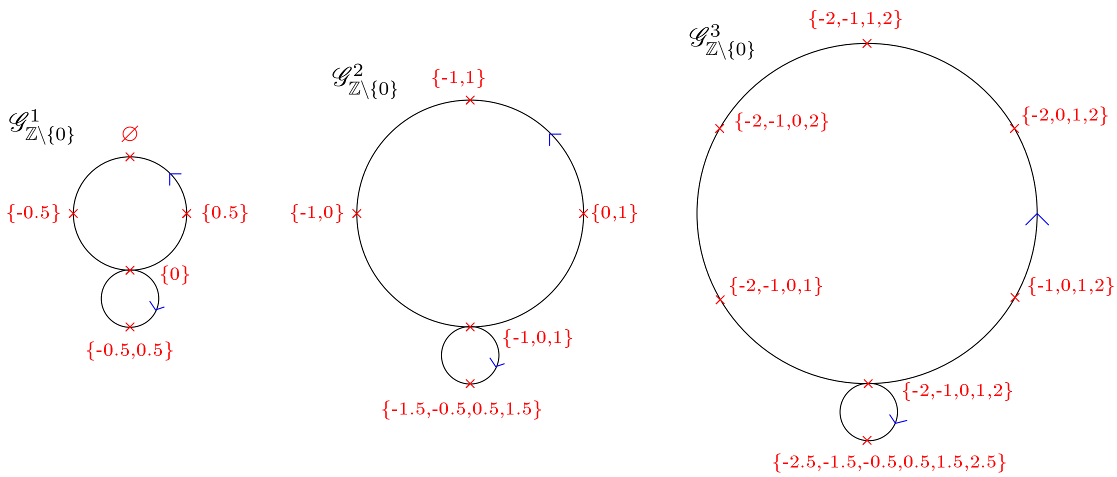


Figure 4: Three first geometric approximants of $\Omega_{\mathbb{Z}\setminus\{0\}}$

The upper circles correspond to the local patterns of $\mathbb{Z}\setminus\{0\}$ where we see the "gap" present in $\mathbb{Z}\setminus\{0\}$. In mathematical language, the upper circles correspond to the set of point sets $(\mathbb{Z}\setminus\{0\} + x) \cap \mathcal{B}(0, n)$ with $x \in]-n, -n[$.

The lower little circles correspond to the local patterns where we do not see the "gap" present in $\mathbb{Z}\setminus\{0\}$, namely in which all points are at distance exactly 1 from the previous and the next ones.

The blue arrows show the orientation inhereted through the action of $\mathbb{R}$ over Λ (from the orientation of $\mathbb{R}$).

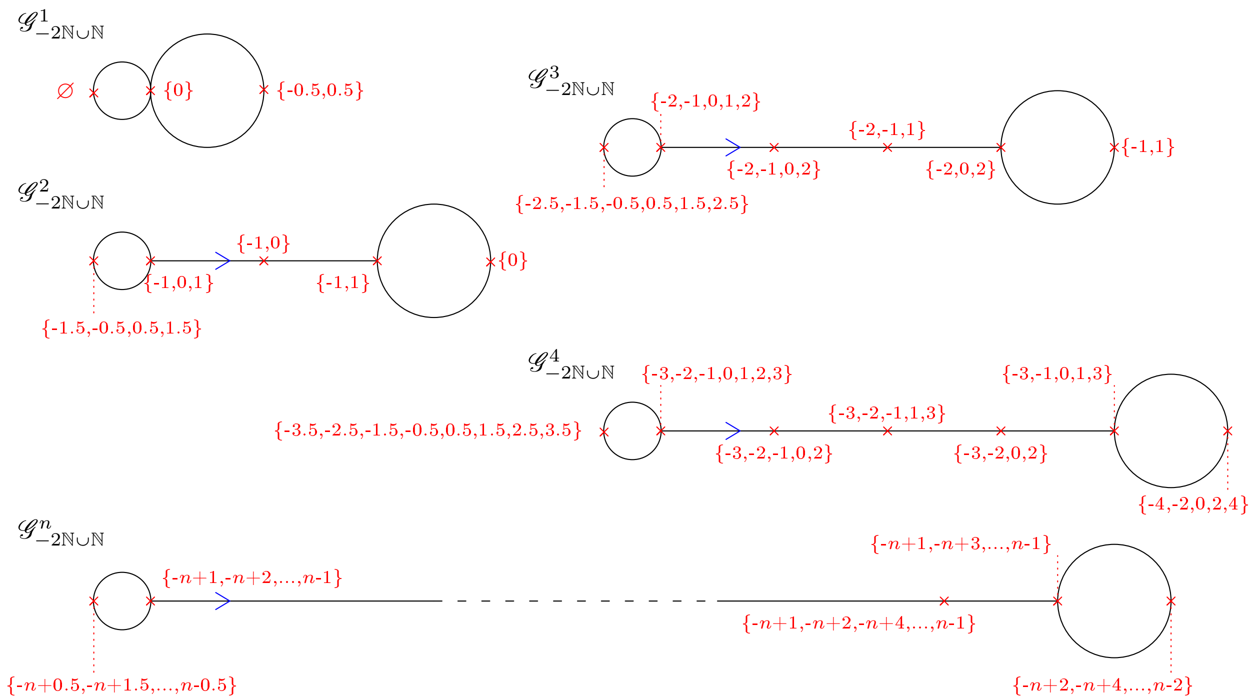


Figure 5: Four first geometric approximants of $\Omega_{-2\mathbb{N}\cup\mathbb{N}}$

The right circles correspond to the local patterns of $-2\mathbb{N}\cup\mathbb{N}$ where we only see the part $-2\mathbb{N}$ of $-2\mathbb{N}\cup\mathbb{N}$, in mathematical language, the right circles correspond to the set of the finite point sets $(-2\mathbb{N}\cup\mathbb{N}+x)\cap\mathcal{B}(0,n)$ with $x\geqslant n$.

The left circles correspond to the local patterns where we only see the part $\mathbb{N}$ of $-2\mathbb{N}\cup\mathbb{N}$, in mathematical language, the left circles correspond to the set of the point sets $(-2\mathbb{N}\cup\mathbb{N}+x)\cap\mathcal{B}(0,n)$ with $x\leqslant -n$.

The line between them corresponds to the local patternss where we see the "switchover" from $-2\mathbb{N}$ to $\mathbb{N}$, namely to the point sets of the form $(-2\mathbb{N}\cup\mathbb{N}+x)\cap\mathcal{B}(0,n)$ with $-n<x<n$.

The blue arrows show the orientation inhereted through the action of $\mathbb{R}$ over Λ (from the orientation of $\mathbb{R}$).

So the approximants are just finite graphs, spaces which are much easier to understand than most of the hulls of FLC Delone sets (see for example the slinkies obtained for $\Lambda = \mathbb{Z}\backslash\{0\}$ or $\Lambda = -2\mathbb{N} \cup \mathbb{N}$ in section 4.2). This is a reason why the approximants are convenient.

Gähler's construction

In the previous construction (geometric approximation of the hull $\Omega_\Lambda^{(loc)}$), a point in $\mathscr{G}_\Lambda^N$ gives us information about how to put tiles on a centered ball B_n. It can be convenient to consider approximants which give information about how to put a *fixed number* of points, no matter how much space they take.
We recall that one can see a point set in $\mathbb{R}$ as a tiling, two successive points then correspond to a tile, or more precisely to the boundary of a tile. The first approximant corresponds to instructions as to how to put one tile around the origin, that is to say which tile should be taken and where exactly around the origin it should be put. In the second approximation, we add the information of the two neighbours of the first tile. In the third one, we add the information of another ring of neighbours, *etc.*
This construction comes from [14], where it is done from a tiling point of view.

Definition 15 (Gähler approximation in $\mathbb{R}$)**.** *Let Λ be a Delone set in $\mathbb{R}$. Let $(x_n)_{n\in\mathbb{Z}}$ be an increasing enumeration of Λ and $\Delta = \{x_{i+1} - x_i \mid i \in \mathbb{Z}\}$ the set of all distances between two consecutive points of Λ. We denote by*

$$\mathscr{N}_\Lambda^n = \left\{ s = (x_{-n}, ..., x_0, ..., x_{n+1}) \in \mathbb{R}^{2n+2} \;\middle|\; \begin{array}{l} x_{-n} < ... < x_0 < x_1 < ... < x_{n+1} \text{ and } x_0 \leqslant 0 \leqslant x_1 \\ \exists x \in \mathbb{R},\ \exists K \subset \mathbb{R} \text{ compact, such that } (\Lambda + x) \cap K = s \end{array} \right\} \Big/ \sim$$

where

$$(x_{-n}, ..., x_0 = 0, ..., x_{n+1}) \sim (x'_{-n}, ..., x'_0, ..., x'_{n+1}) \text{ if and only if } \forall k,\ x'_k = x_k - x_1$$

the n-th Gähler approximation *of* $\Omega_\Lambda^{(loc)}$.

In other words, $\mathscr{N}_\Lambda^n$ tells how to put $2n+1$ consecutive tiles around the origin. We ask that this pattern could be seen somewhere in Λ, that is to say that this arrangement of tiles can be extended to a local translate of Λ. We also ask that the middle tile (*i.e.* the $(n+1)$-th) contains the origin, that is to say that the tiling we construct through the approximants does not grow only on the positive (or the negative) part of $\mathbb{R}$.
Remark 13: The Gähler approximation can also be defined for Delone sets in $\mathbb{R}^d$, which is explained in [14].

The Gähler approximants can be seen as quotients of the geometrical approximants. Indeed, let $r = \min d$ and $R = \max d$ be the infimum and supremum of a symbolic coding d of Λ. We recall that this is equivalent to saying that Λ is an (r, R)-Delone set, or in other words that for all $k \in [\![-n, n]\!]$ and $(x_{-n}, ..., x_0, ..., x_{n+1}) \in \mathscr{N}_\Lambda^n$, $r < x_{k+1} - x_k < R$. Then there exists a canonical map from $\mathscr{G}_\Lambda^{\mathcal{B}(0,(n+1)R)}$ to $\mathscr{N}_\Lambda^n$ and from $\mathscr{N}_\Lambda^n$ to $\mathscr{G}_\Lambda^{\mathcal{B}(0,nr)}$:

$$\begin{array}{ccccc} \mathscr{G}_\Lambda^{\mathcal{B}(0,(n+1)R)} & \longrightarrow & \mathscr{N}_\Lambda^n & \longrightarrow & \mathscr{G}_\Lambda^{\mathcal{B}(0,nr)} \\ (x_{-p}, ..., x_q) & \mapsto & M = (x_{-n}, ..., x_0, ..., x_{n+1}) & \mapsto & M \cap \mathcal{B}(0, nr) \end{array}$$

where $p \geqslant n$ and $q \geqslant n+1$ because of the choice of R and $x_{-n} \leqslant -nr$ and $x_{n+1} \geqslant nr$ because of the choice of r.
In particular, $\mathscr{N}_\Lambda^n$ is a quotient of $\mathscr{G}_\Lambda^{\mathcal{B}(0,(n+1)R)}$.

Proposition 12. *With regard to the quotient topology, the forgetful maps*

$$f_{\mathcal{N},n} : \begin{array}{ccc} \mathscr{N}_\Lambda^{n+1} & \longrightarrow & \mathscr{N}_\Lambda^n \\ (x_{-n-1}, ..., x_0, ..., x_{n+2}) & \longmapsto & (x_{-n}, ..., x_0, ..., x_{n+1}) \end{array}$$

are continuous for all $n \in \mathbb{N}$.

Proof. The following diagram commutes, and we know that the upper arrow, the right and the left ones are continuous. It follows that $f_{\mathcal{N},n}$ is also continuous.

$$\begin{array}{ccc} \mathscr{G}_\Lambda^{\mathcal{B}(0,(n+2)R)} & \xrightarrow{f_{\mathcal{G},n}} & \mathscr{G}_\Lambda^{\mathcal{B}(0,(n+1)R)} \\ \downarrow & & \downarrow \\ \mathscr{N}_\Lambda^{n+1} & \xrightarrow{f_{\mathcal{N},n}} & \mathscr{N}_\Lambda^n \end{array}$$

□

Proposition 13. *Associated with the forgetful maps,* $\varprojlim \mathscr{N}_\Lambda^n$ *is homeomorphic to* Ω_Λ*.*

Proof. Since we have continuous maps

$$\mathscr{N}_\Lambda^N \longrightarrow \mathscr{G}_\Lambda^{\mathcal{B}(0,(n+1)R)} \longrightarrow \mathscr{N}_\Lambda^n \longrightarrow \mathscr{G}_\Lambda^{\mathcal{B}(0,nr)} \qquad \text{(where } Nr \geqslant (n+1)R\text{)}$$

which are compatible with the forgetful maps

$$\mathscr{G}_\Lambda^{\mathcal{B}(0,(n+1)R)} \longrightarrow \mathscr{G}_\Lambda^{\mathcal{B}(0,nr)} \quad \text{and} \quad \mathscr{N}_\Lambda^N \longrightarrow \mathscr{N}_\Lambda^n \ .$$

We know, by universal property, that $\varprojlim \mathscr{N}_\Lambda^n$ is homeomorphic to $\varprojlim \mathscr{G}_\Lambda^n$, namely homeomorphic to the hull Ω_Λ of Λ.

□

Example 6. We represent Gähler approximants for $\mathbb{Z}\backslash\{0\}$.

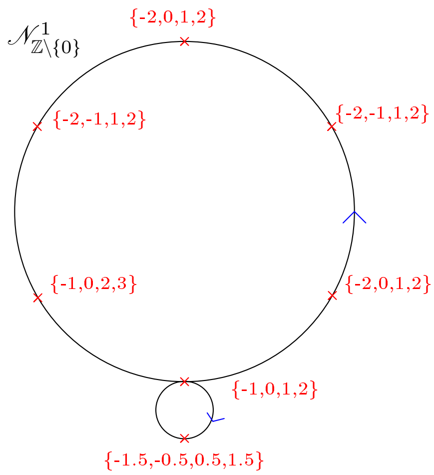


Figure 6: First Gähler approximant of $\Omega_{\mathbb{Z}\backslash\{0\}}$

Remark 14: Like the geometrical approximants, the Gähler approximants are homeomorphic to finite graphs.

We note that the maps for the Fibonacci example are getting more and more complicated between each approximant. The Anderson-Putnam construction allows to avoid this inconvenient.

Aderson-Putnam's construction

This construction comes from [14], where it is done with a tiling point of view. We define it only for a Fibonacci point set Λ but this could be generalized to any *substitution* point set in $\mathbb{R}$, namely a point set whose lexicon is equal to the lexicon of a word being a fixpoint of a substitution map.

Definition 16. *Let* Λ *be a Fibonacci point set. We define*

$$\mathscr{S}_\Lambda^1 = \left\{ (x_{-1}, x_0, x_1, x_2) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_{-1} < x_0 \leqslant 0 < x_1 < x_2 \\ \exists x \in \mathbb{R},\ \exists K \subset \mathbb{R} \text{ compact, such that } (\Lambda + x) \cap K = s \end{array} \right\} = \mathscr{N}_\Lambda^1$$

and

$$\theta_n : \begin{array}{ccccc} \mathscr{S}_\Lambda^n & \longrightarrow & & \theta_n(\mathscr{S}_\Lambda^n) & \\ \{x_k\} & \longmapsto & \{\varphi.x_k\} & \displaystyle\bigcup_{\substack{x_k \text{ such that} \\ x_{k+1} = x_k + \varphi}} & \{\varphi.x_k + \varphi\} \end{array} \qquad \text{with} \qquad \mathscr{S}_\Lambda^{n+1} = \theta_n(\mathscr{S}_\Lambda^n)$$

for all $n \geqslant 1$.

Remark 15: Note that every point in $\mathscr{S}_\Lambda^n$ does not contain information about the same number of tiles, neither about the same length, but they contain information about at least a certain number of tiles, at least a certain length, and at most another certain number of tiles and certain length.

The Anderson-Putnam approximants can be seen as quotients of the geometrical approximants. Indeed there exists a canonical map from $\mathscr{G}_\Lambda^{\mathcal{B}(0,2\varphi^n)}$ to $\mathscr{S}_\Lambda^n$ and from $\mathscr{S}_\Lambda^n$ to $\mathscr{G}_\Lambda^{\mathcal{B}(0,\varphi^{n-1})}$:

$$\begin{array}{ccccc} \mathscr{G}_\Lambda^{\mathcal{B}(0,2\varphi^n)} & \longrightarrow & \mathscr{S}_\Lambda^n & \longrightarrow & \mathscr{G}_\Lambda^{\mathcal{B}(0,\varphi^{n-1})} \\ (x_{-p}, ..., x_q) & \mapsto & M = (x_{-l_n}, ..., x_0, ..., x_{r_n}) & \mapsto & M \cap \mathcal{B}(0, \varphi^{n-1}) \end{array}$$

These maps are well defined because:

- If $(x_{-1}, x_0, x_1, x_2) \in \mathscr{S}_\Lambda^1$, then $-2\varphi < x_1 \leqslant -1$ and $1 < x_1 \leqslant 2\varphi$.
- If all $s = (x_{-l_n}, ..., x_0, ..., x_{r_n}) \in \mathscr{S}_\Lambda^n$ are such that $-b < x_{-l_n} \leqslant -a$ and $a < x_{r_n} \leqslant b$, then all $t = (x_{-l_{n+1}}, ..., x_0, ..., x_{r_{n+1}}) \in \mathscr{S}_\Lambda^{n+1}$ are such that $-b\varphi < x_{-l_n} \leqslant -a\varphi$ and $a\varphi < x_{r_n} \leqslant b\varphi$.
- By induction, we conclude that for all $n \in \mathbb{N}$, for all $s = (x_{-l_n}, ..., x_0, ..., x_{r_n}) \in \mathscr{S}_\Lambda^n$, we have $-2\varphi^n < x_{-l_n} \leqslant \varphi^{n-1}$ and $\varphi^{n-1} < x_{r_n} \leqslant 2\varphi^n$.

In particular, $\mathscr{S}_\Lambda^n$ is a quotient of $\mathscr{G}_\Lambda^{\mathcal{B}(0,2\varphi^n)}$.

Proposition 14. *With regard to the quotient topology, the forgetful maps*

$$\begin{array}{cccc} w_n : & \mathscr{S}_\Lambda^{n+1} & \longrightarrow & \mathscr{S}_\Lambda^n \\ & \{x_k\} & \longmapsto & \left\{\frac{1}{\varphi}.x_k\right\} \setminus \left(\bigcup_{\substack{x_k \text{ such that} \\ x_k + 1 = x_{k+1}}} \left\{\frac{1}{\varphi}.x_k\right\} \right) \end{array}$$

are continuous for all $n \in \mathbb{N}$.

Proof. The following diagram commutes, and we know that the upper arrow, the right and the left ones are continuous. It follows that w_n is also continuous.

$$\begin{array}{ccc} \mathscr{G}_\Lambda^{\mathcal{B}(0,2\varphi^{n+2})} & \xrightarrow{f_{\mathcal{G},n}} & \mathscr{G}_\Lambda^{\mathcal{B}(0,2\varphi^{n+1})} \\ \downarrow & & \downarrow \\ \mathscr{S}_\Lambda^{n+1} & \xrightarrow{w_n} & \mathscr{S}_\Lambda^n \end{array}$$

□

Proposition 15. *Associated with the forgetful maps, $\varprojlim \mathscr{S}_\Lambda^n$ is homeomorphic to Ω_Λ.*

Proof. Since we have continuous maps

$$\mathscr{S}_\Lambda^{n+3} \longrightarrow \mathscr{G}_\Lambda^{\mathcal{B}(0,2\varphi^n)} \longrightarrow \mathscr{S}_\Lambda^n \longrightarrow \mathscr{G}_\Lambda^{\mathcal{B}(0,\varphi^{n-1})}$$

which are compatible with the forgetful maps

$$\mathscr{G}_\Lambda^{\mathcal{B}(0,2\varphi^n)} \longrightarrow \mathscr{G}_\Lambda^{\mathcal{B}(0,\varphi^{n-1})} \quad \text{and} \quad \mathscr{S}_\Lambda^{n+3} \longrightarrow \mathscr{S}_\Lambda^n .$$

We know, by universal property, that $\varprojlim \mathscr{S}_\Lambda^n$ is homeomorphic to $\varprojlim \mathscr{G}_\Lambda^n$, namely homeomorphic to the hull Ω_Λ of Λ.

□

Example 7. We reprensent Anderson-Putnam approximants for $n = 1, 2$, and 3. We write only the symbolic condings associated with the finite point sets.

We note that once more, the approximants are homeomorphic to finite graphs. The advantage of the Anderson-Putnam approximants over geometrical or the Gähler approximants is that the forgetful maps are all similar.

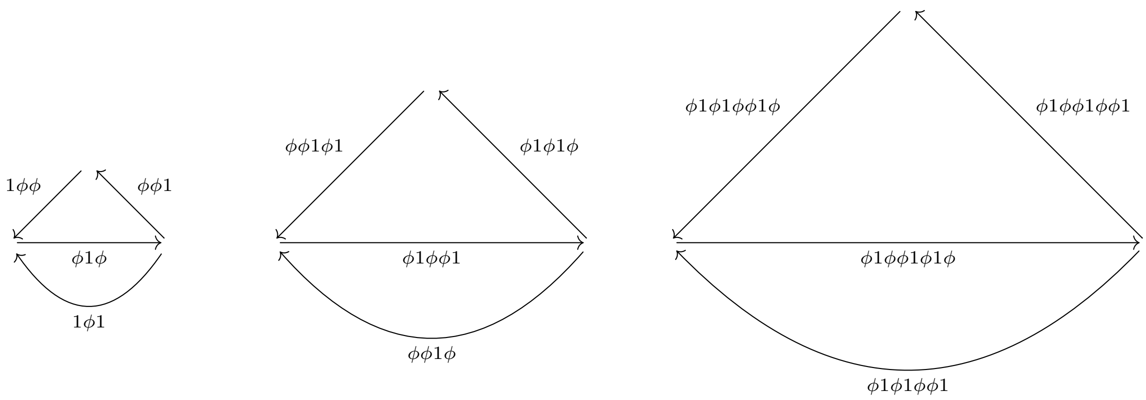


Figure 7: $\mathscr{S}_\Lambda^1$, $\mathscr{S}_\Lambda^2$ and $\mathscr{S}_\Lambda^3$

6 Functions on an inverse limit

We want to study functions on the hull in order to better understand the hull itself. We will denote by $C(X)$ the set of continuous functions on X, and $C_c(X)$ the set of continuous functions with compact support on X.

6.1 A first spanning set

We first recall the definition of pullbacks and the Stone-Weierstraß theorem:

Definition 17 (Pullback)**.** *Let X and Y be two topological spaces, and $\sigma : X \to Y$ a continuous function.*

$$\begin{array}{cccc} \sigma^* : & C(Y) & \to & C(X) \\ & f & \mapsto & f \circ \sigma \end{array}$$

is called pullback *of σ.*

Theorem 6 (Stone-Weierstraß). *Let X be a compact space. Let $C(X)$ be the algebra of the continuous real-valued functions on X.*
If A is a subalgebra of $C(X)$ which separates points[7]*, then either it exists x_0 such that for all $f \in A$, $f(x_0) = 0$ and A is dense in $\{f \in C(X) \mid f(x_0) = 0\}$, or A is dense in $C(X)$.*

For a proof of this theorem, see [9].

Proposition 16. *Let $(X_n)_{n\in\mathbb{N}}$ be compact spaces, with continuous functions $f_n : X_{n+1} \to X_n$. Let $X = \varprojlim X_n$ be an inverse limit of $(X_n)_{n\in\mathbb{N}}$ and let us denote by $\pi_n : X \to X_n$ the collection of projections. The continuous real-valued functions on X are spanned by the pullbacks of the continuous real-valued functions on the spaces X_n:*

$$C(X) = \overline{\bigcup_{n\in\mathbb{N}*} \pi_n^*(C(X_n))}^{\|\cdot\|_\infty}$$

Proof. For all $n \in \mathbb{N}$, $\pi_n^*(C(X_n)) \subset \pi_{n+1}^*(C(X_{n+1}))$, so $\bigcup\limits_{n\in\mathbb{N}} \pi_n^*(C(X_n))$ is a subalgebra of $C(X)$. All we need to prove is that it separates points and contains, for all $x_0 \in X$, a function f_{x_0} which is non-zero on x_0. We can then conclude with the Stone-Weierstraß theorem.
So, let x and y be distinct points in X. It exists $N \in \mathbb{N}$ such that $\pi_N(x) \neq \pi_N(y)$ otherwise x and y would be the same point in the inverse limit.
But using Urysohn's lemma, we know that $C(X_N)$ separates points since X_N is a compact space. So, there exists a function f such that $f(\pi_N(x)) \neq f(\pi_N(y))$, namely $\pi_N^*(f) \in \bigcup\limits_{n\in\mathbb{N}} \pi_n^*(C(X_n))$ is such that $\pi_N^*(f)(x) \neq \pi_N^*(f)(y)$. Hence, $\bigcup\limits_{n\in\mathbb{N}} \pi_n^*(C(X_n))$ separates points.

[7] This means that $\forall x, y \in X$, $[x \neq y \Rightarrow \exists f \in A, f(x) \neq f(y)]$.

Moreover, $\pi_1^*(\mathbb{1}_{X_1}) = \mathbb{1}_X \in \bigcup_{n\in\mathbb{N}} \pi_n^*(C(X_n))$ so for all $x_0 \in X$, $\pi_1^*(\mathbb{1}_{X_1})(x_0) \neq 0$, namely there is no point $x_0 \in X$ such that all functions in $\bigcup_{n\in\mathbb{N}} \pi_n^*(C(X_n))$ is zero on x_0.
We conclude that $\bigcup_{n\in\mathbb{N}} \pi_n^*(C(X_n))$ is dense in $C(X)$.

□

This proposition allows us to study only functions on approximants, instead of all functions on the hull. This is a reason why the approximants are really convenient! In the following section, we define functions on approximants coming from functions on G.

6.2 Periodization

Definition 18 (Periodization)**.** *Let G be an lcsc group and Λ an FLC Delone set.*
The periodization map *of Λ is defined as:*

$$\begin{array}{ccccccc} \mathcal{P}: & C_c(G) & \longrightarrow & C(\Omega_\Lambda) & & & \\ & f & \longmapsto & \mathcal{P}(f): & \Omega_\Lambda & \to & \mathbb{R} \\ & & & & M & \mapsto & \sum\limits_{x\in M} f(x) \end{array}$$

This map is well-defined because on the one hand for every $M \in \Omega_\Lambda$ and $f \in C_c(\mathbb{R})$, the sum defining $\mathcal{P}(f)(M)$ is finite so $\mathcal{P}(f)(M)$ is well-defined and on the other hand $\mathcal{P}(f)$ is continuous, as established in Proposition 5.1 in [5].
Remark 16: This map is called periodization map because if Λ is a discrete subgroup of $\mathbb{R}$, let us say $\Lambda = \alpha\mathbb{Z}$ then:

$$\begin{aligned} \forall M = \Lambda + \tau \in \Omega_\Lambda,\ \forall \lambda = n\alpha \in \Lambda,\ \mathcal{P}(f)(M+\lambda) &= \sum_{x\,\in\, M+n\alpha} f(x) = \sum_{m\,\in\, M} f(m+n\alpha) \\ &= \sum_{k\,\in\,\mathbb{Z}} f(k\alpha+\tau+n\alpha) \\ &= \sum_{k\,\in\,\mathbb{Z}} f(k\alpha+\tau) = \sum_{m\,\in\,\alpha\mathbb{Z}+\tau} f(m) \\ &= \mathcal{P}(f)(M) \end{aligned}$$

We will denote by $\mathcal{A}$ the algebra of functions on Ω_Λ spanned by periodizations of continuous functions with compact supports on G: $\mathcal{A} =< \mathcal{P}(C_c(G)) >$, where $< S >$ denotes the algebra spanned by the set S.

Proposition 17. *Let Λ be an FLC Delone set and Ω_Λ be its hull.*
Then $\mathcal{A}$ is dense in $C(\Omega_\Lambda)$.

Proof. To prove this, we will prove that $\mathcal{A}$ separates points in Ω_Λ and that for every $\omega \in \Omega_\Lambda$, there exists a function $\psi \in \mathcal{A}$ such that $\psi(\omega) \neq 0$. The Stone-Weierstraßtheorem then implies that $\mathcal{A}$ is dense in $C(\Omega_\Lambda)$.

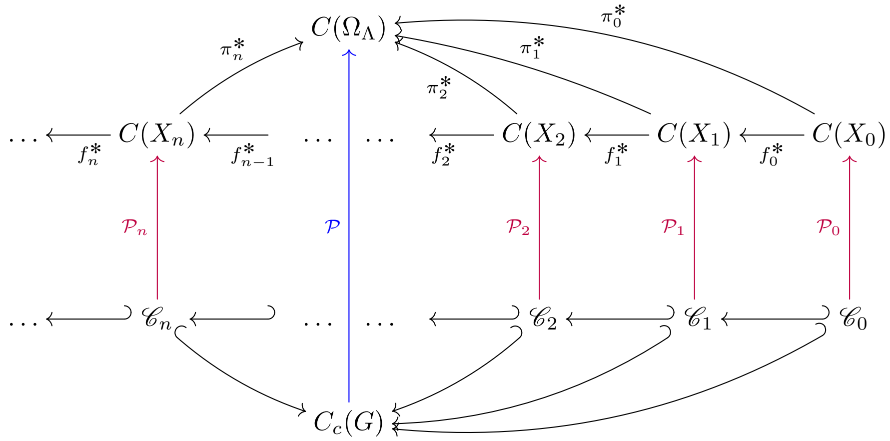
We know that the hull Ω_Λ of Λ is the set of all local translates of Λ. This implies that Ω_Λ is a set of FLC Delone sets, in particular, $\varnothing \notin \Omega_\Lambda$. If $\omega \in \Omega_\Lambda$, then there exists $x \in \omega$ and $r > 0$ such that $]x-r, x+r[\cap\omega = \{x\}$ (because Λ is discrete).
Let $h \in C_c(\mathbb{R})$ be a continuous real-valued function on $\mathbb{R}$ such that $h(x) = 1$ and $supp(h) \subset]x-r, x+r[$. Then $\mathcal{P}(h)(\omega) = 1$ and $\mathcal{P}(h) \in \mathcal{A}$ by definition of $\mathcal{A}$.

If $\omega' \in \Omega_\Lambda$, distinct from ω, then there exists $x \in \omega\backslash\omega'$ or $x \in \omega'\backslash\omega$, say $x \in \omega\backslash\omega'$. The point sets $\omega\backslash\omega'$ and ω' are closed and disjoined, so $d(\omega\backslash\omega', \omega') = \varepsilon > 0$.
Let $h \in C_c(\mathbb{R})$ be a continuous real-valued function on $\mathbb{R}$ such that $h(x) = 1$ and $supp(h) \subset]x-\varepsilon, x+\varepsilon[$ and $h = 1$ on $]x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}[$. Then $\mathcal{P}(h)(\omega) > 0$ but $\mathcal{P}(h)(\omega') = 0$. So $\mathcal{A}$ separates points.

We conclude that $\mathcal{A}$ is dense in $C_c(\Omega_\Lambda)$.

□

Now, we want to define *local periodizations* $\mathcal{P}_n$, namely periodizations on the approximants $(X_n)_{n\in\mathbb{N}}$ of the hull Ω_Λ (geometric, Gähler or Anderson–Putnam approximants). The goal is to have a commutative diagram:



where the $(\mathscr{C}_n)_{n\in\mathbb{N}}$ are subsets of $C_c(G)$.

Definition 19 (Local periodizations)**.** *The* local periodization maps *of an FLC Delone set* Λ *are defined as:*

$$\begin{array}{rccccl} \mathcal{P}_n: & \mathscr{C}_n & \longrightarrow & C(X_n) & & \\ & f & \longmapsto & \mathcal{P}_n(f): & X_n & \rightarrow \quad \mathbb{R} \\ & & & & M & \mapsto \quad \sum_{x\in M} f(x) \end{array}$$

If we want this diagram to commute, then the $(\mathscr{C}_n)_{n\in\mathbb{N}}$ must not contain too many functions.
For instance, we give a counterexample in $\mathbb{R}$, with $\Lambda = \mathbb{Z}\backslash\{0\}$, $\mathscr{C}_n = C_c(\mathbb{R})$, $X_n = \mathscr{G}^n_\Lambda$ associated with $B_n = \mathcal{B}(0,n)$.
Let us consider $h \in C_c(\mathbb{R})$ such that $supp(h) \subset [-n_0 - \frac{3}{4}, n_0 + \frac{3}{4}]$ and $h = 1$ on $[-n_0 - \frac{1}{2}, n_0 + \frac{1}{2}]$, and $M = \left\{-n_0 - \frac{1}{2}, -n_0 + \frac{1}{2}, ..., n_0 + \frac{1}{2}\right\} \in X_{n_0+1}$ and $N = M\backslash\left\{n_0 + \frac{1}{2}\right\} \in X_{n_0+1}$
We have $\mathcal{P}_{n_0+1}(h)(M) = |M| \neq |N| = \mathcal{P}_{n_0+1}(h)(N)$ but $\mathcal{P}_{n_0}(h)(M) \neq \mathcal{P}_{n_0}(h)(N)$ so $f^*_{n_0}(\mathcal{P}_{n_0}(h)) \neq \mathcal{P}_{n_0+1}(h)$ which means that the diagram does not commute.

We will denote by $\mathcal{A}_n$ the algebra of functions on $C(X_n)$ spanned by periodizations of functions in $\mathscr{C}_n$.

We also want $(\mathscr{C}_n)_{n\in\mathbb{N}}$ to be such that $\mathcal{A} = \overline{\bigcup_{n\in\mathbb{N}} \pi^*_n(\mathcal{A}_n)}^{\|\cdot\|_\infty}$. To ensure this condition, $(\mathscr{C}_n)_{n\in\mathbb{N}}$ must contain enough functions. Indeed, taking $\mathscr{C}_n = \mathscr{C}_0 = C_c(\mathcal{B}(0,1))$ for all $n \in \mathbb{N}$ would make the diagram commute but we would have $\mathcal{A} \supsetneq \overline{\bigcup_{n\in\mathbb{N}} \pi^*_n(\mathcal{A}_n)}^{\|\cdot\|_\infty}$. For instance, if $h \in C_c(\mathcal{B}(0,1))$, then $\mathcal{P}(h)$ cannot distinguish $M \in \Omega_\Lambda$ from $N \in \Omega_\Lambda$ (*i.e.* $\mathcal{P}h(M) \neq \mathcal{P}h(N)$) whenever $M \cap \mathcal{B}(0,1) = N \cap \mathcal{B}(0,1)$.

We will study this local periodizations for $G = \mathbb{R}$, with $X_n = \mathscr{G}^n_\Lambda$ and $X_n = \mathscr{S}^n_\Lambda$ and give examples of $(\mathscr{C}_n)_{n\in\mathbb{N}}$ so that the diagram commute and $\mathcal{A} = \overline{\bigcup_{n\in\mathbb{N}} \pi^*_n(\mathcal{A}_n)}^{\|\cdot\|_\infty}$.

6.3 Local periodization in geometric approximants

Fix a sequence of open balls (B_n) in G, say $B_n = \mathcal{B}(0, R_n)$, with increasing radii $R_n < R_{n+1}$. Consider the geometric approximants associated with the sequence of balls (B_n).

Definition 20 (Geometric local periodization)**.** *The* geometric local periodization map *of an FLC Delone set* Λ *is defined as:*

$$\begin{array}{rccccl} \mathcal{P}_{\mathcal{G},n}: & C_c(\mathcal{B}(0,R_n)) & \longrightarrow & C(\mathscr{G}^n_\Lambda) & & \\ & h & \longmapsto & \mathcal{P}_{\mathcal{G},n}(h): & \mathscr{G}^n_\Lambda & \rightarrow \quad \mathbb{R} \\ & & & & M & \mapsto \quad \sum_{x\in M} h(x) \end{array}$$

With the notations of the previous subsection, we are chosing $\mathscr{C}_n = C_c(B_n)$ and $X_n = \mathscr{G}^n_\Lambda$ where the geometrical approximants are associated with the sequence of balls (B_n). Then the diagram commutes because

$$\begin{aligned}
&\forall h \in C_c(B_n) \subset C_c(B_{n+1}),\ \forall p \in \mathscr{G}_\Lambda^{n+1}, \\
&\qquad f_n^* \circ [\mathcal{P}_{\mathcal{G},n}(h)](p) = [\mathcal{P}_{\mathcal{G},n}(h)](p \cap B_n) \\
&\qquad\qquad = \sum_{x \in p \cap B_n} h(x) \\
&\qquad\qquad = \sum_{x \in p} h(x) \qquad \text{because } h \in C_c(B_n) \\
&\qquad\qquad = [\mathcal{P}_{\mathcal{G},n}(h)](p).
\end{aligned}$$

In this sense a periodization is a kind of periodic function.

Example of $\Lambda = -2\mathbb{N} \cup \mathbb{N} \subset G = \mathbb{R}$

We represent the geometric local periodization of a function.

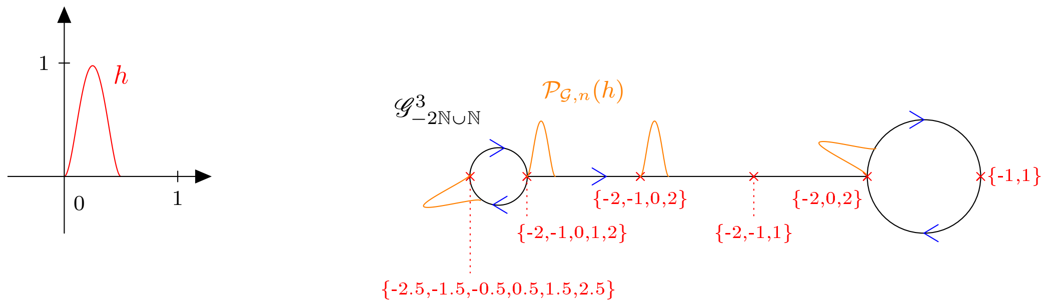


Figure 8: A function h and its periodization on $\mathscr{G}^3_{-2\mathbb{N}\cup\mathbb{N}}$

We note that the shape of h appear each time that a point p of $\mathscr{G}_\Lambda^n$ contain $x \in p \cap supp(h)$.

6.4 Local periodization in Anderson-Putnam's construction

Let Λ be a Fibonacci point set.
Each point of the n-th Anderson-Putnam approximant covers a same ball $\mathcal{B}(0, R_n)$ with R_n depending on the growth's rate of the substitution map σ. In other words, if $(x_{-l_n}, ..., x_{r_n}) \in \mathscr{S}_\Lambda^n$, then $x_{-l_n} \leqslant -R_n$ and $x_{r_n} \geqslant R_n$. For the Fibonacci case, $R_n = \varphi^{n-1}$. We define periodization like before, on these balls $\mathcal{B}(0, R_n)$.

Definition 21 (Anderson-Putnam Periodization). *The* Anderson-Putnam periodization map *of a Fibonacci point set Λ is defined as:*

$$\begin{array}{rccccc}
\mathcal{P}_{\mathcal{S},n}: & C_c(\mathcal{B}(0,\varphi^{n-1})) & \longrightarrow & C(\mathscr{S}_\Lambda^n) & & \\
& f & \longmapsto & \mathcal{P}_{\mathcal{S},n}(f): & \mathscr{S}_\Lambda^n \to & \mathbb{R} \\
& & & & M \mapsto & \sum_{x \in M} f(x)
\end{array}.$$

Remark 17: It might be that two points of $\mathscr{S}_\Lambda^n$ could not be distinguished with periodized functions, in other words it can exist $\alpha, \beta \in \mathscr{S}_\Lambda^n$ such that $\alpha \neq \beta$ but $\forall f \in C_c(\mathcal{B}(0, R_n))$, $\mathcal{P}_{\mathcal{S},n}(f)(\alpha) = \mathcal{P}_{\mathcal{S},n}(f)(\beta)$. However, it exists an $N \in \mathbb{N}$ such that $\sigma^N(\alpha)$ and $\sigma^N(\beta)$ are distinguishable with functions on $\mathscr{S}_\Lambda^{n+N}$.

Remark 18: We could define those periodizations more generally for substitution point sets.

From now on, we assume that the $(\mathscr{C}_n)$ are chosen such that the whole diagram commutes and $\mathcal{A} = \overline{\bigcup_{n\in\mathbb{N}} \pi_n^*(\mathcal{A}_n)}^{\|\cdot\|_\infty}$.

7 Periodization complexity

Let us denote by $\mathcal{A}_n^{(k)}$ the vector space in $C(X_n)$ spanned by products of at most k periodizations of functions in $\mathscr{C}_n$, and by $\mathcal{A}^{(k)}$ the vector space in $C(X)$ spanned by products of at most k periodizations of functions in $C_c(\mathbb{R})$.

Definition 22 (Periodization complexity)**.** *Let Λ be an FLC Delone point set.*
We will say that Λ has periodization complexity *at most N if*

$$C(\Omega_\Lambda) = \overline{\mathcal{A}^{(N)}}^{\|\cdot\|_\infty}$$

and has periodization complexity N if it has periodization complexity at most N but has not periodization complexity at most $N-1$.

To be of periodization complexity N basically means that one can construct every function on the hull with sums of products of at most N periodizations (possibly by approximating it with a sequence of such sums).

Remark 19: Periodization complexity is not a notion from the literature but was introduced for my internship.

7.1 Sufficient conditions

To show that Λ is of periodization complexity at most K, one can prove that for all $n \in \mathbb{N}$, every continuous function on the (geometric, Gähler or Anderson-Putnam) approximant X_n of the hull Ω_Λ of Λ can be written as a sum of products of at most K periodizations of continuous functions on $\mathbb{R}$. But this will not always be the case (actually, very rarely). Though, there are other properties, more common, also implying that Λ is of periodization complexity at most K.

Proposition 18 (Sufficient conditions for periodization complexity at most K)**.**
We have the following chain of implications:

$$\forall n \in \mathbb{N},\ \mathcal{A}_n \subset \mathcal{A}_n^{(K)} \qquad (1.)$$

$$\Rightarrow \forall n \in \mathbb{N},\ \mathcal{A}_n \subset \overline{\mathcal{A}_n^{(K)}}^{\|\cdot\|_\infty} \qquad (2.)$$

$$\Rightarrow \forall n \in \mathbb{N},\ \exists N_n \in \mathbb{N},\ f_{N_n-1}^* \circ \cdots \circ f_n^*(\mathcal{A}_n) \subset \overline{\mathcal{A}_{N_n}^{(K)}}^{\|\cdot\|_\infty} \qquad (3.)$$

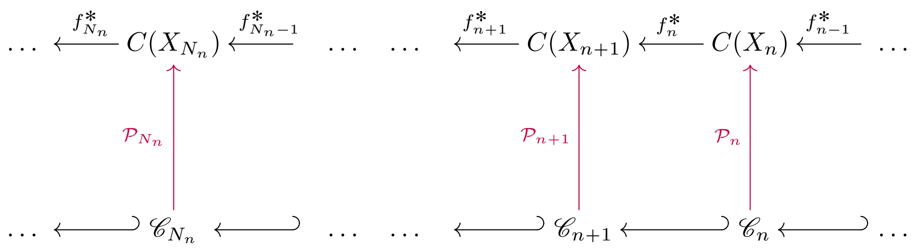
$$\Rightarrow \exists n_0,\ \forall n \geqslant n_0,\ \exists N_n \in \mathbb{N},\ f_{N_n-1}^* \circ \cdots \circ f_n^*(\mathcal{A}_n) \subset \overline{\mathcal{A}_{N_n}^{(K)}}^{\|\cdot\|_\infty} \qquad (4.)$$

$$\Rightarrow \Lambda \textit{ is of periodization complexity at most } K. \qquad (5.)$$

Those properties mean respectively that:

1. For all $n \in \mathbb{N}$, every continuous function on the (geometric, Gähler or Anderson-Putnam) approximant X_n of the hull Ω_Λ of Λ can be written as a sum of products of at most K periodizations on X_n of continuous functions on $\mathbb{R}$.
2. For all $n \in \mathbb{N}$, every continuous function on X_n can be approximated by elements (h_n) which can each be written as a sum of products of at most K periodizations on X_n of continuous functions on $\mathbb{R}$.
3. For all $n \in \mathbb{N}$, every continuous function h on X_n is such that its pullback $f_{N_n-1}^* \circ \cdots \circ f_n^*(h)$ (a continuous function on X_{N_n}) can be approximated by elements (h_n) which can each be written as a sum of products of at most K periodizations on X_{N_n} of continuous functions on $\mathbb{R}$.
4. For all natural number n big enough, every continuous function h on X_n is such that its pullback $f_{N_n-1}^* \circ \cdots \circ f_n^*(h)$ (a continuous function on X_{N_n}) can be approximated by elements (h_n) which can each be written as a sum of products of at most K periodizations on X_{N_n} of continuous functions on $\mathbb{R}$.

Proof. The implication (1.) $\Rightarrow$ (2.) is clear, (2.) $\Rightarrow$ (3.) comes from the fact that



commutes. So a function on $C(X_n)$ can also be seen as a function on $C(X_{n+1})$ or more generally on $C(X_N)$ with $N \geqslant n$.
The implication (3.) ⇒ (4.) is true because periodization complexity is a property about functions on the hull, which we see as an inverse limit of approximants, hence only approximants from a certain rank onwards are meaningful.
It remains to show (4.) ⇒ (5.). Let $h \in C(\Omega_\Lambda)$ be a continuous function on the hull of Λ. Thanks to the choice of $(\mathscr{C}_n)$, $\mathcal{A} = \overline{\bigcup_{n\in\mathbb{N}} \pi_n^*(\mathcal{A}_n)}^{\|\cdot\|_\infty}$, which means that h can be approximated by pullbacks $(\pi_n^*(h_n))$ where for all $n \in \mathbb{N}$, $h_n \in \mathcal{A}_n$. But for n big enough, h_n is approximated by elements in $\mathcal{A}_{N_n}^{(K)} \subset \mathcal{A}^{(K)}$. And since the pullback of the sum of products is the sum of products of pullbacks, we conclude that $h \in \overline{\mathcal{A}^{(K)}}^{\|\cdot\|_\infty}$. In other words, Λ is of periodization complexity at most K.

□

7.2 Case of uniform lattices

Theorem 7. *$\mathbb{Z}$ is of periodization complexity 1.*

Proof. Let $\psi \in C(\Omega_\Lambda)$ a continuous function on the hull. We construct a continuous function h with compact support in $\mathbb{R}$ such that $\mathcal{P}(h) = \psi$.
We know that the hull of Λ is $\Omega_\Lambda = \{(\mathbb{Z}+x) \mid x \in \mathbb{R}\} = \{(\mathbb{Z}+x) \mid x \in [0,1]\}$. Let us denote $\omega_x = (\mathbb{Z}+x)$ for all $x \in [0,1]$ (then $\omega_0 = \omega_1$).
Moreover, $A = \{(\mathbb{Z}+x) \mid x \in [0,0.4[\cup]0.6,1]\}$ and $B = \{(\mathbb{Z}+x) \mid x \in]0.3,0.7[\}$ are two open subsets such that $\Omega_\Lambda = A \cup B$. Thanks to partitions of unity, we can write $\psi = \psi_A + \psi_B$ with $\psi_A \in C_c(A)$ and $\psi_B \in C_c(B)$.
We define $h_A \in C_c(]-0.4,0.4[)$ by:

$$\forall x \in \mathbb{R},\ h_B(x) = \begin{cases} \psi_A(\omega_x) & \text{if } x \in [0,0.4[\\ \psi_A(\omega_{x+1}) & \text{if } x \in]-0.4,0] \\ 0 & \text{else} \end{cases}$$

and $h_B \in C_c(]0.3,0.7[)$ by:

$$\forall x \in \mathbb{R},\ h_B(x) = \begin{cases} \psi_B(\omega_x) & \text{if } x \in]0.3,0.7[\\ 0 & \text{else} \end{cases}$$

Then, $h = h_A + h_B \in C_c(\mathbb{R})$ is such that $\mathcal{P}(h) = \psi$.

□

Remark 20: This statement remains true for $\alpha\mathbb{Z}$, which can be proved the same way.

Actually this is a particular case of a general statement due to A. Selberg:

Theorem 8. *Let G be an lcsc group, and Λ a uniform lattice of G.
Λ is of periodization complexity 1.*

For a proof a this theorem, see [13] (lemma 1.1 of chapter 1, explained with a slightly different vocabulary).

7.3 A useful function

We define here a function which will be useful to prove theorems about periodization complexity.
Given a function f on $\mathbb{R}$, whose support is in $[a,b]$ with $b-a<1$, we define a function $\tilde{f}$ by :

$$\forall x \in \mathbb{R},\ \tilde{f}(x) = f(x) - f(x-1) \qquad \text{(See Figure 9 and 10).}$$

The function $\tilde{f}$ is useful because its periodization is non zero on $M \in \Omega_\Lambda$ only if there exist two successive points in M not being at distance exactly 1 from each other.

Lemma 1 (Behaviour of $\tilde{f}$). *Let Λ an FLC Delone set and $\omega \in \Omega_\Lambda$ a local translate of Λ. Let f be a continuous non-negative function with compact support on $[a,b]$ with $b-a<1$. Then we have:*

$$\mathcal{P}(\tilde{f})(\omega) > 0 \text{ if and only if } \begin{cases}]a,b[\cap \omega \neq \varnothing \\ ]a+1,b+1[\cap \omega = \varnothing \end{cases}$$

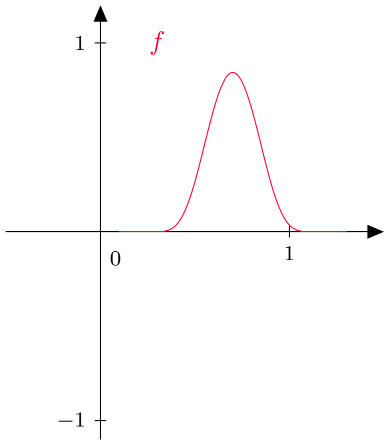


Figure 9: f

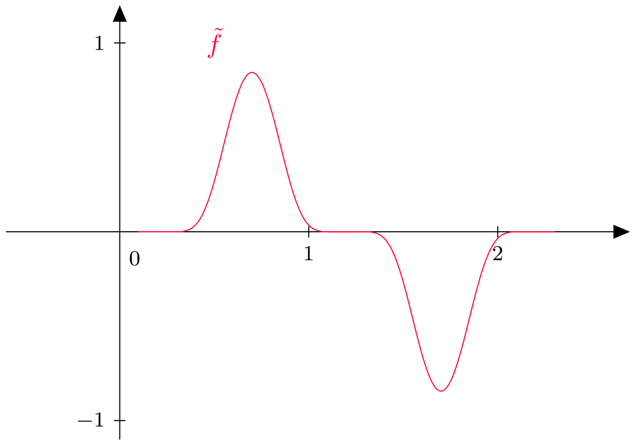


Figure 10: $\tilde{f}$

and

$$\mathcal{P}(\tilde{f})(\omega) < 0 \textit{ if and only if } \begin{cases}]a, b[\cap \omega = \varnothing \\ ]a+1, b+1[\cap \omega \neq \varnothing. \end{cases}$$

In particular, if $\mathbb{Z} \in \Omega_\Lambda$ *(for instance if* $\mathbb{Z} = \mathbb{Z}\backslash\{0\}$*), then:*

$$\forall \tau \in \mathbb{R},\ \mathcal{P}(\tilde{f}_{x_0})(\mathbb{Z} + \tau) = 0.$$

Proof. By definition of periodization, $[\mathcal{P}(h)](\omega) \neq 0$ only if there is $x \in \omega$ such that $h(x) \neq 0$. In particular, for $h = \tilde{f}$, which is non negative on $]a, b[$, negative on $]a+1, b+1[$ and zero everywhere else, we obtain the result stated above.

□

Remark 21: We can say the same about $\omega \in \mathcal{G}^n_\Lambda$ and f with $[a, b] \subset [-n, n-1]$.

7.4 Examples of periodization complexity 2

The following results about examples of periodization complexity are new results that I proved during my internship.

Theorem 9. *Let F be a finite non empty set. $\mathbb{Z}\backslash F$ is of periodization complexity at most 2.*

Before proving the theorem, let us define some notations.
Fix $F = \{y_1, .., y_m\} \subset \mathbb{Z}$ a finite and non empty subset of $\mathbb{Z}$ (with $y_1 < ... < y_m$). Let us denote

$$\{z_1, ..., z_p\} = \mathbb{Z}\backslash F \cap [\![y_1 - 1, y_m + 1]\!] = [\![y_1 - 1, y_m + 1]\!] \backslash F$$

with $z_1 < ... < z_p$ and

$$d_i = z_{i+1} - z_i.$$

Fix $n \in \mathbb{N}$ such that $n > y_m - y_1 + 1$ and let $B_n = \mathcal{B}(0, n)$ be an open ball.
The geometric approximation $\mathcal{G}^n_{\mathbb{Z}\backslash F}$ of $\mathbb{Z}\backslash F$ associated with B_n is like the one of $\mathbb{Z}\backslash\{0\}$. Indeed they are made of two parts:

$$X_n = \{\ (\mathbb{Z}\backslash F + x) \cap]-n, n[\ \mid\ -n - y_m \leqslant x \leqslant n - y_1\ \}$$

and

$$\begin{aligned} Y_n &= \{\ (\mathbb{Z}\backslash F + x) \cap]-n, n[\ \mid\ x \leqslant -n - y_m \ \text{ or } \ x \geqslant n - y_1\ \} \\ &= \{\ (\mathbb{Z} + x) \cap]-n, n[\ \mid\ x \in \mathbb{R}\ \} \\ &= \{\ (\mathbb{Z} + x) \cap]-n, n[\ \mid\ x \in [0, 1]\ \}. \end{aligned}$$

Let us denote

$$\forall x \in [-n - y_m, n - y_1],\ \alpha^{(n)}_x = (\mathbb{Z}\backslash F + x) \cap]-n, n[$$

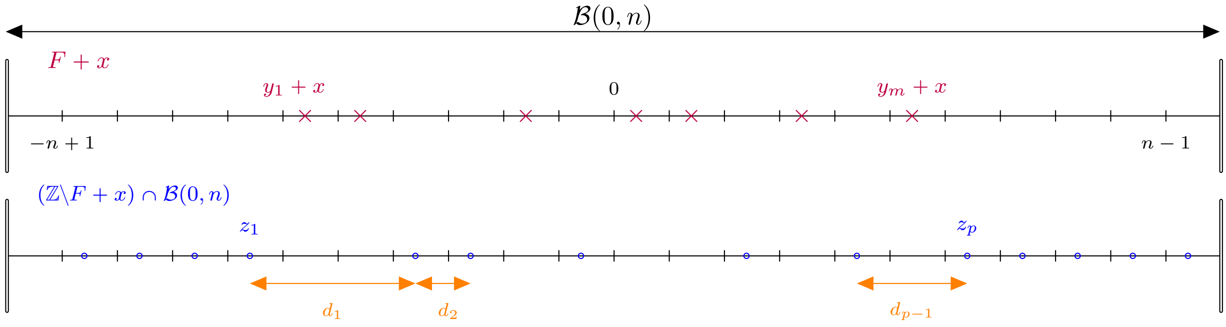


Figure 11: $F+x$ and $(\mathbb{Z}\backslash F+x)\cap\mathcal{B}(0,n)$ for a certain $x\in\mathbb{R}$

and

$$\forall x\in[0,1],\ \beta_x^{(n)}=(\mathbb{Z}+x)\cap]-n,n[$$

and

$$s^{(n)}=\mathbb{Z}\cap B_n.$$

We note that

$$\alpha_{-n-f_k}^{(n)}=\alpha_{n-f_1}^{(n)}=\beta_0^{(n)}=\beta_1^{(n)}=s^{(n)}$$

and

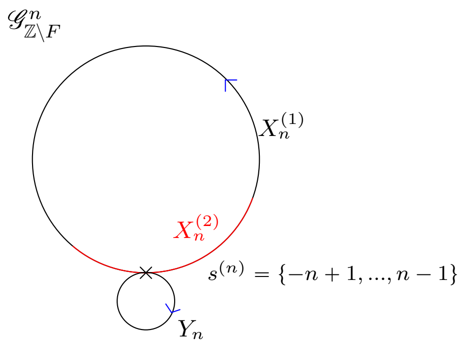
$$X_n\cap Y_n=\left\{s^{(n)}\right\}\quad\text{and}\quad \mathscr{G}^n_{\mathbb{Z}\backslash F}=X_n\cup Y_n.$$

It will also be convenient to define:

$$\begin{aligned}X_n^{(1)}&=\left\{\alpha_x^{(n)}\ \middle|\ -n<x+y_1-1\ \text{ and }\ x+y_m+1<n\right\}\\&=\left\{\alpha_x^{(n)}\ \middle|\ -n-y_1+1<x<n-y_m-1\right\}\end{aligned}$$

and

$$X_n^{(2)}=\left\{\alpha_x^{(n)}\ \middle|\ -n-y_m\leqslant x\leqslant -n-y_1+1\ \text{ or }\ n-y_m-1\leqslant x\leqslant n-y_1\right\}=X_n\backslash X_n^{(1)}.$$



The shapes of $X_n^{(1)}$ and $X_n^{(2)}$ depend on Λ, but $X_n^{(2)}$ is always a neighbourhood of $s^{(n)}$.

The points $\alpha_x^{(n)}\in X_n^{(1)}$ are such that $y_1+x-1\in\alpha_x^{(n)}$ and $y_m+x+1\in\alpha_x^{(n)}$, with words: we can see all the "gaps made by F" in any point of $X_n^{(1)}$, like in figure 8.
We define a collection of open subset of X_n:

$$\begin{aligned}\forall k\in[\![-n,n-1]\!],\ U_k&=\{\alpha_x\mid\ k<x<k+1\}\\ \forall k\in[\![-n,n-2]\!],\ V_k&=\{\alpha_x\mid\ k+\tfrac{1}{2}<x<k+\tfrac{3}{2}\}\\ V_{n-1}&=\{\alpha_x\mid\ x\in[-n,-n+\tfrac{1}{2}[\cup]n-\tfrac{1}{2},n]\}\end{aligned}$$

Then $X_n=\bigcup\limits_{k\in[\![-n,n-1]\!]}U_k\cup V_k$.

And finally we chose $N\in\mathbb{N}$ such that:

$$\forall x\in]-n-y_m,n-y_1[,\ \left\{\begin{array}{l}x+y_1-1>-N\\ x+y_m+1<N\end{array}\right.,\quad\text{namely}\quad\left\{\begin{array}{l}-N\leqslant -n-y_m+y_1-1\\ N\geqslant n-y_1+y_m+1\end{array}\right..$$

In other words, we take $N \geqslant n + y_m - y_1 + 1$.

We have all the notations we need to prove that $\mathbb{Z}\backslash F$ is of periodization complexity at most 2. We will need the following lemma:

Lemma 2. *With the fixed n and N and the introduced notations, we have:*

$$\forall \alpha_x^{(n)} \in X_n,\ \exists! \alpha_{x'}^{(N)} \text{ such that } \alpha_{x'}^{(N)} \cap B_n = \alpha_x^{(n)}.$$

Moreover, this $\alpha_{x'}^{(N)}$ is actually $\alpha_x^{(N)}$ and is in $X_N^{(1)}$.

Proof. By defintion, $\alpha_x^{(n)} = (\mathbb{Z}\backslash F + x) \cap B_n$ where $-n - y_m \leqslant x \leqslant n - y_1$, hence $-N - y_m \leqslant x \leqslant N - y_1$.
So $\alpha_x^{(N)}$ is well-defined and $\alpha_x^{(N)} = (\mathbb{Z}\backslash F + x) \cap B_N$, in particular $\alpha_x^{(N)} \cap B_n = \alpha_x^{(n)}$.
Then, $\alpha_x^{(N)}$ is in $X_N^{(1)}$ if and only if $-N < x + y_1 - 1$ and $x + y_m + 1 < N$, which exactly the conditions on which we chose N.
It remains to show that for $x' \neq x$, we have $\alpha_{x'}^{(N)} \cap B_n \neq \alpha_x^{(n)}$.
If $\alpha_x^{(n)} \in X_n^{(1)}$, then we see all gaps made by F in $\alpha_x^{(n)}$ so there is only one x such that $\alpha_x^{(n)} = \mathbb{Z}\backslash F + x$. Else, $\alpha_x^{(n)} \in X_n^{(2)}$, which means that $y_1 + x \leqslant -n < y_m + x$ or $y_1 + x < n \leqslant y_m + x$. With words: there is at lest one gap in $\alpha_x^{(n)}$, but not all the gaps that F make. But with our choice of $n > y_m - y_1 + 1$, this leads to $y_m + x < -1$ or $1 < y_1 + x$. So there are at least n consecutive points in $\alpha_x^{(n)}$ which are at distance 1 from each others. This allow us to identify from which translate of $\mathbb{Z}\backslash F$ $\alpha_x^{(n)}$ is a part.

□

Now, we prove that $\mathbb{Z}\backslash F$ is of periodization at most 2.
Proof. Let $\psi \in C(\mathcal{G}_\Lambda^n)$ a continuous function on the hull. We can assume without loss of generality that ψ is non-negative (else we write $\psi = \psi_+ + \psi_-$).
Let us show that $\psi \in \mathcal{A}_N^{(2)}$ for a certain N.
First step: Construct $h \in C_c(B_n)$ such that $\mathcal{P}_n(h) = \psi$ on Y_n. $Y_n = \{(\mathbb{Z} + x) \mid x \in [0, 1]\}$ so we can construct h as we did for the proof of $\mathbb{Z}$ being of complexity 1. But $\mathcal{P}_n(h)$ is not necessarily equal to ψ on X_n. We need to do the second step, with $\psi - \mathcal{P}_n(h)$.

Second step: Construct $h \in C_c(B_n)$ such that $\mathcal{P}(h) = \psi$ on Y_n, when $\psi = 0$ on Y_n.
Thanks to partitions of unity, we can write ψ as $\psi = \sum_{k \in [\![-n, n-1]\!]} \psi_{U_k} + \psi_{V_k}$ where $\psi_{U_k} \in C_c(U_k)$ and $\psi_{V_k} \in C_c(V_k)$ are non-negative.
Since $\psi(s^{(n)}) = 0$, we can assume that $\psi_{V_{n-1}}(s^{(n)}) = \psi_{U_{n-1}}(s^{(n)}) = 0$.
From now, we assume, without loss of generality because periodizations are linear, that $supp(\psi) \subset U_k$ or $supp(\psi) \subset V_k$. Say that $supp(\psi) \subset U_{k_0}$.

(i) We first assume that $supp(\psi) \subset X_n^{(1)}$.
We define $h, g \in C_c(\mathbb{R})$ by, for all $x \in \mathbb{R}$:

$$h(x) = \begin{cases} \sqrt{\psi(\alpha_{x+1-y_1}^{(n)})} & \text{if } -n \leqslant x \leqslant n-1 \\ 0 & \text{else} \end{cases}$$

$$g(x) = \begin{cases} \sqrt{\psi(\alpha_{x-y_m}^{(n)})} & \text{if } -n+1 \leqslant x \leqslant n \\ 0 & \text{else} \end{cases}.$$

In particular, for $x \in [-n - y_m, n - y_1]$ we have:

$$h(y_1 - 1 + x) = \sqrt{\psi(\alpha_x^{(n)})}$$
$$g(y_m + x) = \sqrt{\psi(\alpha_x^{(n)})}.$$

and h and g are actually with support of length at most 1 because ψ has its support in a U_{k_0}. More precisely, the previous equations tell us that $supp(h) \subset [y_1 - 1 + k_0, y_1 + k_0]$ and $supp(g) \subset [y_m + k_0, y_m + k_0 + 1]$. More precisely, $g(x + y_m - y_1 + 1) = h(x)$.
Let us verify that $\mathcal{P}_{\mathcal{G},n}(\tilde{h})\mathcal{P}_{\mathcal{G},n}(-\tilde{g}) = \psi$.
From lemma 1 about the "useful function", we know that $[\mathcal{P}_{\mathcal{G},n}(\tilde{h})](\beta_x^{(n)}) = 0 = [\mathcal{P}_{\mathcal{G},n}(\tilde{g})](\beta_x^{(n)})$ for every $x \in [0, 1]$. So $\mathcal{P}_{\mathcal{G},n}(\tilde{h})\mathcal{P}_{\mathcal{G},n}(-\tilde{g}) = \psi$ on Y_n.

For $\alpha_x^{(n)} \in X_n$, we have:

$$\begin{aligned}
\mathcal{P}_{\mathcal{G},n}(\tilde{h})\mathcal{P}_{\mathcal{G},n}(-\tilde{g})(\alpha_x^{(n)}) &= \left(\sum_{i \in \alpha_x^{(n)}} h(i) - h(i-1)\right)\left(\sum_{i \in \alpha_x^{(n)}} -g(i) + g(i-1)\right) \\
&= \left(\sum_{i \in (\mathbb{Z}\setminus F+x)\cap B_n} h(i) - h(i-1)\right)\left(\sum_{i \in (\mathbb{Z}\setminus F+x)\cap B_n} -g(i) + g(i-1)\right) \\
&= \left(\sum_{\substack{i=-n-\lfloor x\rfloor \\ i\notin F}}^{n-\lfloor x\rfloor} h(i+x) - h(i+x-1)\right)\left(\sum_{\substack{i=-n-\lfloor x\rfloor \\ i\notin F}}^{n-\lfloor x\rfloor} -g(i+x) + g(i+x-1)\right) \\
&= \left(\sum_{\substack{i=y_1-1+k-\lfloor x\rfloor \\ i\notin F}}^{y_1+k+1-\lfloor x\rfloor} h(i+x) - h(i+x-1)\right)\left(\sum_{\substack{i=y_m+k-\lfloor x\rfloor \\ i\notin F}}^{y_m+k+1-\lfloor x\rfloor} -g(i+x) + g(i+x-1)\right)
\end{aligned}$$

But we have:

$$\left(\sum_{\substack{i=y_1-1+k-\lfloor x\rfloor \\ i\notin F}}^{y_1+k+1-\lfloor x\rfloor} h(i+x) - h(i+x-1)\right) = \sum_{i \text{ such that } i\notin F \text{ and } i\in F} h(i+x) \quad - \sum_{i \text{ such that } i\in F \text{ and } i\notin F} h(i+x)$$

and there is at most one term in these sums which can be non zero because $supp(h)$ is of length 1. We can write a similar thing for the sum of $\mathcal{P}_{\mathcal{G},n}(-\tilde{g})$. If $x \in]k_0, k_0+1[$, namely if $\alpha_x^{(n)} \in U_{k_0}$ and:

$$\begin{aligned}
\sum_{i \text{ such that } i\notin F \text{ and } i\in F} h(i+x) \quad - \sum_{i \text{ such that } i\in F \text{ and } i\notin F} h(i+x) &= h(y_1 - 1 + x) \\
&= \sqrt{\psi(\alpha_x^{(n)})}.
\end{aligned}$$

For the sum of $\mathcal{P}_{\mathcal{G},n}(-\tilde{g})$, we obtain:

$$\begin{aligned}
\left(\sum_{\substack{i=y_m+k-\lfloor x\rfloor \\ i\notin F}}^{y_m+k+1-\lfloor x\rfloor} -g(i+x) + g(i+x-1)\right) &= g(y_m + x) \\
&= \sqrt{\psi(\alpha_x^{(n)})}.
\end{aligned}$$

So we have, for $\alpha_x^{(n)} \in U_{k_0}$, we have

$$[\mathcal{P}_{\mathcal{G},n}(\tilde{h})\mathcal{P}_{\mathcal{G},n}(-\tilde{g})](\alpha_x^{(n)}) = \psi(\alpha_x^{(n)}).$$

Conversely, if $[\mathcal{P}_{\mathcal{G},n}(\tilde{h})\mathcal{P}_{\mathcal{G},n}(-\tilde{g})](\alpha_x^{(n)}) \neq 0$, then there must be $i \in \mathbb{N}$ such that $i-1+x \in \alpha_x^{(n)}$, $i+x \notin \alpha_x^{(n)}$, $i+y_m-y_1+1+x \in \alpha_x^{(n)}$ and $i+y_m-y_1+x \notin \alpha_x^{(n)}$, which is possible only if $\alpha_x^{(n)} \in X_n^{(1)}$.
In this case, we have

$$[\mathcal{P}_{\mathcal{G},n}(\tilde{h})\mathcal{P}_{\mathcal{G},n}(-\tilde{g})](\alpha_x^{(n)}) = h(i-1+x)g(i+y_m-y_1+x)$$

which is non-zero only if $i+x \in]k_0, k_0+1[$, namely only if $\alpha_x^{(n)} \in U_{k_0}$.
So, for $\alpha_x^{(n)} \in X_n \setminus U_{k_0}$, we have:

$$[\mathcal{P}_{\mathcal{G},n}(\tilde{h})\mathcal{P}_{\mathcal{G},n}(-\tilde{g})](\alpha_x^{(n)}) = 0 = \psi(\alpha_x^{(n)}).$$

Then

$$\mathcal{P}_{\mathcal{G},n}(\tilde{h})\mathcal{P}_{\mathcal{G},n}(-\tilde{g}) = \psi$$

so $\psi \in \mathcal{A}_n^{(2)}$.

(ii) If $supp(\psi) \cap X_n^{(2)} \neq \varnothing$, we pullback ψ in $\mathscr{G}_\Lambda^N$

But because of lemma 2, the pullback $f_N^* \circ ... \circ f_n^*(\psi)$ has its support in $X_n^{(1)}$ and we can apply (i) to it.
We obtain $h, g \in C_c(]-N, N[)$ such that:

$$\mathcal{P}_N(\tilde{h})\mathcal{P}_N(-\tilde{g}) = f_N^* \circ ... \circ f_n^*(\psi).$$

So $f_N^* \circ ... \circ f_n^*(\psi) \in \mathcal{A}_N^{(2)}$

We have shown the property (5.) of proposition 17, we can conlude that $\mathbb{Z} \backslash F$ is of periodization complexity at most 2.

□

We can use the same ideas as for $\mathbb{Z} \backslash F$ to prove that:

Theorem 10. *The FLC Delone set $-2\mathbb{N} \cup \mathbb{N}$ is of periodization complexity at most 2.*

Remark 22: This statement remains true for $-k\alpha\mathbb{N} \cup \alpha\mathbb{N}$ ($k \in \mathbb{N}^*$ and $\alpha \neq 0$).

8 Conclusion

The hull is a space related to point sets. Studying the hull and its properties gives information about the point set itself, though we did not establish anything of the form "if the hull of Λ has such property X, then Λ has some property Y". What we did establish though are properties about the hull or its approximants given a fixed point set Λ. This is a first step to understand the links between point sets and their hulls.

Periodization complexity could be a relevant tool, telling how "complicated" continuous functions on the hull are, hence how "complicated" the hull is, hence how "complicated" a point set is. I just had time to begin to study this notion and lots of questions remain. I would say as a conjecture that a Fibonacci point set is of periodization complexity at most 3, which is in some sense striking, because a Fibonacci point set requires much more technics to be defined than $\mathbb{Z} \backslash \{0\}$ does but it still has a finite periodization complexity. We can ask ourselves whether, given an $n \in \mathbb{N}$, it is possible to construct point sets of periodization complexity exactly n. Does there exist a point set with no finite periodization complexity? Is it more common to have finte periodization complexity or not to ?

Finally, having a finite periodization complexity also has consequences for measures on the hull, namely in ergodic theory.

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