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Group of interval exchange transformations

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Abstract
We study the group of all interval exchange transformations. We first prove a result of Dahmani, Fujiwara and Guirardel ([DFG13]): the group generated by a generic pair of elements of IET([0;1[) is not free (assuming a suitable irreducibility condition on the underlying permutation). Then we extend this result to a more general meaning of "generic pairs".
Additionally, we discuss some possible generalisations of IET([0;1[). We give an example of a two-generated subgroup of the group of all affine interval exchange transformations that contains an isomorphic copy of every finite group.

Key words: interval exchange transformations, free group of rank 2, affine interval exchange transformations.

Contents

1 Introduction

During this internship I have worked on a question of Katok: Is there a rank 2 free subgroup in the group of interval exchange transformations ?
If so, then the group would not be amenable. If not, we cannot conclude about amenability. The von Neumann's conjecture (a group is amenable if and only if it does not contain a rank 2 free subgroup) was indeed disproved in 1980 by Alexender Ol'shanskii. Nicolas Monod gave a simple counterexample in [Mon13]. On the other hand we still do not know whether the first potential counterexample, the Thompsoon group F ([CF11]), is amenable or not. Both interval exchange transformations and elements of the Thompson group F are piecewise affine maps and generalise circle diffeomorphisms. Interval exchange transformations (IETs) are continuous but at finitely many points and are local isometries whereas elements of the Thompson group F are piecewise linear homeomorphisms.
These groups have been studed first by Keane and then more widely since 1980. They have given first examples of minimal and non uniquely ergodic maps. They arise in many areas of mathematics such as dynamical systems, polygonal billiards, geometry and flows over flat surfaces.

I will first introduce some notations and basic results about IETs and then present the proof of ([DFG13]) of the fact that "lots" of couples of IETs do not generate a rank 2 free group in section 3.

Theorem 1 (Dahmani-Fujiwara-Guirardel). *There exists a dense open set*

$$\Omega \subset IET([0;1[) \times IET_{irred}([0;1[)$$

such that for every $(S,T) \in \Omega$, $\langle S,T \rangle$ is not free of rank 2.

I will then extend the result to wider sets of IETs in section 4.

Theorem 2. *Let S be any IET on $[0;1[$. There exists a dense open set $\Omega_{irred}(S) \subset IET_{irred}([0;1[)$ such that for every $T \in \Omega_{irred}(S)$, $\langle S,T \rangle$ is not free of rank 2.*

Finally, we discuss some generalisations of the group of IETs. The generalisation into *orientable flipped interval exchange transformations* turns out to be an interval exchange transformations group (all its elements could be seen as IETs). However the generalisation into affine interval exchange transformations (AIETs) leads to other behaviours such as the existence wandering intervals ([BHM10]). Like the group of IETs (Theorem 8.1 of [DFG13]), the group of AIETs has a two-generated subgroup that contains an isomorphic copy of every finite group.

Theorem 5. *There is a subgroup $F < AIET([0;1[)$ generated by two elements that contains an isomorphic copy of all finite groups and a free semigroup.*

2 Interval exchange transformations

2.1 Definition

Definition 1 (IET). *An* interval exchange transformation *(IET) on $[0;1[$ is a bijection from $[0;1[$ onto itself which is everywhere continuous on the right, continuous except on a finite number of points and differentiable where it is continuous with differential equal to 1.*
One denotes by $\Delta(T)$ the set of discontinuities of T.

One defines analogously an IET on another interval, a circle or a union of intervals and circles.
Roughly speaking, an IET on $[0;1[$ cuts the interval $[0;1[$ into finitely many intervals and shuffles them. Let us introduce some examples.
Example 1. Let $\theta \in]0;1[$ and define R by:

$$\forall x \in [0;1[,\ R(x) = \begin{cases} x+1-\theta & \text{if } 0 \leqslant x < \theta \\ x-\theta & \text{if } \theta \leqslant x < 1 \end{cases}.$$

Then R is an IET. One can think of it as the rotation of angle $1-\theta$ on the circle $\mathbb{R}/\mathbb{Z}$. See Figure 1.

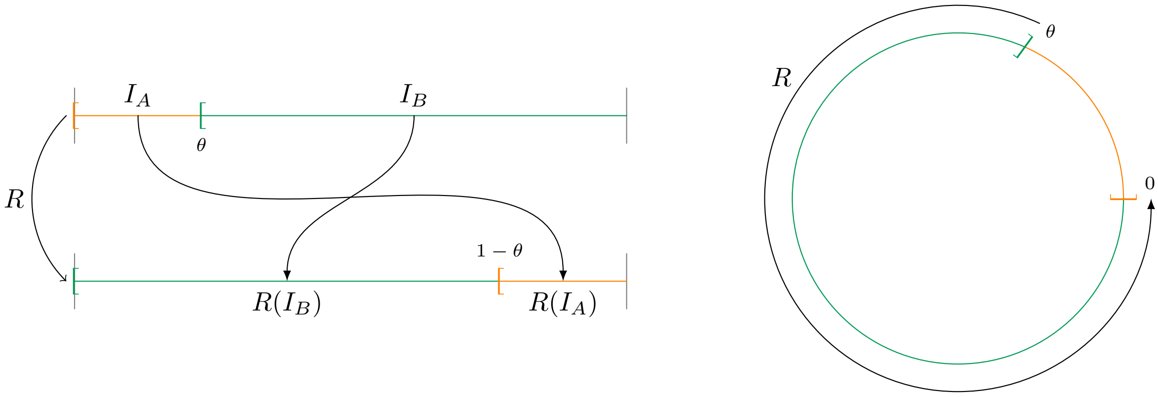


Figure 1: Example of an IET which is a rotation

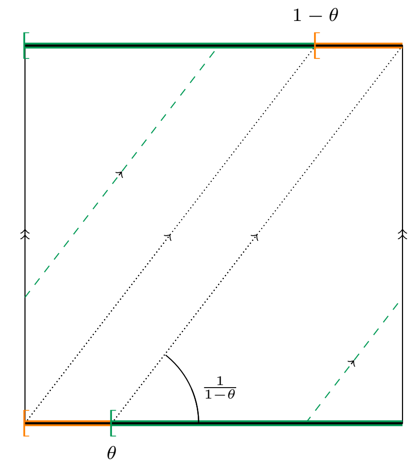


Figure 2: The IET R seen as a first recurrence map on the torus

We can also see it as the first recurrence map of the flow of angle $\frac{1}{1-\theta}$ on the torus. See Figure 2.

Example 2. Let T defined by:

$$\forall x \in [0;1[,\ T(x) = \begin{cases} x + \frac{4}{5} & \text{if } x \in A = \left[0; \frac{2}{10}\right[\\ x + \frac{1}{2} & \text{if } x \in B = \left[\frac{2}{10}; \frac{3}{10}\right[\\ x + \frac{1}{10} & \text{if } x \in C = \left[\frac{3}{10}; \frac{6}{10}\right[\\ x - \frac{3}{5} & \text{if } x \in D = \left[\frac{6}{10}; 1\right[\end{cases}.$$

One easily checks that T is an IET, and A, B, C and D are its intervals of continuity. Figure 3 illustrates T and the composition of T with the rotation R.

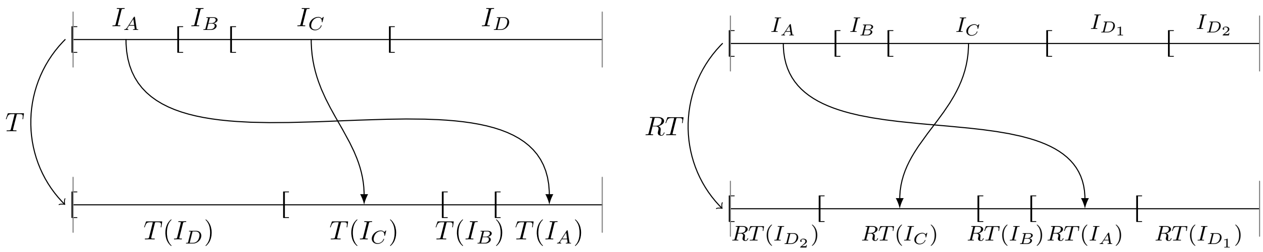


Figure 3: Composition of IETs

Proposition 1. *The set of IETs on a domain $\mathcal{D}$, endowed with the law of composition, is a group.*

Proof.

- ⋆ The identity on $\mathcal{D}$ is an IET.
- ⋆ If T is an IET, then T^{-1} is well defined, everywhere continuous on the right, continuous except on a finite number of points and differentiable where it is continuous with differential equal to 1.
- ⋆ The composition of two interval exchange transformations S and T is a bijection, which is everywhere continuous on the right. Moreover, if T is continuous at $x \in \mathcal{D}$ (and so differentiable at x) and if S is continuous at $T(x)$ (and so differentiable at $T(x)$), then $S \circ T$ is continuous and differentiable at x, with differential equal to 1. If $S \circ T$ is continuous at $x \in \Delta(T) \cup T^{-1}(\Delta(S))$, then it is continuous on a neighbourhood of x (by discreteness of $\Delta(T) \cup T^{-1}(\Delta(S))$) and differentiable on this neighbourhood, with differential 1, so is $S \circ T$ at x. Hence $S \circ T$ is an IET.

□

2.2 Description of an IET

We use the formalism introduced by Marmi, Moussa and Yoccoz ([MMY05]) to describe IETs. Let T be an IET on $[0;1[$. Let $\mathcal{A}$ be a finite alphabet (of cardinal n) and $[0;1[= \bigcup_{a \in \mathcal{A}} I_a$ be a partition of $[0;1[$ such that T is continuous on each interval I_a. For every $a \in \mathcal{A}$, let $\lambda_a = |I_a|$ be the length of I_a. Let $\pi_0 : \mathcal{A} \to \{1, ..., n\}$ be the one-to-one map sending a to i if and only if a is the letter of the i-th interval (from the left to the right) of $[0;1[$ with regard to the partition $[0;1[= \bigcup_{a \in \mathcal{A}} I_a$. Let $\pi_1 : \mathcal{A} \to \{1, ..., n\}$ be the one-to-one map sending a to i if and only if a is the letter of the i-th interval (from the left to the right) of $[0;1[$ with regard to the partition $[0;1[= \bigcup_{a \in \mathcal{A}} T(I_a)$. Then $\pi = \pi_1 \circ \pi_0^{-1} \in \mathfrak{S}_n$ describes the action of T on the set $(I_a)_{a \in \mathcal{A}}$.

We represent π by the table $\begin{pmatrix} \pi_0^{-1}(1) & \pi_0^{-1}(2) & ... & \pi_0^{-1}(n) \\ \pi_1^{-1}(1) & \pi_1^{-1}(2) & ... & \pi_1^{-1}(n) \end{pmatrix}$.

Remark 1: Given a permutation π, there are many pairs (π_0, π_1) such that $\pi = \pi_1 \circ \pi_0^{-1}$, each one corresponding to a different labelling of the intervals. In most examples in this report, we take the latin alphabet for $\mathcal{A}$ and π_0 such that $\pi_0(A) = 1$, $\pi_0(B) = 2$, $\pi_0(C) = 3$ ...

Remark 2: If there exists $i \in \{1, ..., n\}$ such that $\pi(i+1) = \pi(i) + 1$, then T is continuous on the union $I_{\pi_0^{-1}(i)} \cup I_{\pi_0^{-1}(i+1)}$ of the i-th and the $(i+1)$-th intervals. The converse also holds. In other words, the intervals $(I_a)_{a \in \mathcal{A}}$ are exactly the intervals of continuity of T if and only if $\forall i \in [1; n[, \ \pi(i+1) \neq \pi(i) + 1$. In this case we call π the *underlying permutation* of T.

Proposition 2. *An IET T is defined by the set $\lambda(T)$ of the lengths of its intervals of continuity, and the underlying permutation $\pi(T)$.*

Example 3. For the IET T of the previous example, we have $\lambda(T) = \left(\lambda_A = \frac{2}{10}, \lambda_B = \frac{1}{10}, \lambda_C = \frac{3}{10}, \lambda_D = \frac{4}{10}\right)$ and the underlying permutation is $\pi(T) = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$.

Remark 3: Given $\lambda(T) = (\lambda_a(T))_{a \in \mathcal{A}}$ and $\pi(T) = \pi_1 \circ \pi_0^{-1}$, one easily computes $\lambda(T^{-1}) = \lambda(T)$ and $\pi(T^{-1}) = \pi_0 \circ \pi_1^{-1}$. In our example, it gives $\pi(T^{-1}) = \begin{pmatrix} D & C & B & A \\ A & B & C & D \end{pmatrix}$. See Figure 4.

If S is another IET, the data $\lambda(S \circ T)$ and $\pi(S \circ T)$ are less straightforward to compute. See Figure 5.

There are other ways to describe T: one can replace the data of the lengths of the intervals of continuity by the data of the points of discontinuity, or replace the data of the underlying permutation by the data of the translation lengths. Let us introduce some maps to navigate from one of these points of view to the other.
First we define four maps from the set of IETs that give respectively the lengths of the intervals of continuity, the points of discontinuity, the underlying permutation and the translation lengths. Here, the labelling of the intervals with letters does not appear. By convention l_i (resp. t_i) denotes the length of (resp. translation length on) the i-th interval of continuity, and b_i the i-th point of discontinuity.

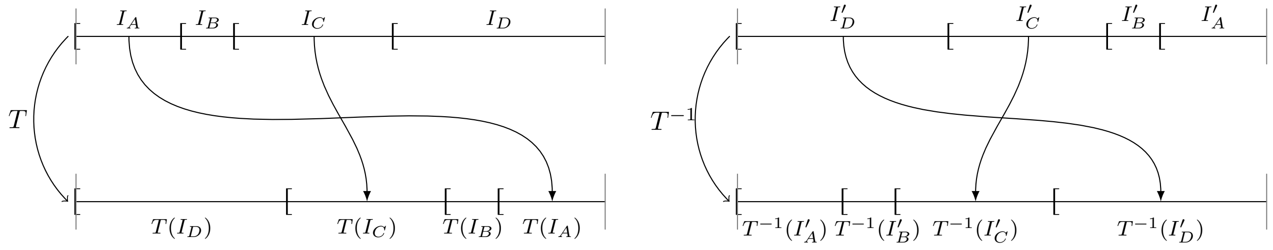


Figure 4: Example of an IET and its inverse

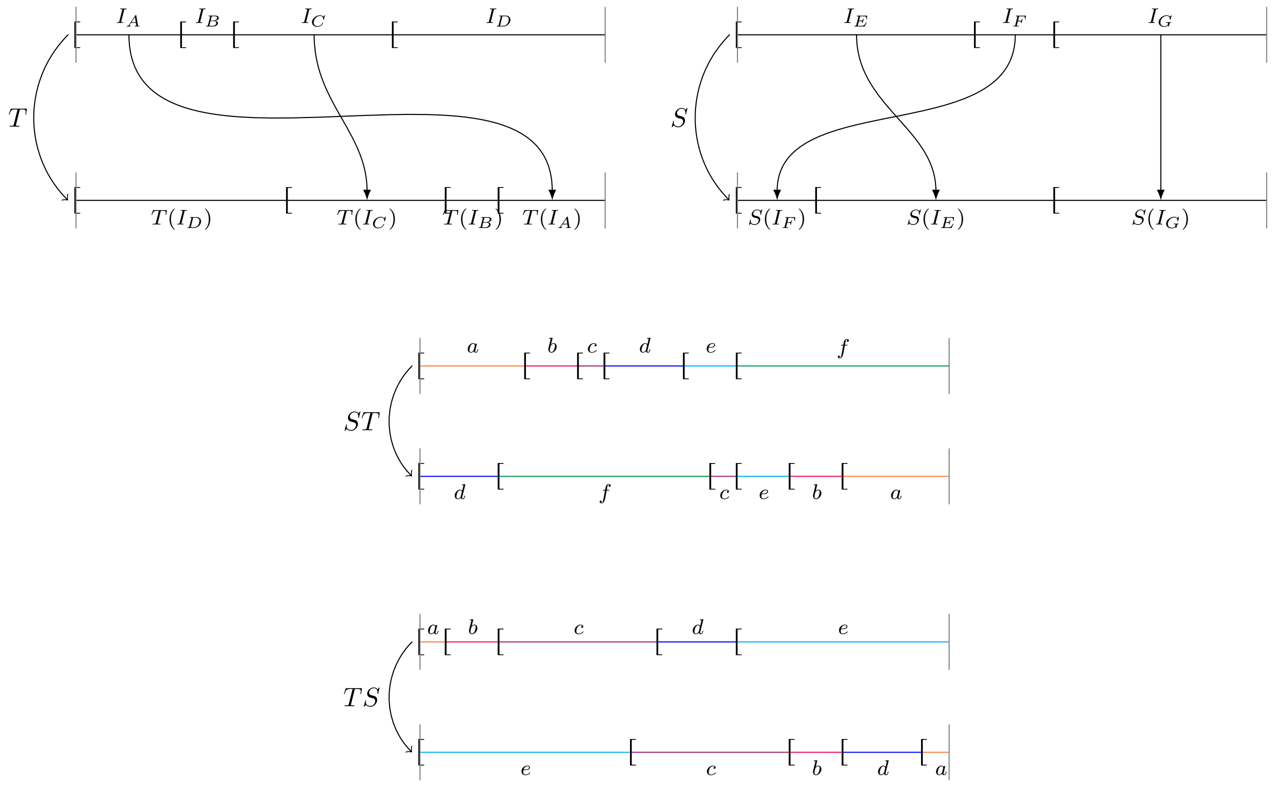


Figure 5: Composition of two IETs

$$\begin{array}{rccl} \lambda: & IET([0;1[) & \longrightarrow & \{(a_1,...,a_n) \mid n \in \mathbb{N},\, \forall i \in \{1,...,n\},\, a_i \in \mathbb{R}_+^*,\, \sum_{i=1}^n a_i = 1\} \\ & T & \mapsto & \lambda(T) = (l_1,...,l_n) \end{array}$$

$$\begin{array}{rccl} \beta: & IET([0;1[) & \longrightarrow & \{(a_1,...,a_{n-1}) \mid n \in \mathbb{N},\, 0 < a_1 < ... < a_{n-1} < 1\} \\ & T & \mapsto & \beta(T) = (b_1,...,b_{n-1}) \end{array}$$

$$\begin{array}{rccl} \pi: & IET([0;1[) & \longrightarrow & \mathfrak{S}_n \\ & T & \mapsto & \pi(T) \end{array}$$

$$\begin{array}{rccl} \tau: & IET([0;1[) & \longrightarrow & \{(a_1,...,a_n) \mid n \in \mathbb{N},\, \forall i, a_i \in]-1;1[\} \\ & T & \mapsto & \tau(T) = (t_1,...,t_n) \end{array}$$

There is an obvious relation between $(l_1,...,l_n) = \lambda(T)$ and $(b_1,...,b_{n-1}) = \beta(T)$:

$$\forall i \in \{1,...,n\},\, l_i = b_i - b_{i-1}$$

with the natural convention $b_0 = 0$ and $b_n = 1$. Equivalently:

$$\forall i \in \{1,...,n-1\},\, b_i = \sum_{j=1}^{i} l_j.$$

The relation between $\tau(T)$ and $(\pi(T), \lambda(T))$ is almost as straightforward as the relation between $\lambda(T)$ and $\beta(T)$. We denote by Φ the corresponding map:

$$\Phi(\pi, (l_1, ..., l_n)) = (t_1, ..., t_n)$$

where

$$\forall i \in \{1, ..., n\},\ t_i = -\sum_{j=1}^{i-1} l_j + \sum_{j=i}^{\pi(i)-1} l_{\pi^{-1}(j)}.$$

Given a permutation $\sigma \in \mathfrak{S}_n$, we will denote by Φ_σ the map

$$\Phi_\sigma : \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ (l_1, ..., l_n) & \mapsto & \Phi(\sigma, (l_1, ..., l_n)) = (t_1, ..., t_n) \end{array}.$$

In this report, we will use $(l_1, l_2, ..., l_n)$ for the set of lengths of intervals of continuity of an IET (l_1 denotes the length of the leftmost interval in the partition of $[0; 1[$ and l_n the length of the rightmost interval), $(b_1, ..., b_{n-1})$ for the set of discontinuities (always supposed in the increasing order, with $b_0 = 0$ and $b_n = 1$ when needed), and $(t_1, ..., t_n)$ for the translation lengths. So if T is an IET with $\beta(T) = (b_1, ..., b_n)$ and $\tau(T) = (t_1, ..., t_n)$ then T is such that:

$$\forall i \in \{1, ..., n\},\ \forall x \in [b_{i-1}; b_i[,\ T(x) = x + t_i.$$

Definition 2 (irreducible permutation). *A permutation $\sigma \in \mathfrak{S}_n$ is said to be* irreducible *if for all $k < n$, the set $\{1, ..., k\}$ is not σ-invariant.*

If the underlying permutation π of an IET T is reducible, say $\{1, ..., k\}$ is π-invariant, then T preserves $I_1 \cup ... \cup I_k$ (hence $I_{k+1} \cup ... \cup I_n$ too). We can study the dynamics of both restrictions instead of studying the dynamics of T.
But if we want to study the group spanned by two interval exchange transformations, assuming they have irreducible underlying permutation *is* a loss of generality. See example in Figure 5 of S and T where S has a reducible underlying permutation.

2.3 Topology

One equips IET($[0; 1[$) with the following distance:

$$d: \begin{array}{ccc} \text{IET}([0;1[) \times \text{IET}([0;1[) & \longrightarrow & \mathbb{R}_+ \\ (S,T) & \longmapsto & \begin{cases} ||\lambda(S) - \lambda(T)||_1 & \text{if } \pi(S) = \pi(T) \\ \infty & \text{otherwise} \end{cases} \end{array}.$$

Definition 3. *Let $\pi \in \mathfrak{S}_n$. We denote by IET_π the set of interval exchange transformations with underlying permutation π.*

Note that IET_π is not a group. We can picture it as a simplex of dimension $n-1$, because once the underlying permutation is fixed, the IET is determined by the lengths of its intervals of continuity (or by its points of discontinuity). Since we are working with normalized IETs (defined over $[0; 1[$) the lengths of the first $n-1$ intervals of continuity are sufficient to determine an IET with n intervals of continuity.

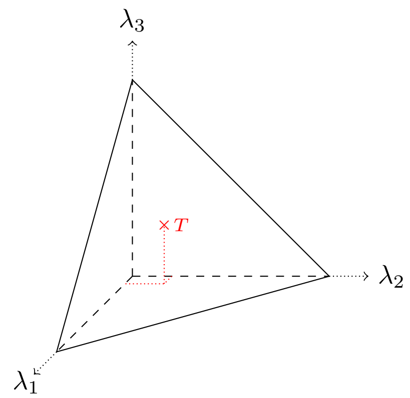


Figure 6: The simplex representing IET_π and $T = (\pi, \lambda(T))$ from examples 2 and 3

Proposition 3. *If T and T' are at distance δ from each other, then for all i:*

1. $|l_i - l'_i| \leqslant \frac{\delta}{2}$;
2. $|b_i - b'_i| \leqslant \frac{\delta}{2}$;
3. $|t_i - t'_i| \leqslant 2\delta$.

Proof. If T and T' are at distance $\delta < \infty$, then they have the same underlying permutation, call it $\pi \in \mathfrak{S}_n$.

1. We prove this by contrapositive. First note that $|\sum_{j\neq i} l'_j - \sum_{j\neq i} l_j| = |l_i - l'_i|$ (because $\sum_j l'_j = \sum_j l_j = 1$). If $|l_i - l'_i| > \frac{\delta}{2}$, then

$$\begin{aligned} d(T,T') = \|\, \lambda(T) - \lambda(T')\,\|_1 &\geqslant |l_i - l'_i| + \left| \sum_{j\neq i} l_j - \sum_{j\neq i} l'_j \right| \\ &\geqslant |l_i - l'_i| + |l_i - l'_i| \\ &> \delta. \end{aligned}$$

2. By contraposition, if $|b_i - b'_i| > \frac{\delta}{2}$, then $i \neq n$ (because $b_n = b'_n = 1$) and

$$\begin{aligned} d(T,T') = \|\, \lambda(T) - \lambda(T')\,\|_1 &\geqslant \left| \sum_{j=1}^{i} l_j - \sum_{j=1}^{i} l'_j \right| + \left| \sum_{j=i+1}^{n} l_j - \sum_{j=i+1}^{n} l'_j \right| \\ &\geqslant |b_i - b'_i| + |-b_i + b'_i| \\ &> \delta. \end{aligned}$$

3. One has

$$\begin{aligned} |t_i - t'_i| &= \left| -\sum_{j=1}^{i-1} l_j + \sum_{j=i}^{\pi(i)-1} l_{\pi^{-1}(j)} + \sum_{j=1}^{i-1} l'_j - \sum_{j=i}^{\pi(i)-1} l'_{\pi^{-1}(j)} \right| \\ &\leqslant \left| \sum_{j=1}^{i-1} -l_j + l'_j \right| + \left| \sum_{j=i}^{\pi(i)-1} l_{\pi^{-1}(j)} - l'_{\pi^{-1}(j)} \right| \\ &\leqslant \|\, l - l'\,\|_1 + \|\, l - l'\,\|_1 = 2d(T,T') \\ &\leqslant 2\delta. \end{aligned}$$

□

3 Katok's question

Katok asked whether there is a free subgroup of rank 2 in IET([0;1[). If we had some relation(s) between any two IETs S and T, then we could answer Katok's question (by no). We do not know so many relations but we can still build a relation between two IETs taken in a dense subset. We produce such relations following the ideas of [DFG13]:

1. (Basic relation). Pick two obvious relations between S_0 and T_0, both of finite order, with discontinuities on some rationals with the same denominator q. Say $r_1(S_0, T_0) = id$ and $r_2(S_0, T_0) = id$.
2. (Small pertubation). Pick S and T close to S_0 and T_0. Then $r_1(S,T)$ and $r_2(S,T)$ induce translations with small translation length on every interval of continuity which are not too close to $\frac{1}{q}\mathbb{N}$.
3. (An IET with small support). The commutator $U = [r_1(S,T), r_2(S,T)]$ has a small support, located in a neighbourhood of $\frac{1}{q}\mathbb{N}$.
4. (Drifting the support). With an additional condition on T_0, some power k of T can drift the support of U such that $T^k(supp(U)) \cap supp(U) = \varnothing$.
5. (Relation). Then U commutes with T^kUT^{-k}, which gives the relation $[U, T^kUT^{-k}] = id$.

In the following we explain each step with more details. We first introduce some notations.

3.1 q-rationality and obvious relations

Definition 4 (q-rationality). *Let $q \in \mathbb{N}$. An IET is called q-*rational *if all its discontinuity points are in $\frac{1}{q}\mathbb{N}$.*

Proposition 4. *If T is q-rational, then its interval lengths $(l_i)_{1\leqslant i\leqslant n} = \lambda(T)$, and translation lengths $(t_i)_{1\leqslant i\leqslant n} = \tau(T)$ are all in $\frac{1}{q}\mathbb{N}$.*

Proof. It comes directly from the relations between discontinuity points, interval lengths and translation lengths:

$$\forall i, l_i = b_i - b_{i-1}$$
$$\forall i, t_i = -\sum_{j=1}^{i-1} l_j + \sum_{j=1}^{\sigma(i)-1} l_{\sigma^{-1}(j)}$$

where σ is the underlying permutation of T.

□

Proposition 5. *If T is q-rational, then it is of finite order dividing $q!$.*

Proof. Let $\sigma \in \mathfrak{S}_q$ be such that for all $k < q$, $I_k = [\frac{k}{q}; \frac{k+1}{q}[$ is sent to $I_{\sigma(k)}$ by T.
We underline that σ is not necessarily the underlying permutation of T. It is the case if and only if T has q intervals of continuity, namely if the set $\Delta(T)$ of points at which T is discontinuous is equal to (and not only included in) $\frac{1}{q}\mathbb{N} \cap [0; 1[$.
For all $n \in \mathbb{N}$, T^n sends I_k to $I_{\sigma^n(k)}$. Hence T has same order than σ: a finite order dividing $\#\mathfrak{S}_q = q!$. □

In particular one has $T^{q!} = id$ whenever T is q-rational.

3.2 Steps 2 and 3: Building an IET with a small support

Definition 5 ($\epsilon - neighbourhood$). *Let $\epsilon > 0$ and $X \subset \mathbb{R}$. We denote by*

$$\mathcal{N}_\epsilon(X) = \{y \in [0; 1[\mid \exists x \in X, |x - y| < \epsilon\}$$

*the set of points at distance stricly less than ϵ from X, and we call it the ϵ-*neighbourhood *of X.*

Lemma 1. *For all $\epsilon > 0$, $m, q \in \mathbb{N}$, there exists $\eta > 0$ such that if S and T are η-close to q-rational IETs S_0 and T_0 respectively and if w is a word of length at most m over the letters $s^{\pm 1}, t^{\pm 1}$ such that $w(S_0, T_0) = id$ then $w(S, T)$ acts on each connected component of $[0; 1[\setminus \mathcal{N}_\epsilon(\frac{1}{q}\mathbb{N})$ with translation lengths strictly smaller than ϵ.*

Proof. Let $\epsilon > 0$, $m, q \in \mathbb{N}$.
Let $\eta > 0$ to be fixed later on. Let S, T η-close to S_0, T_0 two q-rational IETs.
We show by induction that for all $i \leqslant m$, for all word w_i of length i, for all $p < q$, the image of $]\frac{p}{q} + 2m\eta; \frac{p+1}{q} - 2m\eta[$ by $w_i(S, T)$ is included in $]\frac{p'}{q} + 2(m-i)\eta; \frac{p'+1}{q} - 2(m-i)\eta[$, where $p' = w_i(S_0, T_0)(p)$.
Base case: The property is true for $i = 0$, since the identity sends $]\frac{p}{q} + 2m\eta; \frac{p+1}{q} - 2m\eta[$ to itself.
Inductive step: Let $i \leqslant m$ and assume that the property is true for $i - 1$. Let us show it for i.
Let $p < q$ and w_i a word of length i, and w_{i-1} its suffix of length $i - 1$.
Then either $w_i(S, T) = Sw_{i-1}(S, T)$ or $w_i(S, T) = Tw_{i-1}(S, T)$.
By hypothesis $w_{i-1}(S, T)$ sends $]\frac{p}{q} + 2m\eta; \frac{p+1}{q} - 2m\eta[$ to $I =]\frac{p'}{q} + 2(m-i+1)\eta; \frac{p'+1}{q} - 2(m-i+1)\eta[$, where $p' = w_{i-1}(S_0, T_0)(p)$.
The interval I is included in $]\frac{p'}{q} + \frac{\eta}{2}; \frac{p'+1}{q} - \frac{\eta}{2}[$, on which S (resp. T) is continuous (Proposition 3). Moreover S (resp. T) has translation lengths differing by at most 2η (Proposition 3) from the ones of S_0 (resp. T_0). So the image of I by S (resp. T) is included in some $]\frac{p''}{q} + 2(m-i)\eta; \frac{p''+1}{q} - 2(m-i)\eta[$, where $p'' = S(p') = w_i(S, T)(p)$ (resp. $p'' = T(p') = w_i(S, T)(p)$). Hence the property is true for i.

We conclude that for every word w of length m such that $w(S_0, T_0) = id$, $w(S,T)$ sends every interval of type $]\frac{p}{q} + 2m\eta; \frac{p+1}{q} - 2m\eta[$ within $]\frac{p}{q}; \frac{p+1}{q}[$. In other words $w(S,T)$ acts on $[0;1[\backslash \mathcal{N}_{2m\eta}(\frac{1}{q}\mathbb{N})$ with translation lengths smaller than $2m\eta$.
Choosing $\eta < \frac{\epsilon}{2m}$ gives the lemma. □

Lemma 2. *For all $\epsilon > 0$, $q \in \mathbb{N}$, there exists $\eta > 0$ such that if S and T are η-close to q-rational IETs S_0 and T_0 respectively, then $[S^{q!}, TS^{q!}T^{-1}]$ induces the identity on each interval of $[0;1[\backslash \mathcal{N}_\epsilon(\frac{1}{q}\mathbb{N})$.*

Proof. Apply the previous lemma for $\frac{\epsilon}{2}$ to the words $s^{q!}$ and $ts^{q!}t^{-1}$. Whenever S and T are closer than $\frac{\epsilon}{4(q!+2)}$ to S_0 and T_0, the IETs $S^{q!}$ and $TS^{q!}T^{-1}$ induce translations of translation lengths strictly smaller than $\frac{\epsilon}{2}$ on each connected component of $[0;1[\backslash \mathcal{N}_{\frac{\epsilon}{2}}(\frac{1}{q}\mathbb{N})$.
Let $x \in [0;1[\backslash \mathcal{N}_\epsilon(\frac{1}{q}\mathbb{N})$.
Denote by I the connected component of x in $[0;1[\backslash \mathcal{N}_{\frac{\epsilon}{2}}(\frac{1}{q}\mathbb{N})$ and by $t_I, t'_I \in [-\frac{\epsilon}{2}; \frac{\epsilon}{2}]$ the translation lengths of I by $S^{q!}$ and $TS^{q!}T^{-1}$ respectively.
On the one hand:

$$S^{q!} \circ TS^{q!}T^{-1}(x) = S^{q!}(x + t'_I) = x + t'_I + t_I$$

because $x + t'_I \in I$ since $d(x + t'_I, \frac{1}{q}\mathbb{N}) \geqslant d(x, \frac{1}{q}\mathbb{N}) - |t'_I| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$.
On the other hand:

$$TS^{q!}T^{-1} \circ S^{q!}(x) = TS^{q!}T^{-1}(x + t_I) = x + t_I + t'_I$$

because $x + t_I \in I$ since $d(x + t_I, \frac{1}{q}\mathbb{N}) \geqslant d(x, \frac{1}{q}\mathbb{N}) - |t_I| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$.
Hence $S^{q!}$ and $TS^{q!}T^{-1}$ commute on $[0;1[\backslash \mathcal{N}_\epsilon(\frac{1}{q}\mathbb{N})$. We conclude that $[S^{q!}, TS^{q!}T^{-1}]$ induces the identity on each interval of $[0;1[\backslash \mathcal{N}_\epsilon(\frac{1}{q}\mathbb{N})$. □

In other words the IET $U = [S^{q!}, TS^{q!}T^{-1}]$ has a small support included in $\mathcal{N}_\epsilon(\frac{1}{q}\mathbb{N})$. We have reached the goal of the first three steps. In the following section we conjugate U by a well chosen IET to get an IET that commutes with U.

3.3 Step 4: Drifting the support

Recall that Φ_σ is the map that sends lengths of the intervals of continuity to the corresponding translation lengths (see definition page 7).

Definition 6 (Driftable permutation). *A permutation $\sigma \in \mathfrak{S}_n$ is said to be* driftable *if there exists a vector $dl = (dl_1, ..., dl_n) \in \mathbb{R}^n$ with $\sum dl_i = 0$ such that the vector $dr = \Phi_\sigma(dl) \in \mathbb{R}^n$ has only positive coordinates.*
We call dl a drifting direction *and dr a* drifting vector.
We denote by $dr_{min} = \underset{1 \leqslant i \leqslant n}{min}\, dr_i > 0$ and $dr_{max} = \underset{1 \leqslant i \leqslant n}{max}\, dr_i$ the maximal and minimal drift of dr.

Proposition 6. *A permutation is driftable if and only if it is irreducible.*

See Proposition 5.12 of [DFG13] for a proof of this fact.

Proposition 7. *The set of drifting directions or drifting vectors are cones.*

Proof. Let $\sigma \in \mathfrak{S}_n$ be a driftable permutation. For every $i \in \{1, ..., n\}$, denote by H_i^+ the vectorial half-space

$$H_i^+ = \mathbb{R}^{i-1} \times \mathbb{R}_+^* \times \mathbb{R}^{n-i}$$

and by

$$L_i^+ = \Phi_\sigma^{-1}(H_i^+)$$

its preimages by Φ_σ, which is also a vectorial half-space because Φ_σ is linear.
The set of drifting vectors (resp. directions) is the intersection of all the vectorial half-spaces H_i^+ (resp. L_i^+), hence is a cone. □

Definition 7. *Let T_0 be an IET and $\sigma = \pi(T) \in \mathfrak{S}_n$ be its underlying permutation. Let $dl \in \mathbb{R}^n$. We define $T_\theta \in IET_\sigma$ by $\lambda(T_\theta) = \lambda(T_0) + \theta dl$, where θ is small enough to ensure that all the lengths of the intervals of continuity $l_i(T_\theta)$ are positive.*

Lemma 3. *Let $\sigma \in \mathfrak{S}_n$ be a driftable permutation, and dl and dr be associated the drifting direction and drifting vector. Let $q \in \mathbb{N}$ and $T_0 \in IET_\sigma([0;1[)$ be q-rational. Let θ be such that $0 < \theta dr_{max} < \frac{1}{q}$ and small enough so that T_θ is well defined.*
Then all the translation lengths of T_θ are in $[\theta dr_{min}; \theta dr_{max}]$ mod $\frac{1}{q}$.
And if T is μ-close to T_θ, where $2\mu < \theta dr_{min}$ and $2\mu < \frac{1}{q} - \theta dr_{max}$, then all the translation lengths of T are in $[\theta dr_{min} - 2\mu; \theta dr_{max} + 2\mu]$ mod $\frac{1}{q}$.

Proof. One computes the translation lengths of T_θ using the linear map Φ_σ:

$$\tau(T_\theta) = \Phi_\sigma(\lambda(T_\theta)) = \Phi_\sigma(\lambda(T_0) + \theta dl) = \Phi_\sigma(\lambda(T_0)) + \theta\Phi_\sigma(dl) = \tau(T_0) + \theta dr$$

which reduces to:

$$\tau(T_\theta) \equiv \theta dr \mod \frac{1}{q}$$

because T_0 is q-rational. Since $0 < \theta dr_{\min} \leqslant \theta dr_{\max} < \frac{1}{q}$, one can conclude that all the coordinates of $\tau(T_\theta)$ (i.e. all the translation lengths of T_θ) are in $[\theta dr_{\min}; \theta dr_{\max}]$ mod $\frac{1}{q}$.
To prove the second point, one writes:

$$\lambda(T) = \lambda(T_0) + \theta dl + \epsilon$$

where $\epsilon = (\epsilon_i)_{1 \leqslant i \leqslant n} \in \mathbb{R}^n$ is such that $||\epsilon||_1 < \mu$.
Applying Φ_σ and reducing modulo $\frac{1}{q}$ one gets:

$$\tau(T) \equiv \theta dr + \Phi_\sigma(\epsilon) \mod \frac{1}{q}$$

where

$$\Phi_\sigma(\epsilon) = \left(-\sum_{j=1}^{i} \epsilon_j + \sum_{j=1}^{\sigma(i)-1} \epsilon_{\sigma^{-1}(j)} \right)_{1 \leqslant i \leqslant n}$$

so every coordinate of $\Phi_\sigma(\epsilon)$ is smaller than $2||\epsilon||_1$ hence than 2μ. Since $0 < \theta dr_{\min} - 2\mu \leqslant \theta dr_{\max} + 2\mu < \frac{1}{q}$, one can conclude that all the coordinates of $\tau(T)$ (i.e. all the translation lengths of T) are in the interval $[\theta dr_{\min} - 2\mu; \theta dr_{\max} + 2\mu]$ mod $\frac{1}{q}$. □

Lemma 4. *Within the same setting as in the previous lemma, let $\rho = \frac{dr_{max}}{dr_{min}}$, $\epsilon < \frac{1}{11q\rho}$, $\theta < \frac{\epsilon}{dr_{min}}$ and $\mu < \frac{\theta dr_{min}}{4}$. Let T be μ-close to T_θ.*
Then there exists $k \in \mathbb{N}$ such that all the translation lengths of T^k are in $[2\epsilon; \frac{1}{q} - 2\epsilon]$ mod $\frac{1}{q}$. In particular, if U is an IET such that $supp(U) \subset \mathcal{N}_\epsilon(\frac{1}{q})$ then $[U, T^k U T^{-k}] = id$.

Proof. First, one has

$$\theta dr_{\min} < \epsilon < \frac{1}{11q\rho} \leqslant \frac{1}{11q} < \frac{1}{q}$$

and

$$2\mu < \frac{1}{2}\theta dr_{\min} < \theta dr_{\min}$$

and

$$\theta dr_{\max} + 2\mu < \theta dr_{\max} + \theta dr_{\min} \leqslant 2\theta dr_{\max} = 2\theta dr_{\min}\rho < 2\epsilon\rho < \frac{2}{11q} < \frac{1}{q}$$

so one can apply the previous lemma: all the translation lengths of T are in $[\theta dr_{\min} - 2\mu; \theta dr_{\max} + 2\mu]$ mod $\frac{1}{q}$.
This interval is included in $[\frac{1}{2}\theta dr_{\min}; \frac{3}{2}\theta dr_{\max}]$.
For every $k \in \mathbb{N}$ such that $\frac{3k}{2}\theta dr_{\max} < \frac{1}{q}$, the IET T^k has all its translation lengths in $[\frac{k}{2}\theta dr_{\min}; \frac{3k}{2}\theta dr_{\max}]$ mod $\frac{1}{q}$.
We want an integer k such that both $2\epsilon < \frac{k}{2}\theta dr_{\min}$ and $\frac{3k}{2}\theta dr_{\max} < \frac{1}{q} - 2\epsilon$.
Take k such that $4\epsilon < k\theta dr_{\min} \leqslant 5\epsilon$ (this is possible because $\theta dr_{\min} < \epsilon$). Then one has:

$$\frac{3k}{2}\theta dr_{\max} = \frac{3k}{2}\theta dr_{\min}\rho \leqslant \frac{3}{2} \times 5\epsilon\rho < \frac{15}{2}\frac{1}{11q} = \frac{1}{q} - \frac{7}{22q} < \frac{1}{q} - \frac{2}{11q} < \frac{1}{q} - 2\epsilon\rho \leqslant \frac{1}{q} - 2\epsilon.$$

So T^k has all its translation lengths in $[2\epsilon; \frac{1}{q} - 2\epsilon]$ mod $\frac{1}{q}$.

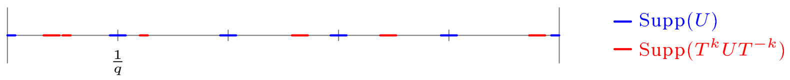


Figure 7: Drifting small support

Finally, one has $supp(T^kUT^{-k}) = T^k(supp(U))$ so if $supp(U) \subset \mathcal{N}_\epsilon(\frac{1}{q})$ then $supp(T^kUT^{-k}) \subset [0;1[\backslash \mathcal{N}_\epsilon(\frac{1}{q})$ and $supp(T^kUT^{-k}) \cap T^k(supp(U)) = \varnothing$. Hence U and T^kUT^{-k} commute. □

We have found an element that can drift a small support: step 4 is done! Let us highlight the three key points in this construction:

1. the "driftability" of T_0;
2. the localisation of the small support of U around $\frac{1}{q}\mathbb{N}$;
3. the q-rationality of T_0.

3.4 Step 5: Conclusion

We now have all the elements to prove the following proposition. It is only a matter of putting together the pieces of a jigsaw puzzle and choosing constants appropriately.

Proposition 8. *Assume S_0 and T_0 to be q-rational and the permutation $\sigma \in \mathfrak{S}_n$ associated to T_0 to be driftable.*
Then there exist a neighbourhood $\mathcal{U}$ of S and an open set $\mathcal{V}$ which accumulates on T_0 such that $\langle S, T\rangle$ is not free of rank 2 whenever $(S,T) \in \mathcal{U} \times \mathcal{V}$.

Proof. Let dl be a drifting direction for σ let dr be the corresponding drifting vector. Let $\rho = \frac{dr_{\max}}{dr_{\min}}$ and take

$$\epsilon < \frac{1}{11q\rho}; \qquad \eta < \frac{\epsilon}{4(q!+2)}; \qquad \theta < \min(\frac{\epsilon}{dr_{\min}}, \frac{\eta}{2||dl||_1}); \qquad \mu < \min(\frac{\theta dr_{\min}}{4}, \frac{\eta}{2}).$$

Define $\mathcal{U} = \mathcal{B}(S_0, \eta)$ the set of IETs that are at distance strictly less than η from S_0.
Take $T_\theta \in IET_\sigma$ such that $\lambda(T_\theta) = \lambda(T_0) + \theta dl$. Define $\mathcal{V}_{dl,\theta}$ the set of IETs that are at distance strictly less than μ from T_θ. Then every $T \in \mathcal{V}_{dl,\theta}$ verifies:

$$d(T, T_0) \leqslant d(T, T_\theta) + d(T_\theta, T_0) < \mu + \theta||dl||_1 < \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

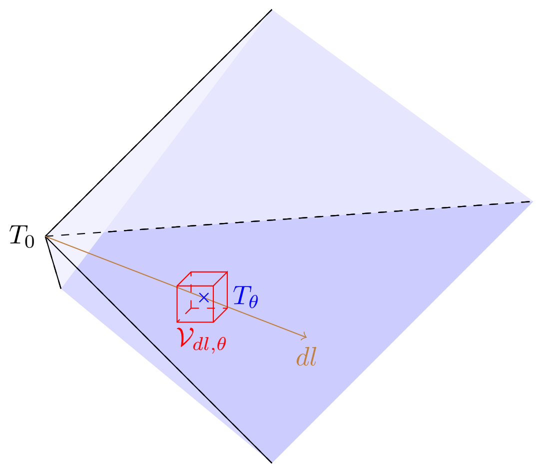


Figure 8: T_θ within the cone of drifting directions

From Lemma 2 one knows that for every $(S,T) \in \mathcal{U} \times \mathcal{V}_{dl,\theta}$ the IET $U = [S^{q!}, TS^{q!}T^{-1}]$ has its support included in $\mathcal{N}_\epsilon(\frac{1}{q}\mathbb{N})$. From Lemma 4 one knows that there exists $k \in \mathbb{N}$ such that $[U, T^kUT^{-k}] = id$.
It remains to check that this relation is not trivial. Denote by $u = s^{q!}ts^{q!}t^{-1}s^{-q!}ts^{-q!}t^{-1}$ the (non-trivial) word over the letters $s^{\pm 1}$ and $t^{\pm 1}$ such that $u(S,T) = U$. The word $w = ut^kut^{-k}u^{-1}t^ku^{-1}t^{-k}$ is equal to:

$$(s^{q!}ts^{q!}t^{-1}s^{-q!}ts^{-q!}\mathbf{t^{-1}})\mathbf{t^k}(s^{q!}ts^{q!}t^{-1}s^{-q!}ts^{-q!}\mathbf{t^{-1}})\mathbf{t^{-k}}(\mathbf{t}s^{q!}t^{-1}s^{q!}ts^{-q!}t^{-1}s^{-q!})\mathbf{t^k}(\mathbf{t}s^{q!}t^{-1}s^{q!}ts^{-q!}t^{-1}s^{-q!})t^{-k}$$

which reduces to:

$$s^{q!}ts^{q!}t^{-1}s^{-q!}t\mathbf{s^{-q!}t^{k-1}s^{q!}}ts^{q!}t^{-1}s^{-q!}ts^{-q!}\mathbf{t^{-k}}s^{q!}t^{-1}s^{q!}ts^{-q!}t^{-1}s^{-q!}\mathbf{t^{k+1}}s^{q!}t^{-1}s^{q!}ts^{-q!}t^{-1}s^{-q!}t^{-k}.$$

This word is reduced, except if $k = 1$, in which case it reduces to:

$$s^{q!}ts^{q!}t^{-1}s^{-q!}\mathbf{t^{2}}s^{q!}t^{-1}s^{-q!}ts^{-q!}\mathbf{t^{-k}}s^{q!}t^{-1}s^{q!}ts^{-q!}t^{-1}s^{-q!}\mathbf{t^{k+1}}s^{q!}t^{-1}s^{q!}ts^{-q!}t^{-1}s^{-q!}t^{-k}.$$

So the relation found is not trivial. Hence for every $(S,T) \in \mathcal{U} \times \mathcal{V}_{dl,\theta}$ the subgroup $\langle S,T\rangle$ is not free of rank 2. To have an open set $\mathcal{V}$ which accumulates on T_0 we take the union of all the convenient $\mathcal{V}_{dl,\theta}$. More precisely, we define:

$$\mathcal{V}_{dl} = \bigcup_{\epsilon<\frac{1}{11q\rho}} \bigcup_{\eta<\frac{\epsilon}{4(q!+2)}} \bigcup_{\theta<\min(\frac{\epsilon}{dr_{\min}},\frac{\eta}{2\|dl\|_1})} \mathcal{V}_{dl,\theta} \quad \text{and} \quad \mathcal{V} = \bigcup_{dl \text{ drifting direction}} \mathcal{V}_{dl}.$$

□

Theorem 1 (Dahmani-Fujiwara-Guirardel). *There exists a dense open set $\Omega \subset IET([0;1[) \times IET_{irred}([0;1[)$ such that for every $(S,T) \in \Omega$, $\langle S,T\rangle$ is not free of rank 2.*

Proof. For every q-rational IETs S_0 and T_0, with T_0 driftable, denote by $\mathcal{U}(S_0)$ and $\mathcal{V}(T_0)$ the open sets given by the previous proposition and define:

$$\Omega = \bigcup_{q\geqslant 2} \bigcup_{2\leqslant n\leqslant q} \bigcup_{\sigma\in\mathfrak{S}_n \text{ driftable}} \bigcup_{T_0\in IET_\sigma\ q-\text{rational}} \bigcup_{S_0\ q-\text{rational}} \mathcal{U}(S_0) \times \mathcal{V}(T_0).$$

The set Ω is a union of open sets hence it is open. And the previous proposition says that for every $(S,T) \in \Omega$, $\langle S,T\rangle$ is not free of rank 2.
Let us show that it is dense in $\text{IET}([0;1[) \times \text{IET}_{irred}([0;1[)$. Let $\mathcal{O} \subset \text{IET}([0;1[) \times \text{IET}_{irred}([0;1[)$ be an open set.
By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a couple of rational IETs (S_0,T_0) in $\mathcal{O}$. Let q be a common denominator for all the points in $\Delta(S_0)$ and $\Delta(T_0)$.
The open set $\mathcal{O}$ intersects every set that accumulates on (S_0,T_0) so:

$$\mathcal{O} \cap \Omega \supset \mathcal{O} \cap (\mathcal{U}(S_0) \times \mathcal{V}(T_0)) \neq \varnothing$$

which means that Ω is dense in $\text{IET}([0;1[) \times \text{IET}_{irred}([0;1[)$. □

4 Extending to generalised setting

We would like to find relations between more IETs, to change the dense subset of Theorem 1 into a full-measure subset. We do not have such a result but we still have an in between one: full-measure for the first component, density for the second one. In order to prove it using the same main ideas, we must dispense with the q-rationality of S_0. This hypothesis plays a key role to build the basic relation we have started with and to localise the small support of U around the points in $\frac{1}{q}\mathbb{N}$.
We avoid the first problem by taking $[T_0^{q!}, ST_0^{q!}S^{-1}] = id$ as the basic relation (instead of $[S_0^{q!}, T_0S_0^{q!}T_0^{-1}] = id$), which is true whenever T_0 is q-rational (whatever properties S may have).
We fix the second problem by taking into account the location of the discontinuity points of S_0. We introduce some notations in this purpose.

4.1 Notations

Definition 8. *Let S be any IET. Let q be a positive integer.*
Let us define

$$X_q(S) = \Delta(S^{-1}) \cup S\left(\frac{1}{q}\mathbb{N}\right) \cup \{0,1\}$$

and

$$Y_q(S) = \pi_q(X_q(S))$$

where $\pi_q : [0;1] \to [0;\frac{1}{q}[$ is the canonical projection modulo $\frac{1}{q}$.

Define also

$$Z_q(S) = Y_q(S) \sqcup \left(\frac{1}{q} + Y_q(S)\right) \sqcup \left(\frac{2}{q} + Y_q(S)\right) \sqcup \cdots \sqcup \left(\frac{q-1}{q} + Y_q(S)\right) \sqcup \{1\}.$$

Denote by

$$\alpha_q(S) = \min_{C \text{ connected component of } [0;\frac{1}{q}[\setminus Y_q(S)} diam(C)$$

the length of the smallest interval of $[0; \frac{1}{q}[\setminus Y_q(S)$.
And finally define

$$\mathcal{U}_\alpha^q = \{R \in IET([0;1[) \,|\, \alpha_q(R) > \alpha\}.$$

Note that $X_q(S) \subset Z_q(S)$ and that $\alpha_q(S) > 0$ is well defined because the set $X_q(S)$ is finite. See Figure 9.

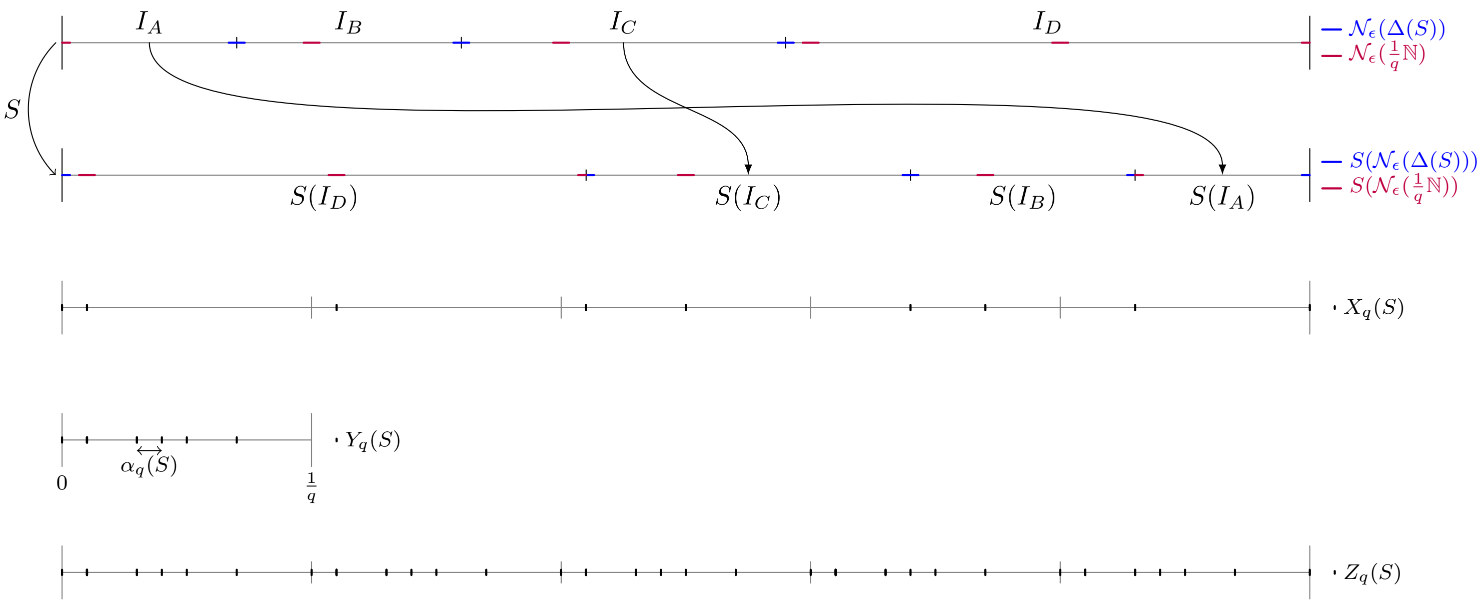


Figure 9: Illustration of the notations (here, $q = 5$)

Lemma 5. *Let* $\epsilon > 0$ *and* S *be an IET in* $\mathcal{U}_\epsilon^q$. *Then:*

$$S\left(\mathcal{N}_\epsilon\left(\Delta(S) \cup \frac{1}{q}\mathbb{N}\right)\right) \subset \mathcal{N}_\epsilon(X_q(S)).$$

Proof. Let us prove that $S(\mathcal{N}_\epsilon(\Delta(S))) = \mathcal{N}_\epsilon(S(\Delta(S)) \cup \{1\})$. Denote by $\{b_1, ..., b_n\} = \Delta(S)$ the set of discontinuity points of S (in increasing order).
If $y \in S(\mathcal{N}_\epsilon(\Delta(S)))$, say $y = S(x)$ with $0 < x - b_k < \epsilon$ (resp. $0 < b_k - x < \epsilon$), then x is in the same interval of continuity of S than b_k (resp. b_{k-1}). In the first case one has $|S(x) - S(b_k)| < \epsilon$, namely $y \in \mathcal{N}_\epsilon(S(\Delta(S)))$. In the second case, $y = S(x)$ is ϵ-close to 1 (if $\sigma \in \mathfrak{S}_n$, the underlying permutation of S, is such that $\sigma(k-1) = n$) or to $S(b_i)$ (if $\sigma(k-1) + 1 = i$), namely $y \in \mathcal{N}_\epsilon(\{1\}) \cup \mathcal{N}_\epsilon(S(\Delta(S)))$.
Conversely, assume that $y \in \mathcal{N}_\epsilon(S(\Delta(S)) \cup \{1\})$. If $y \in \mathcal{N}_\epsilon(\{1\})$, then $|S^{-1}(y) - b_i| < \epsilon$ where $\sigma(i) = n$ and σ is the underlying permutation of S. Else $y \in \mathcal{N}_\epsilon(S(\Delta(S)))$, say $0 < y - S(b_k) < \epsilon$ (resp. $0 < S(b_k) - y < \epsilon$), then $|S^{-1}(y) - b_k| < \epsilon$ (resp. $|S^{-1}(y) - b_i| < \epsilon$ where $\sigma(i) = k - 1$). In both cases $y \in S(\mathcal{N}_\epsilon(\Delta(S)))$.
Let us prove that $S(\mathcal{N}_\epsilon(\frac{1}{q}\mathbb{N})) \subset \mathcal{N}_\epsilon(S(\frac{1}{q}\mathbb{N}))$.
If $y \in S(\mathcal{N}_\epsilon(\frac{1}{q}\mathbb{N}))$, say $y = S(x)$ with $|x - \frac{m}{q}| < \epsilon \leqslant \alpha_q(S)$, then x is in the same interval of continuity of S than $\frac{m}{q}$. This implies that $|S(x) - S(\frac{m}{q})| < \epsilon$, namely $y \in \mathcal{N}_\epsilon(S(\frac{1}{q}))$.
Note that the other inclusion does not necessarily hold.

We can conclude:

$$\begin{aligned} S\left(\mathcal{N}_\epsilon\left(\Delta(S)\cup\frac{1}{q}\mathbb{N}\right)\right) &= S\left(\mathcal{N}_\epsilon(\Delta(S))\cup\mathcal{N}_\epsilon\left(\frac{1}{q}\mathbb{N}\right)\right) = S\left(\mathcal{N}_\epsilon(\Delta(S))\right)\cup S\left(\mathcal{N}_\epsilon\left(\frac{1}{q}\mathbb{N}\right)\right) \\ &\subset \mathcal{N}_\epsilon(S(\Delta(S)))\cup\mathcal{N}_\epsilon(\{1\})\cup\mathcal{N}_\epsilon\left(S\left(\frac{1}{q}\mathbb{N}\right)\right) \\ &\subset \mathcal{N}_\epsilon(\underbrace{S(\Delta(S))\cup\{0,1\}}_{=\Delta(S^{-1})\cup\{0,1\}})\cup\mathcal{N}_\epsilon\left(S\left(\frac{1}{q}\mathbb{N}\right)\right) \\ &\subset \mathcal{N}_\epsilon(X_q(S)). \end{aligned}$$

Indeed, $S(\Delta(S))\cup\{0,1\} = \Delta(S^{-1})\cup\{0,1\}$ (see Remark 3). □

We are now ready to extend the results of section 3. Let us start to build an IET with small support.

4.2 Building an IET with small support

The following lemmas are quite the same as the ones of steps 2 and 3, with little changes to overcome the loss of q-rationality of S_0.

> **Lemma 6.** *For all $\epsilon > 0$, $q \in \mathbb{N}$, there exists $\eta > 0$ such that if T_0 is q-rational and if T is η-close to T_0, then for all IET S, the IET $ST^{q!}S^{-1}$ induces translations on $[0;1[\setminus S(\mathcal{N}_\epsilon(\Delta(S)\cup\frac{1}{q}\mathbb{N}))$ with translation lengths strictly less than ϵ.*

Proof. Let $\eta < \frac{\epsilon}{2q!}$ and $x \notin S(\mathcal{N}_\epsilon(\Delta(S)\cup\frac{1}{q}\mathbb{N}))$.
Then $S^{-1}(x) \notin \mathcal{N}_\epsilon(\Delta(S)\cup\frac{1}{q}\mathbb{N})$. Since $T^{q!}$ is a translation on $\mathcal{N}_\epsilon(\frac{1}{q}\mathbb{N})$ with translation length strictly less than ϵ (Lemma 1), one gets that $|T^{q!}S^{-1}(x) - S^{-1}(x)| < \epsilon$. Since $d(S^{-1}(x),\Delta(S)) \geqslant \epsilon$, this gives that $T^{q!}S^{-1}(x)$ is in the same interval of continuity of S than $S^{-1}(x)$. Applying the local isometry S to the inequality $|T^{q!}S^{-1}(x) - S^{-1}(x)| < \epsilon$ one gets $|ST^{q!}S^{-1}(x) - x| < \epsilon$. □

> **Lemma 7.** *For all $\epsilon > 0$, $q \in \mathbb{N}$, there exists $\eta > 0$ such that if T_0 is q-rational and if T is η-close to T_0, then for all IET S, the IET $[T^{q!}, ST^{q!}S^{-1}]$ induces the identity on each interval of $[0;1[\setminus\mathcal{N}_\epsilon(X_q(S))$.*

Proof. Let us write $A_\epsilon = [0;1[\setminus\mathcal{N}_\epsilon(X_q(S))$ and $A_{\frac{\epsilon}{2}} = [0;1[\setminus\mathcal{N}_{\frac{\epsilon}{2}}(X_q(S))$.
Note that $\mathcal{N}_{\frac{\epsilon}{2}}(\frac{1}{q}\mathbb{N}) \subset \mathcal{N}_{\frac{\epsilon}{2}}(X_q(S)) \subset \mathcal{N}_\epsilon(X_q(S))$ and $S(\mathcal{N}_{\frac{\epsilon}{2}}(\Delta(S)\cup\frac{1}{q}\mathbb{N})) \subset \mathcal{N}_{\frac{\epsilon}{2}}(X_q(S)) \subset \mathcal{N}_\epsilon(X_q(S))$.
Apply Lemma 6 with $\frac{\epsilon}{2}$: whenever T is closer than $\frac{\epsilon}{4q!}$ to T_0, the IET $ST^{q!}S^{-1}$ induces translations of translation lengths strictly smaller than $\frac{\epsilon}{2}$ on each connected component of $[0;1[\setminus S(\mathcal{N}_{\frac{\epsilon}{2}}(\Delta(S)\cup\frac{1}{q}\mathbb{N})) \supset A_{\frac{\epsilon}{2}}$. Moreover, Lemma 1 says that the IET $T^{q!}$ induces translations of translation lengths strictly smaller than $\frac{\epsilon}{2}$ on each connected component of $[0;1[\setminus\mathcal{N}_{\frac{\epsilon}{2}}(\frac{1}{q}\mathbb{N}) \supset A_{\frac{\epsilon}{2}}$.
Let $x \in A_\epsilon$.
Denote by $I = [a_1;a_2]$ the connected component of x in $A_{\frac{\epsilon}{2}}$ and by $t, t' \in [-\frac{\epsilon}{2};\frac{\epsilon}{2}]$ the translation lengths of I by $T^{q!}$ and $ST^{q!}S^{-1}$ respectively.
On the one hand:

$$T^{q!}\circ ST^{q!}T^{-1}(x) = T^{q!}(x+t') = x+t'+t$$

because $x+t' \in I$ since $\frac{\epsilon}{2} < x-a_1$ and $\frac{\epsilon}{2} < a_2 - x$.
On the other hand:

$$ST^{q!}S^{-1}\circ T^{q!}(x) = ST^{q!}S^{-1}(x+t) = x+t+t'$$

because $x+t \in I$ since $\frac{\epsilon}{2} < x-a_1$ and $\frac{\epsilon}{2} < a_2 - x$.
Hence $T^{q!}$ and $ST^{q!}S^{-1}$ commute on $A_\epsilon = [0;1[\setminus\mathcal{N}_\epsilon(X_q(S))$. We conclude that $[T^{q!}, ST^{q!}S^{-1}]$ induces the identity on each interval of $[0;1[\setminus\mathcal{N}_\epsilon(X_q(S))$. □

In other words, the IET $U = [T^{q!}, ST^{q!}S^{-1}]$ has its support included in $\mathcal{N}_\epsilon(X_q(S))$ and so in $\mathcal{N}_\epsilon(Z_q(S))$.
This is useful information because it says that the support is both small and included in the "periodic" set $\mathcal{N}_\epsilon(Z_q(S)) = \mathcal{N}_\epsilon(Y_q(S)) \sqcup \left(\mathcal{N}_\epsilon(Y_q(S)) + \frac{1}{q}\right) \sqcup \ldots \sqcup \left(\mathcal{N}_\epsilon(Y_q(S)) + \frac{q-1}{q}\right)$. This is really important because the only control we have on the drift is modulo $\frac{1}{q}$.

4.3 Drifting the support

The following lemma allows us to drift the support of U to $[0;1[\setminus \mathcal{N}_\epsilon(Z_q)$. We use Lemma 3 to get a lemma analogous to Lemma 4. The only difference is the choice of constants we make.

> **Lemma 8.** *Within the same setting as in Lemma 3, let* $\rho = \frac{dr_{max}}{dr_{min}}$, $\alpha < \frac{1}{2q}$, $\epsilon < \frac{\alpha}{11\rho}$, $\theta < \frac{\epsilon}{dr_{min}}$ *and* $\mu < \frac{\theta dr_{min}}{4}$. *Let* T *be* μ*-close to* T_θ.
> *Then there exists* $k \in \mathbb{N}$ *such that all the translation lengths of* T^k *are in* $[2\epsilon; \alpha - 2\epsilon]$ *mod* $\frac{1}{q}$.

Proof. First, one has

$$\theta dr_{\min} < \epsilon < \frac{\alpha}{11\rho} \leqslant \frac{1}{22q\rho} < \frac{1}{q}$$

and

$$2\mu < \frac{1}{2}\theta dr_{\min} < \theta dr_{\min}$$

and

$$\theta dr_{\max} + 2\mu < \theta dr_{\max} + \theta dr_{\min} \leqslant 2\theta dr_{\max} = 2\theta dr_{\min}\rho < 2\epsilon\rho < \frac{1}{11q} < \frac{1}{q}$$

so one can apply Lemma 3: all the translation lengths of T are in $[\theta dr_{\min} - 2\mu; \theta dr_{\max} + 2\mu]$ mod $\frac{1}{q}$. This interval is included in $[\frac{1}{2}\theta dr_{\min}; \frac{3}{2}\theta dr_{\max}]$.
For every $k \in \mathbb{N}$ such that $\frac{3k}{2}\theta dr_{\max} < \frac{1}{q}$, the IET T^k has all its translation lengths in $[\frac{k}{2}\theta dr_{\min}; \frac{3k}{2}\theta dr_{\max}]$ mod $\frac{1}{q}$.
We want an integer k such that both $2\epsilon < \frac{k}{2}\theta dr_{\min}$ and $\frac{3k}{2}\theta dr_{\max} < \alpha - 2\epsilon\,(< \frac{1}{q})$.
Take k such that $4\epsilon < k\theta dr_{\min} \leqslant 5\epsilon$ (this is possible because $\theta dr_{\min} < \epsilon$). Then one has:

$$\frac{3k}{2}\theta dr_{\max} = \frac{3k}{2}\theta dr_{\min}\rho \leqslant \frac{3}{2} \times 5\epsilon\rho < \frac{15}{2}\frac{\alpha}{11} = \alpha - \frac{7\alpha}{22} < \alpha - \frac{2\alpha}{11} < \alpha - 2\epsilon.$$

So T^k has all its translation lengths in $[2\epsilon; \alpha - 2\epsilon]$ mod $\frac{1}{q}$. □

4.4 Conclusion

There is only one technical lemma (which is a bit painful, sorry for that) to prove before stating a generalisation of Proposition 8.

> **Lemma 9.** *Let* S *be an IET such that for every distinct* $x, y \in \Delta(S^{-1}) \cup \{0\}$, *one has* $x + \frac{1}{q}\mathbb{N} \neq y + \frac{1}{q}\mathbb{N}$. *If* $\alpha < \alpha_q(S)$, *then the set* $\mathcal{U}_\alpha^q$ *is a neighbourhood of* S.

Let us first give an example of an IET S such that $\mathcal{U}_\alpha(S)$ is not a neighbourhood of S to show the importance of the hypothesis on S.

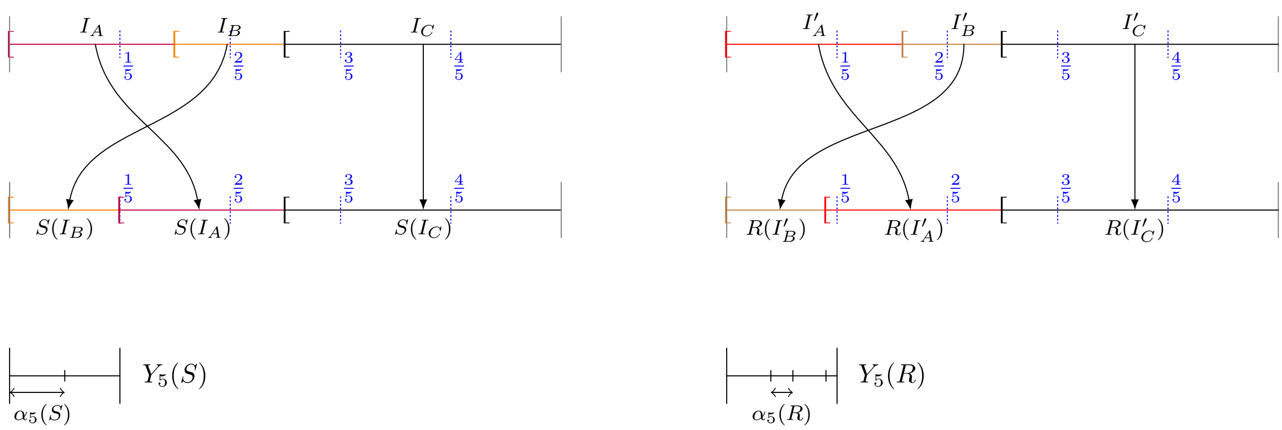


Figure 10: Example where $\mathcal{U}_\alpha$ is not a neighbourhood of S

Example 4. Let $q \geqslant 3$ and consider S defined by $\lambda(S) = \left(\lambda_A = \frac{3}{2q}, \lambda_B = \frac{1}{q}, \lambda_C = \frac{2q-5}{2q}\right)$ and $\pi(S) = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$.
Let $\alpha \in]0; \frac{1}{2q}[$, $\epsilon \in]0; \alpha[$ and define R by $\lambda(R) = \lambda(S) + (\epsilon, -\epsilon, 0)$ and $\pi(R) = \pi(S)$. See Figure 10.
Then $\alpha_q(R) < \epsilon < \alpha < \frac{1}{2q} = \alpha_q(S)$, so $R \notin \mathcal{U}_\alpha$ though R is at distance 2ϵ from S.

Proof. Let S be an IET such that for every distinct $x, y \in \Delta(S^{-1}) \cup \{1\}$, one has $x + \frac{1}{q}\mathbb{N} \neq y + \frac{1}{q}\mathbb{N}$, and let $\alpha < \alpha_q(S)$.
By definition, $S \in \mathcal{U}^q_\alpha$.
Take $\delta = \frac{\alpha_q(S)-\alpha}{8}$ and show that $\mathcal{B}(S,\delta) \subset \mathcal{U}^q_\alpha$. Let $R \in \mathcal{B}(S,\delta)$.
For every x, denote by $\overline{x}$ the only point in $\in [0; \frac{1}{q}[$ such that $x \in \frac{1}{q}\mathbb{N} + \overline{x}$.
Denoting by $\{b_0, ..., b_n\} = \{0\} \cup \Delta(S) \cup \{1\}$ (resp. $\{b'_0, ..., b'_n\}$) the discontinuity points of $S \cup \{0,1\}$ (resp. of R), we claim that:

1. for every $\frac{p}{q}, \frac{p'}{q} \in [0; 1[$, if $\overline{S(\frac{p}{q})} = \overline{S(\frac{p'}{q})}$, then $\frac{p}{q}$ and $\frac{p'}{q}$ are in the same interval of continuity of S;
2. for every $\frac{p}{q} \in [0; 1[$, if $\frac{p}{q}$ is in $[b_{k-1}; b_k[$ the k-th interval of continuity of S, then $\frac{p}{q}$ is also in $[b'_{k-1}; b'_k[$ the k-th interval of continuity of R;
3. for every $\frac{p}{q}, \frac{p'}{q} \in [0; 1[$, if $\overline{R(\frac{p}{q})} \neq \overline{R(\frac{p'}{q})}$, then $\overline{S(\frac{p}{q})} \neq \overline{S(\frac{p'}{q})}$;
4. for every $\frac{p}{q} \in [0; 1[$, $|\overline{R(\frac{p}{q})} - \overline{S(\frac{p}{q})}| \leqslant 2\delta$;
5. for every $x \in \Delta(R^{-1}) \cup \{0,1\}$, one has $d(\overline{x}, \pi_q(\Delta(S^{-1}))) \leqslant \frac{\delta}{2} + 2\delta$;
6. for every $\overline{x} \in Y_q(S)$, there exists a unique $\overline{y} \in Y_q(R)$ such that $|\overline{x} - \overline{y}| \leqslant \frac{5\delta}{2}$.

Let us prove the claims.

1. Let $\frac{p}{q}, \frac{p'}{q} \in [0; 1[$ be such that $\overline{S(\frac{p}{q})} = \overline{S(\frac{p'}{q})}$.
   Let $i, i' \in \{0, ..., n-1\}$ be such that $\frac{p}{q} \in [b_i; b_{i+1}[$ and $\frac{p'}{q} \in [b_{i'}; b_{i'+1}[$.
   The inverse of S is discontinuous at the points $S(b_i)$ and $S(b_{i'})$ unless they are equal to 0, i.e. $S(b_i), S(b_{i'}) \in \Delta(S^{-1}) \cup \{0\}$. By hypothesis $\overline{S(b_i)} = \overline{S(b_{i'})}$ if and only if $S(b_i) = S(b_{i'})$, which means if and only if $b_i = b_{i'}$. This shows that if $\overline{S(\frac{p}{q})} = \overline{S(\frac{p'}{q})}$, then $i = i'$ and $\frac{p}{q}, \frac{p'}{q} \in [b_i, bi+1[$ are in the same interval of continuity of S.
2. Let $\frac{p}{q} \in [0; 1[$ and $k \in \mathbb{N}$ be such that $\frac{p}{q}$ is in the k-th interval of continuity of S: $\frac{p}{q} \in [b_{k-1}; b_k[$.
   Then $b'_k \in [b_k - \frac{\delta}{2}; b_k + \frac{\delta}{2}]$ (see Proposition 3), so:
   $$b'_k - \frac{p}{q} = b'_k - b_k + b_k - \frac{p}{q} \geqslant -\frac{\delta}{2} + \alpha_q(S) = \frac{15\alpha_q(S) + \alpha}{16} > \alpha.$$
   Analogously, one has
   $$\frac{p}{q} - b'_{k-1} > \alpha.$$
   This shows that $\frac{p}{q}$ is in $[b'_{k-1}; b'_k[$ the k-th interval of continuity of R.
3. Let $\frac{p}{q}, \frac{p'}{q} \in [0; 1[$ and assume $\overline{R(\frac{p}{q})} \neq \overline{R(\frac{p'}{q})}$.
   Then $\frac{p}{q}, \frac{p'}{q}$ are not in the same interval of continuity of R and thus cannot be in the same interval of continuity of S, which leads to $\overline{S(\frac{p}{q})} \neq \overline{S(\frac{p'}{q})}$.
4. Since the translation lengths of R differ at most by 2δ from that of S, one gets $|R(\frac{p}{q}) - S(\frac{p}{q})| \leqslant 2\delta$. Say $S(\frac{p}{q}) \in [\frac{m}{q}; \frac{m+1}{q}[$. Either $S(\frac{p}{q}) = \frac{m}{q}$ and the hypothesis on S implies that $S(\frac{p}{q}) = 0$ (because $\overline{\frac{m}{q}} = 0$) and then $R(\frac{p}{q}) \in [0; 2\delta[$ so $|\overline{R(\frac{p}{q})} - \overline{S(\frac{p}{q})}| \leqslant 2\delta$. Either $S(\frac{p}{q}) \neq \frac{m}{q}$ and by definition of $\alpha_q(S)$ one has:
   $$\begin{cases} S\left(\frac{p}{q}\right) - \frac{m}{q} \geqslant \alpha_q(S) > 2\delta \\ \frac{m+1}{q} - S\left(\frac{p}{q}\right) \geqslant \alpha_q(S) > 2\delta \end{cases}$$
   which implies that $R(\frac{p}{q}) \in [\frac{m}{q}; \frac{m+1}{q}[$ and $|\overline{R(\frac{p}{q})} - \overline{S(\frac{p}{q})}| \leqslant 2\delta$.

5. Let $x \in \Delta(R^{-1}) \cup \{0,1\}$ and write $x = R(b'_{k-1})$. Then $|b'_{k-1} - b_{k-1}| \leqslant \frac{\delta}{2}$ and $|R(b'_{k-1}) - S(b_{k-1})| \leqslant \frac{\delta}{2} + 2\delta$ (because of Proposition 3).

6. Let $\overline{x} \in Y_q(S)$.

   Then
$$\overline{y} = \begin{cases} \overline{R(\frac{p}{q})} & \text{if } \overline{x} = \overline{S(\frac{p}{q})} \in \pi_q \circ S(\frac{1}{q}\mathbb{N}) \\ \overline{R(b'_k)} & \text{if } \overline{x} = \overline{S(b_k)} \in \pi_q(\Delta(S^{-1}) \cup \{0,1\}) \end{cases}$$
   is in $Y_q(R)$ and at distance at most $\frac{5\delta}{2}$ from $\overline{x}$.

   Let $\overline{z} \in Y_q(R) \setminus \{\overline{y}\}$. Then:
$$\overline{z} = \begin{cases} \overline{R(\frac{p'}{q})} \in \pi_q \circ R(\frac{1}{q}\mathbb{N}) \\ \text{or} \\ \overline{R(b'_j)} \in \pi_q(\Delta(R^{-1}) \cup \{0,1\}) \end{cases}.$$

   In the first case, if $\overline{x} = \overline{S(\frac{p}{q})}$ then $\overline{S(\frac{p'}{q})} \neq \overline{S(\frac{p}{q})}$ because of claim 3 (because $\overline{z} \neq \overline{y}$). If $\overline{x} = \overline{S(b_k)}$ then $\overline{S(\frac{p'}{q})} \neq \overline{S(b_k)}$. Indeed if $\overline{S(\frac{p'}{q})} = \overline{S(b_k)}$ then $b_k = 0$ and $\frac{p'}{q} \in [b_k; b_{k+1}[= [0; b_1[$, and $\frac{p'}{q} \in [0; b'_1[$ because of claim 2, which leads to $\overline{R(\frac{p'}{q})} = \overline{R(b'_k)}$, i.e. $\overline{z} = \overline{y}$, which is excluded.

   In the second case, if $\overline{x} = \overline{S(\frac{p}{q})}$ then $\overline{S(\frac{p}{q})} \neq \overline{S(b_j)}$. Indeed if $\overline{S(\frac{p}{q})} = \overline{S(b_j)}$ then $b_j = 0$ and $\frac{p}{q} \in [b_k; b_{k+1}[=$ $[0; b_1[$, and $\frac{p}{q} \in [0; b'_1[$ because of claim 2, which leads to $\overline{R(\frac{p}{q})} = \overline{R(b'_j)}$, i.e. $\overline{y} = \overline{z}$, which is excluded. If $\overline{x} = \overline{S(b_k)}$, then one has $\overline{S(b_j)} \neq \overline{S(b_k)}$ because of the hypothesis on S.

   Take
$$z' = \begin{cases} \frac{p'}{q} & \text{if } \overline{z} = \overline{R(\frac{p'}{q})} \in \pi_q \circ R(\frac{1}{q}\mathbb{N}) \\ b_j & \text{if } \overline{z} = \overline{R(b'_j)} \in \pi_q(\Delta(R^{-1})) \end{cases}.$$

   The previous paragraph shows that $\overline{S(z')} \neq \overline{x}$. Thus one has $|\overline{S(z')} - \overline{x}| \geqslant \alpha_q(S)$ and:
$$\begin{aligned} |\overline{z} - \overline{x}| &\geqslant ||\overline{z} - \overline{S(z')}| - |\overline{S(z')} - \overline{x}|| && \text{(triangular inequality)} \\ &\geqslant \alpha_q(S) - \frac{5\delta}{2} = \frac{11\alpha_q(S) + 5\alpha}{16} && \text{because } \begin{cases} |\overline{S(z')} - \overline{x}| \geqslant \alpha_q(S) \\ |\overline{z} - \overline{S(z')}| \leqslant \frac{5\delta}{2} < \alpha_q(S) \end{cases} \\ &> \frac{5\alpha_q(S) - 5\alpha}{16} = \frac{5\delta}{2}. \end{aligned}$$

   So $\overline{y}$ is the only point in $Y_q(R)$ at distance at most $\frac{5\delta}{2}$ from $\overline{x}$.

Finally, we prove that $R \in \mathcal{U}_\alpha^q$. Let $\overline{x}, \overline{y} \in Z_q(R)$ be two distinct points. For each $z \in \{\overline{x}, \overline{y}\}$, let
$$z_S = \begin{cases} \overline{S(\frac{p}{q})} & \text{if } z = \overline{R(\frac{p}{q})} \in \pi_q \circ R(\frac{1}{q}\mathbb{N}) \\ \overline{S(b_k)} & \text{if } z = \overline{R(b'_k)} \in \pi_q(\Delta(R^{-1}) \cup \{0,1\}) \end{cases}.$$

Then $z_S \in Y_q(S)$ is at distance at most $\frac{5\delta}{2}$ from z and $\overline{x}_S \neq \overline{y}_S$ because of claim 6 (because $\overline{x} \neq \overline{y}$). So one has:
$$\begin{aligned} |\overline{x} - \overline{y}| &\geqslant |\overline{x}_S - \overline{y}_S| - |\overline{x}_S - \overline{x}| - |\overline{y}_S - \overline{y}| && \text{(triangular inequality)} \\ &\geqslant \alpha_q(S) - \frac{5\delta}{2} - \frac{5\delta}{2} = \frac{3\alpha_q(S) + 5\alpha}{8} \\ &> \alpha \end{aligned}$$

So $\alpha_q(R) > \alpha$. This means that $R \in \mathcal{U}_\alpha^q$.
So $\mathcal{B}(S, \delta) \subset \mathcal{U}_\alpha^q$ and $\mathcal{U}_\alpha^q$ is a neighbourhood of S. □

<u>Remark 4:</u> The condition on S corresponds to being outside a finite union of hyperplans. This means that for every S in a full-measure set $\mathcal{A}$, $\mathcal{U}_\alpha(S)$ is a neighbourhood of S (where $\alpha < \alpha_q(S)$).

Proposition 9. *Assume T_0 to be q-rational and its underlying permutation $\sigma \in \mathfrak{S}_n$ to be driftable. Let S be any IET.*
Then there exist a set $\mathcal{U} \subset IET([0;1[)$ that contains S and an open set $\mathcal{V} \subset IET_\sigma([0;1[)$ which accumulates on T_0 such that $\langle S, T\rangle$ is not free of rank 2 whenever $T \in \mathcal{V}$.
Moreover if S is such that for every distinct $x, y \in \Delta(S)$, one has $x + \frac{1}{q}\mathbb{N} \neq y + \frac{1}{q}\mathbb{N}$, then the set $\mathcal{U}$ is a neighbourhood of S.

The proof is almost the same as the one of Proposition 8. We just have to adjust the constants to satisfy all the hypotheses of the lemmas in the generalised setting.

Proof. Let $\alpha < \alpha_q(S)$. Note that $\alpha_q(S) \leqslant \frac{1}{2q}$ because S is not q-rational.
Let dl and dr be the drifting direction and vector for σ. Let $\rho = \frac{dr_{\max}}{dr_{\min}}$ and take

$$\epsilon < \frac{\alpha}{11\rho}; \qquad \eta < \frac{\epsilon}{4(q!+2)}; \qquad \theta < \min(\frac{\epsilon}{dr_{\min}}, \frac{\eta}{2||dl||_1}); \qquad \mu < \min(\frac{\theta dr_{\min}}{4}, \frac{\eta}{2}).$$

Take $\mathcal{U} = \mathcal{U}^q_\alpha$. If S is such that for every distinct $x, y \in \Delta(S)$, one has $x + \frac{1}{q}\mathbb{N} \neq y + \frac{1}{q}\mathbb{N}$, then the set $\mathcal{U}$ is a neighbourhood of S (Lemma 9).
Take $T_\theta \in IET_\sigma$ such that $\lambda(T_\theta) = \lambda(T_0) + \theta dl$. Define $\mathcal{V}_{dl,\theta}$ the set of IETs that are at distance strictly less than μ from T_θ. Then every $T \in \mathcal{V}_{dl,\theta}$ verifies:

$$d(T, T_0) \leqslant d(T, T_\theta) + d(T_\theta, T_0) < \mu + \theta||dl||_1 < \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

From Lemma 7 one knows that for every $(R, T) \in \mathcal{U} \times \mathcal{V}_{dl,\theta}$ the IET $U = [T^{q!}, RT^{q!}R^{-1}]$ has its support included in $\mathcal{N}_\epsilon(Z_q(R))$. By definition of $\alpha_q(R)$, the smallest connected component of $[0;1[\setminus \mathcal{N}_\epsilon(Z_q(R))$ has length at least $\alpha_q(R) - 2\epsilon$, hence at least $\alpha - 2\epsilon$ because $R \in \mathcal{U} = \mathcal{U}^q_\alpha$.
So it is enough to drift the support of U with a drift in $[2\epsilon, \alpha - 2\epsilon]$ modulo $\frac{1}{q}\mathbb{Z}$. And from Lemma 8, there exists $k \in \mathbb{N}$ such that $[U, T^kUT^{-k}] = id$.
It remains to check that this relation is not trivial. Denote by $u = t^{q!}rt^{q!}r^{-1}t^{-q!}rt^{-q!}r^{-1}$ the (non-trivial) word over the letters $r^{\pm 1}$ and $t^{\pm 1}$ such that $u(S, T) = U$. The word $w = ut^kut^{-k}u^{-1}t^ku^{-1}t^{-k}$ is equal to:

$$(t^{q!}rt^{q!}r^{-1}t^{-q!}rt^{-q!}r^{-1}).t^k.(t^{q!}rt^{q!}r^{-1}t^{-q!}rt^{-q!}r^{-1}).t^{-k}.(rt^{q!}r^{-1}t^{q!}rt^{-q!}r^{-1}t^{-q!}).t^k.(rt^{q!}r^{-1}t^{q!}rt^{-q!}r^{-1}t^{-q!}).t^{-k}$$

which is reduced, except if $k = q!$ in which case it reduces to the reduced word:

$$t^{q!}rt^{q!}r^{-1}t^{-q!}rt^{-q!}r^{-1}t^kt^{q!}rt^{q!}r^{-1}t^{-q!}rt^{-q!}r^{-1}t^{-k}rt^{q!}r^{-1}t^{2q!}rt^{-q!}r^{-1}t^{-q!}t^{-k}.$$

So the relation is not trivial. Hence for every $(R, T) \in \mathcal{U} \times \mathcal{V}_{dl,\theta}$ the subgroup $\langle R, T\rangle$ is not free of rank 2.
In order to have an open set $\mathcal{V}$ which accumulates on T_0 we take the union of all the convenient $\mathcal{V}_{dl,\theta}$. More precisely, we define:

$$\mathcal{V}_{dl} = \bigcup_{\epsilon < \frac{\alpha}{11\rho}} \bigcup_{\eta < \frac{\epsilon}{4(q!+2)}} \bigcup_{\theta < \min(\frac{\epsilon}{dr_{\min}}, \frac{\eta}{2||dl||_1})} \mathcal{V}_{dl,\theta} \quad \text{and} \quad \mathcal{V} = \bigcup_{dl \text{ drifting direction}} \mathcal{V}_{dl}.$$

□

This leads to the following theorem, analogous to Theorem 1.

Theorem 2. *Let S be any IET on [0; 1[. There exists a dense open set $\Omega_{irred}(S) \subset IET_{irred}([0;1[)$ such that for every $T \in \Omega_{irred}(S)$, $\langle S, T\rangle$ is not free of rank 2.*

Proof. Let S be an IET on $[0;1[$.
For every q-rational driftable IET T_0, fix an $\alpha < \alpha_q(S)$ and denote by $\mathcal{V}(T_0)$ the open sets given by the previous proposition and define:

$$\Omega_{irred}(S) = \bigcup_{q \geqslant 2} \bigcup_{2 \leqslant n \leqslant q} \bigcup_{\sigma \in \mathfrak{S}_n \text{ driftable}} \bigcup_{T_0 \in IET_\sigma \ q-\text{rational}} \mathcal{V}(T_0).$$

The set $\Omega_{irred}(S)$ is a union of open sets hence is open. And the proof of the previous proposition shows that for every $T \in \Omega_{irred}(S)$, $\langle S, T\rangle$ is not free of rank 2.
Let us show that it is dense in $\text{IET}_{irred}([0;1[)$. Let $\mathcal{O} \subset \text{IET}_{irred}([0;1[)$ be an open set.
By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational (and irreducible) IET T_0 in $\mathcal{O}$.
The open set $\mathcal{O}$ intersects every set that accumulates on T_0 so:

$$\mathcal{O} \cap \Omega_{irred}(S) \supset \mathcal{O} \cap \mathcal{V}(T_0) \neq \varnothing$$

which means that $\Omega_{irred}(S)$ is dense in $\text{IET}_{irred}([0;1[)$. □

4.5 Small improvements

We would like to extend the size of the set of couples of IETs which share a relation. Here are two simple remarks that allow to get slightly bigger sets :

1. We have defined a driftable permutation as a permutation σ such that the set $\Phi_\sigma^{-1}\{(\mathbb{R}_+^*)^n\}$ of preimages of $(\mathbb{R}_+^*)^n$ by Φ_σ is nonempty. If σ is driftable then the set of preimages of $(\mathbb{R}_-^*)^n$ by Φ_σ is also non empty (by linearity of Φ_σ). All the proofs work the same way if we consider the set of *negative drifting directions* $dl = (dl_1, ..., dl_n) \in \Phi_\sigma^{-1}\{(\mathbb{R}_-^*)^n\}$ (with $\sum dl_i = 0$) instead of the set of (positive) drifting directions. They are associated to *negative drifting vectors* $dr \in (\mathbb{R}_-^*)^n$ instead of (positive) drifting vectors. We could then build an IET T that drifts the small support of U to the left instead of the right. Given any IET S, this "trick" doubles the size of the set $\mathcal{V}(T_0)$ (built in the proof of Proposition 9) of IETs around T_0 sharing a relation with S.
2. The size of the set $\mathcal{V}(T_0)$ is inversely proportional to the size of the basic relation we take. If the q-rational IET T_0 has order k (recall that k divides $q!$), then we can take $T_0^k = id$ instead of $T_0^{q!} = id$ as the basic relation.

We discuss a bit more the second remark in the next section.

4.6 Looking for smaller relations

The open set $\mathcal{V}(T_0)$ that we built around a q-rational IET T_0 has a diameter which is bounded by $\frac{1}{q!q}$ times a constant (depending only on the underlying permutation σ). The factor $\frac{1}{q!}$ comes from the length of the basic relation we used ($T_0^{q!} = id$). This means that we have relations only for driftable IET that are *very very* close to a q-rational IET. Indeed the following Liouville's theorem says that an algebraic number cannot be too close to a q-rational number:

Theorem 3 (Liouville). *Let x be an algebraic number of degree $d > 1$. There exists $A > 0$ such that for every rational number $\frac{p}{q}$:*

$$\left|x - \frac{p}{q}\right| \geqslant \frac{A}{q^d}.$$

Proof. Let $P \in \mathbb{Z}[X]$ be an irreducible polynomial in $\mathbb{Q}[X]$ (namely that has no root in $\mathbb{Q}$) such that $P(x) = 0$. If $|x - \frac{p}{q}| > 1$, then $|x - \frac{p}{q}| > \frac{1}{q^d}$.
Else denote $M = \max_{t \in [x-1;x+1]} |P'(t)|$, then

$$\begin{aligned} M\left|x - \frac{p}{q}\right| &\geqslant \left|P(x) - P\left(\frac{p}{q}\right)\right| && \text{(mean value theorem)} \\ &\geqslant \frac{1}{q^d} && \text{since } q^d\left|P(x) - P\left(\frac{p}{q}\right)\right| = q^d\left|P\left(\frac{p}{q}\right)\right| \in \mathbb{Z} \end{aligned}$$

so one gets the theorem by taking $A = \min\{1, \frac{1}{M}\}$. □

Arnoux-Yoccoz's example

Let us study an example from Arnoux and Yoccoz (see [AY81] and [Arn88]).
Let a be the only real number such that $a^3 + a^2 + a = 1$. Define the IET g by

$$\lambda(g) = \left(\lambda_A = \frac{a}{2}, \lambda_{A'} = \frac{a}{2}, \lambda_B = \frac{a^2}{2}, \lambda_{B'} = \frac{a^2}{2}, \lambda_C = \frac{a^3}{2}, \lambda_{C'} = \frac{a^3}{2}\right)$$

and

$$\pi(g) = \begin{pmatrix} A & A' & B & B' & C & C' \\ A' & A & B' & B & C' & C \end{pmatrix}.$$

Let h be the rotation of angle $\frac{1}{2}$, namely $\lambda(h) = (\lambda_A = \frac{1}{2}, \lambda_B = \frac{1}{2})$ and $\pi(h) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$. Define finally $f = h \circ g$. One has:

$$\lambda(f) = \left(\lambda_A = \frac{1-a}{2}, \lambda_B = a - \frac{1}{2}, \lambda_C = \frac{a}{2}, \lambda_D = \frac{a^2}{2}, \lambda_E = \frac{a^2}{2}, \lambda_F = \frac{a^3}{2}, \lambda_G = \frac{a^3}{2}\right)$$

and

$$\pi(f) = \begin{pmatrix} A & B & C & D & E & F & G \\ B & E & D & G & F & C & A \end{pmatrix}.$$

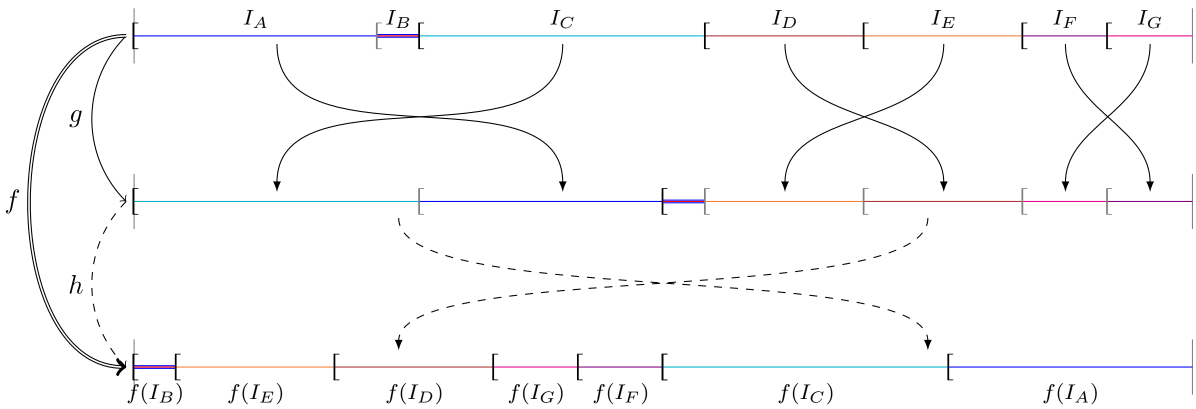


Figure 11: The example of the IET f from Arnoux and Yoccoz

We first show that the IET f is too far from rational IETs: it is in none of the $\mathcal{V}(T_0)$ built in the proof of Theorem 1 for a rational IET T_0.
The following table gives the constants A given by Liouville's theorem for the lengths of the intervals of continuity of f (which are all algebraic of order 3), so that for every rational number $\frac{p}{q}$, $\left|x - \frac{p}{q}\right| \geqslant \frac{A(x)}{q^d}$.

x	$A(x)$
$\lambda_A \quad = \frac{1}{2} - \frac{1}{2}a$	25
$\lambda_B \quad = a - \frac{1}{2}$	89
$\lambda_C \quad = \frac{1}{2}a$	51
$\lambda_D = \lambda_E \quad = \frac{1}{2}a^2$	46
$\lambda_F = \lambda_G \quad = \frac{1}{2}a^3$	71

We can conclude that a q-rational IET T_0 is far from f by at least:

$$d(f, T_0) \geqslant \sum_{X \in \{A,B,\ldots,G\}} \frac{A(\lambda_X)}{q^3} = \frac{282}{q^2} > \frac{1}{q!}.$$

On the other hand f should be η-close to T_0 in order to apply the technics of Proposition 8, where η is less than $\frac{1}{q!}$. This proves that f is in none of the $\mathcal{V}(T_0)$.
In order to overcome this issue, we can use smaller basic relations ($T_0^k = id$ for k the order of T_0 instead of $T_0^{q!} = id$). This gives bigger open sets $\mathcal{V}(T_0)$ around q-rational IETs T_0 that have "small" order. Indeed the size of the set $\mathcal{V}(T_0)$ is inversely proportional to the size of the basic relation we take.
We want to know if it is enough. For every q between 20 and 20000, we have computed the closest q-rational IET to f, its distance δ to f, its order $\mathfrak{o}$ and the bound $\mathfrak{b} = 40q(\mathfrak{o}+2)\delta$. The condition $\mathfrak{b} < 1$ is necessary so that f is in an enlarged $\mathcal{V}(T_0)$.
The distance $\delta(q)$ between f and the closest q-rational IET seems to decrease approximately like $\frac{1}{q}$ (and not faster). See Figure 12.
Remark 5: The distance between f and the closest q-rational IET to f is at most equal to $\frac{5}{q}$. The set of lengths $(l'_1, l'_2, \ldots, l'_7) \in \left(\frac{1}{q}\mathbb{N}\right)^7$ that is the closest to $(l_1, l_2, \ldots, l_7) = \lambda(f)$ is such that for every $i \in \{1, \ldots, 7\}$, $|l'_i - l_i| \leqslant \frac{1}{2q}$. Then we do not necessarily have $\sum_{i=1}^7 l'_i = 1$, only $\frac{7}{2q} \leqslant \sum_{i=1}^7 l'_i \leqslant \frac{7}{2q}$. Knowing that $\sum_{i=1}^7 l'_i \in \frac{1}{q}\mathbb{N}$, we conclude that $\sum_{i=1}^7 l'_i = \frac{k}{q}$ whith $k \in \{-3, \ldots, 3\}$. Thus the lengths of the intervals of continuity of the closest IET to f are given by $(l''_1, \ldots, l''_7)$ where $l''_i = l'_i$ except for the k worst approximations l'_j greater (resp. smaller) than l_j if k is positive (resp. negative). For these worst aproximations, $l''_j = l_j - \frac{1}{q}$ (resp. $l''_j = l_j + \frac{1}{q}$). Hence $\sum_{i=1}^7 |\, l''_i - l_i \,| \leqslant \frac{4}{2q} + \frac{3}{q} = \frac{5}{q}$.

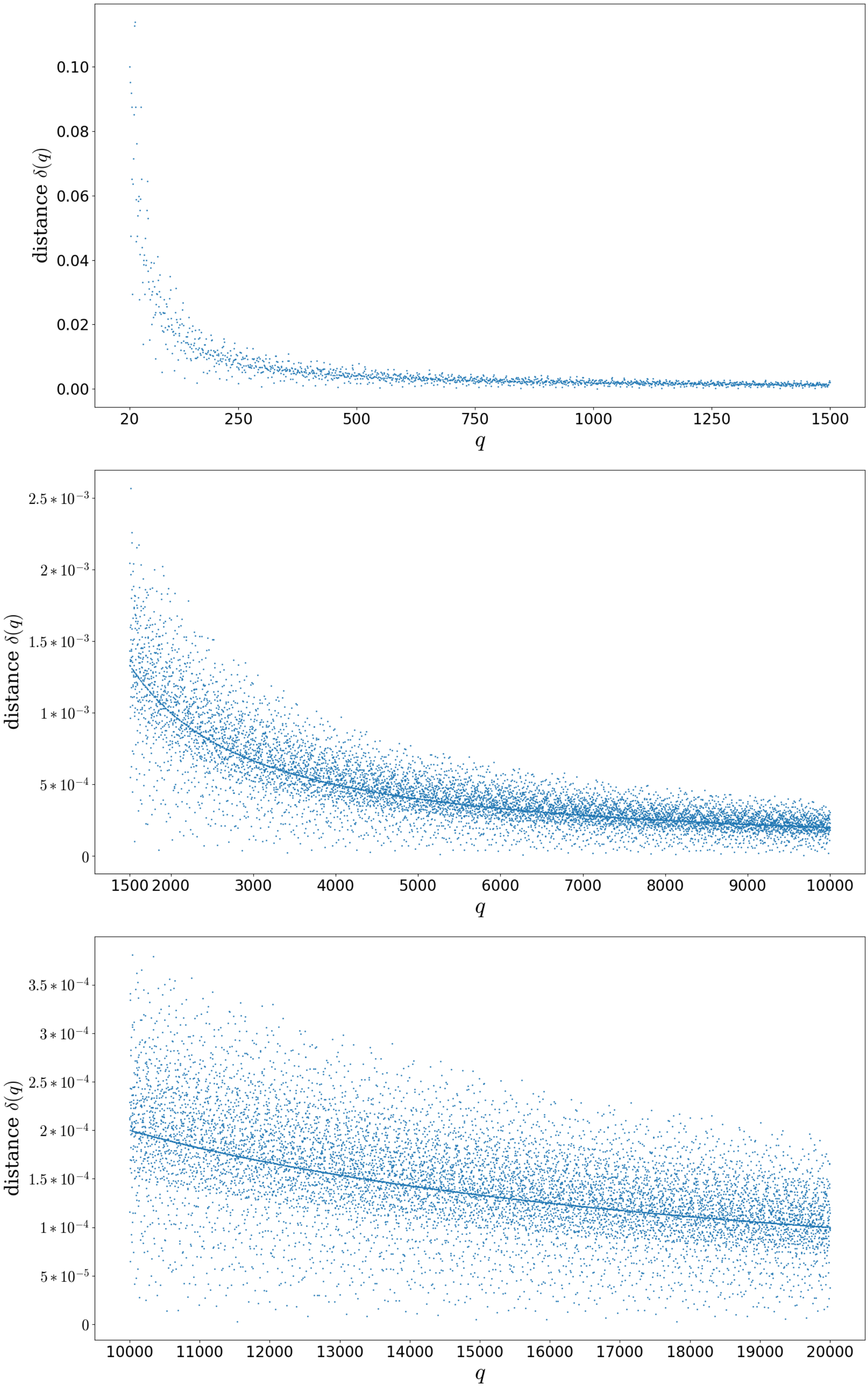


Figure 12: Distance $\delta(q)$ between f and the closest q-rational IET to f

For some values of q, the order $\mathscr{o}(q)$ of this q-rational IET closest to f increases linearly. See Figure 13.

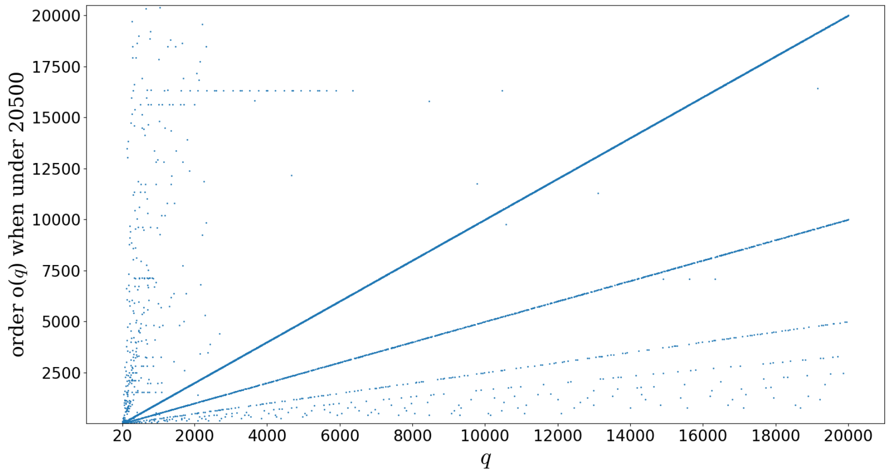
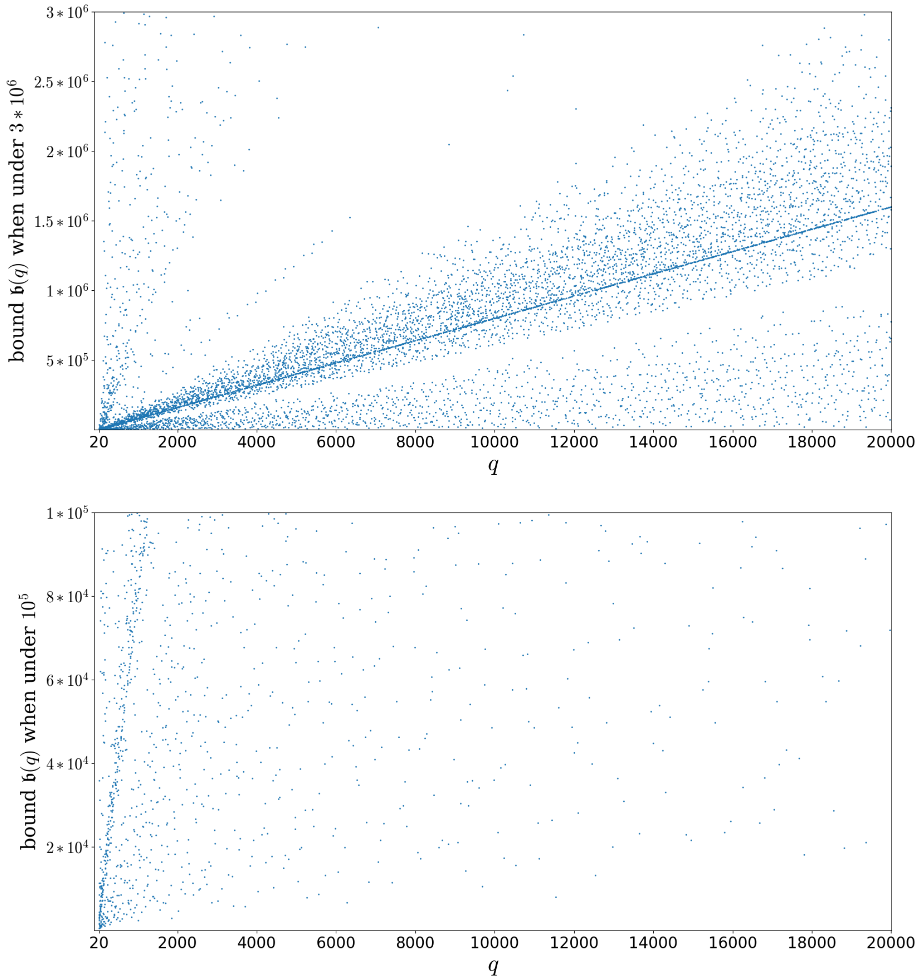


Figure 13: Order $\mathscr{o}(q)$ of the closest q-rational IET to f

As a result, the bound $\mathfrak{b}(q)$ increases at least linearly so it seems that there is no rational IET T_0 close enough to f so that $f \in \mathcal{V}(T_0)$. See Figure 14.



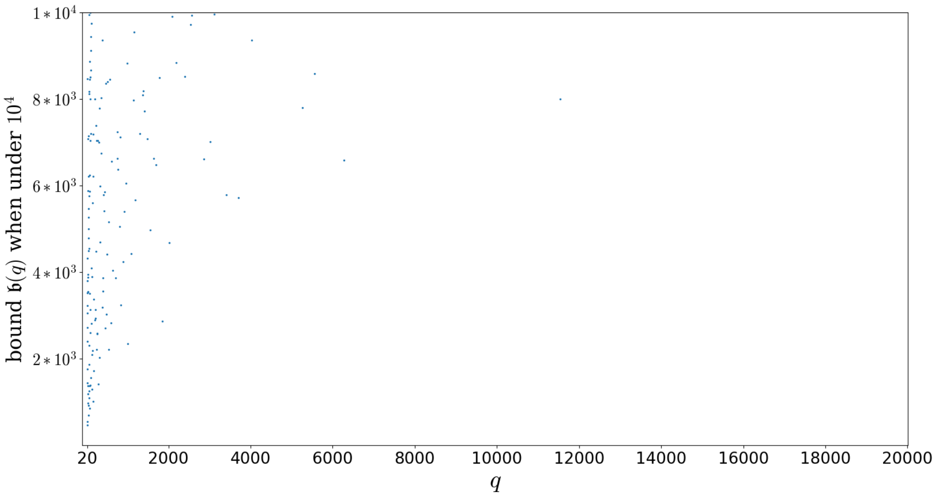


Figure 14: Bound $\mathfrak{b}(q)$

There are algebraic IETs with relations

We can still find algebraic IETs in $\mathcal{V}(T_0)$. We give here an example (even a family of examples), but it is not specific to the chosen T_0 (we can find as many analogous examples as we want for other T_0).

Example 5. Let $q = 10$ and T_0 be the IET defined by

$$\lambda(T_0) = \left(\lambda_A = \frac{2}{10}, \lambda_B = \frac{1}{10}, \lambda_C = \frac{3}{10}, \lambda_D = \frac{4}{10}\right)$$

and

$$\pi(T_0) = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$$

(see Figure 3).

Let S be any IET. We look for an IET T with algebraic lengths of intervals of continuity, such that $\langle S, T\rangle$ is not free of rank 2.

Fix $\alpha < \alpha_q(S)$.

Let $\xi > 0$ to be fixed later on. Define the IET T by $\lambda(T) = \lambda(T_0) + (-\xi, -\xi, \xi, \xi)$ and $\pi(T) = \pi(T_0)$.

Then $d(T, T_0) = 4\xi$, $\beta(T) = \beta(T_0) + (-\xi, -2\xi, -\xi)$ and $\tau(T) = \tau(T_0) + (\xi, 3\xi, 3\xi, \xi)$. So T is in the drifted cone of T_0, in the drifting direction $dl = (-1, -1, 1, 1)$ and the associated drifting vector $dr = (1, 3, 3, 1)$. So $\rho = \frac{dr_{\max}}{dr_{\min}} = 3$.

Following the proof of Proposition 9, we take:

$$\epsilon < \tfrac{\alpha}{11\rho} = \tfrac{\alpha}{33} \qquad \text{and} \qquad \eta < \tfrac{\epsilon}{4(q!+2)} = \tfrac{\alpha}{4\times 33\times(10!+2)} = \tfrac{\alpha}{479001864}.$$

If $\xi < \frac{\eta}{2||dl||_1} = \frac{\eta}{8}$, then $\xi < \frac{\epsilon}{dr_{\min}} = \epsilon$, and T is close enough from T_0 so that $\langle S, T\rangle$ is not free of rank 2.

Moreover if ξ is algebraic of degree $d > 1$ then all the lengths of the intervals of continuity of T are algebraic of degree at least 2.

For instance, we can take $\xi = \frac{\sqrt{2}}{2} r$ where r is a rational number in $]0; \frac{\eta}{8}[$.

5 Generalisations of interval exchange transformations

There are many ways to generalise the notion of interval exchange transformations (IET). For instance, one can allow countably many points of discontinuity, or allow intervals to be flipped, namely allow the derivative to be equal to -1, or allow the differential to be positive (but not necessarily equal to 1). We give here a quick overview of a special case of the second and third possibilities.

5.1 Orientable flipped interval exchange transformations

Definition 9 (OFIET). *An orientable flipped interval exchange transformation (OFIET) on* $[0;1[$ *is a bijection* T *between* $I = [0;1[\times\{0\}\sqcup]0;1]\times\{1\}$ *and itself, which is continuous but on a finite number of points and such that:*

1. *For every* $(x,\epsilon) \in I$*, if* T *is continuous at* (x,ϵ)*, then it is differentiable with regard to the first component. Moreover, if* $T(x,\epsilon) \in [0;1[\times\{\epsilon\}$ *then the derivative of* T *at* (x,ϵ) *is equal to 1:* $T'(x,\epsilon) = 1$*. Conversely, if* $T(x,\epsilon) \in [0;1[\times\{1-\epsilon\}$ *then the derivative of* T *at* (x,ϵ) *is equal to* -1*:* $T'(x,\epsilon) = -1$*.*

2. *For every* (x,ϵ)*,* T *is continuous on the right (resp. on the left) at* (x,ϵ) *if* $\epsilon = 0$ *(resp.* $\epsilon = 1$*).*

One still denotes by $\Delta(T)$ *the set of discontinuities of* T*.*

One defines analogously an OFIET on another interval, a circle or a union of intervals and circles.
Roughly speaking, an OFIET on $[0;1[$ cuts two copies of $[0;1[$ into a finite number of pieces and shuffles them, with the condition that a piece that changes of connected component of I is flipped and a piece that stays in the same connected component of I is not.
Like an IET, an OFIET is equivalent to the data of the interval lengths and the underlying permutation (see [BL] for a complete formalism).
Remark 6: Boissy and Lanneau have studied OFIETs that can be written as the composition of two involutions (these OFIETs are called linear involutions), but their set is not a group. The group they generate is strictly included in OFIET.

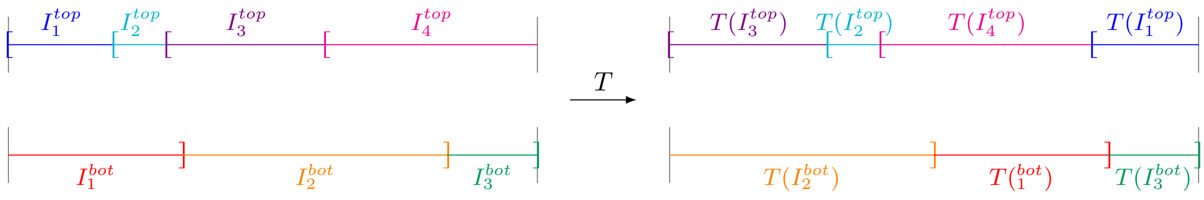


Figure 15: Example of an OFIET without flip

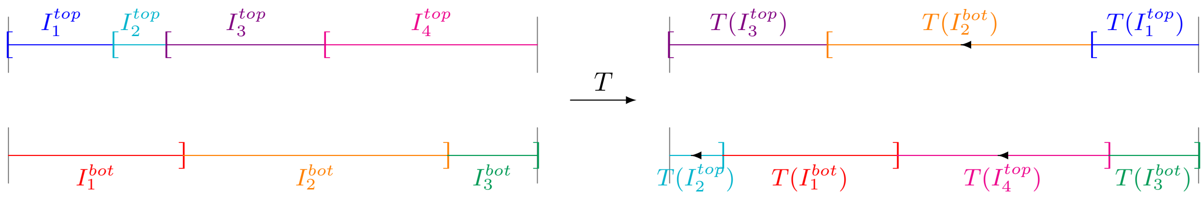


Figure 16: Example of an OFIET with flips

An IET can be seen as an OFIET. Let R be an IET. Roughly speaking, we the dynamic of R the component $[0;1[\times\{0\}$ and the dynamics of R^{-1} in the component $]0;1]\times\{1\}$. See Figure 17.

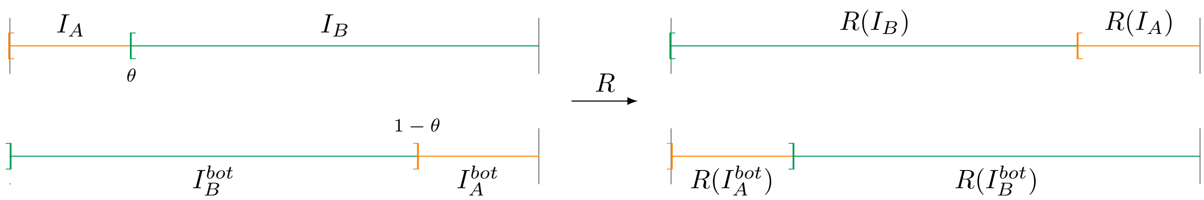


Figure 17: The IET R of Example 1 seen as an OFIET

Proposition 10. *The set of OFIETs on a domain* $\mathcal{D}$*, endowed with the law of composition, is a group.*

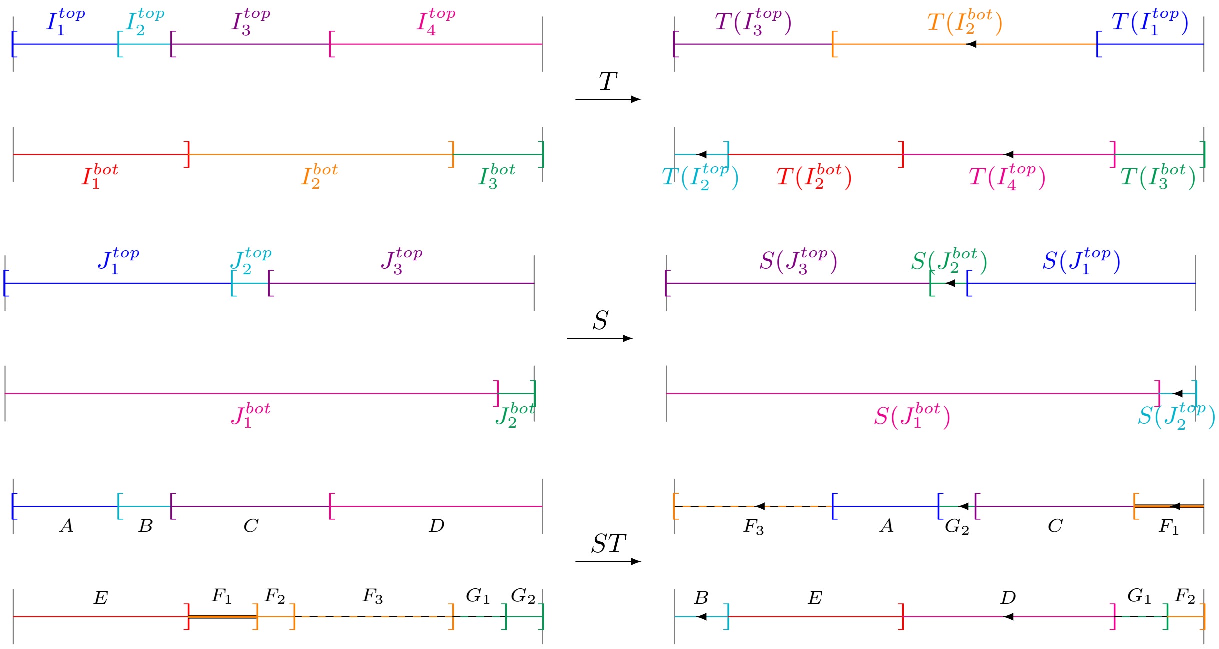


Figure 18: Example of composition of OFIETs

OFIETs are IETs

Theorem 4. *There exists an isomorphism between $OFIET([0;1[)$ and $IET([-1;1[)$.*

Proof. Set:

$$g: \begin{array}{ccc} [-1;1[& \longrightarrow & [0;1[\times\{1\}\cup]0;1]\times\{0\} \\ x & \mapsto & \begin{cases} (-x,0) & \text{if } -1 \leqslant x < 0 \\ (x,1) & \text{if } 0 \leqslant x < 1 \end{cases} \end{array}$$

a bijection between the domain of IET([-1;1[) and the domain of OFIET([0;1[), of inverse:

$$g^{-1}: \begin{array}{ccc} [0;1[\times\{1\}\cup]0;1]\times\{0\} & \longrightarrow & [-1;1[\\ (x,\epsilon) & \mapsto & \begin{cases} -x & \text{if } \epsilon = 0 \\ x & \text{if } \epsilon = 1 \end{cases} \end{array}.$$

We define Ψ to be the following map:

$$\Psi: \begin{array}{ccc} OFIET([0;1[) & \longrightarrow & IET([-1;1[) \\ T & \mapsto & g^{-1}\circ T\circ g \end{array}.$$

The Figure 19 illustrates an example of an OFIET on [0;1[and its image by Ψ.
Of course, we have:

$$\Psi(S\circ T) = g^{-1}\circ S\circ T\circ g = g^{-1}\circ S\circ g\circ g^{-1}\circ T\circ g = \Psi(S)\circ\Psi(T)$$

for every pair (S,T) of OFIETs on [0;1[, and

$$\Psi(g\circ\tilde{T}\circ g^{-1}) = \tilde{T}$$

for every IET $\tilde{T}$ on [-1;1[.
It remains to show that given an OFIET T on [0;1[, $\Psi(T)$ is an IET on [-1;1[, and conversely that given an IET $\tilde{T}$ on [-1;1[, $g\circ\tilde{T}\circ g^{-1}$ is an OFIET on [0;1[.
Let T be an OFIET on [0;1[.
The transformation $\Psi(T)$ is a composition of local isometries, thus it is a local isometry too. The transformations g, T and g^{-1} have finitely many points of discontinuities, so has $\Psi(T)$. We must check that it preserves the orientation.

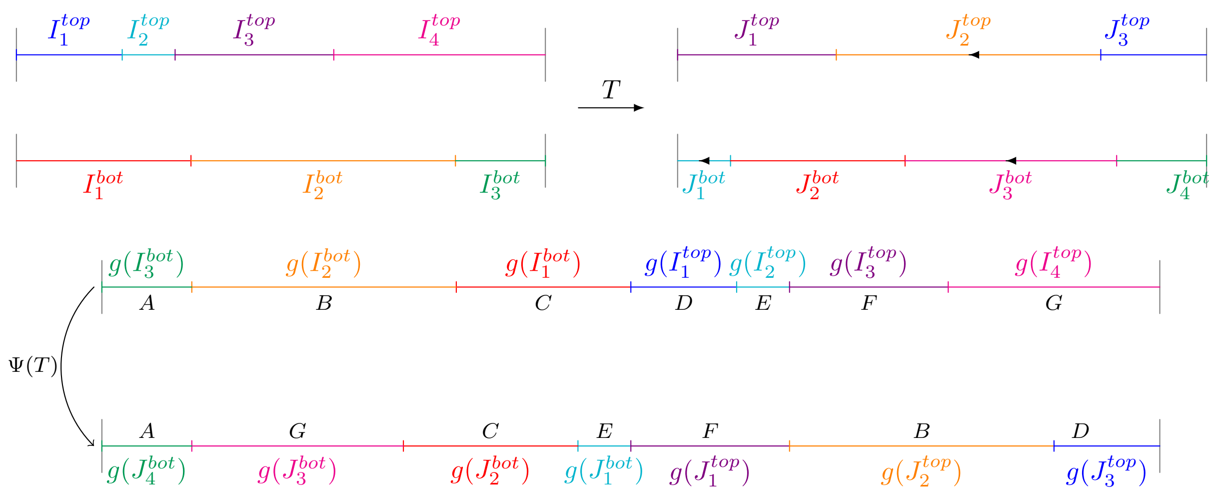


Figure 19: Illustration of the isomorphism Ψ between OFIET([0;1[) and IET([0;1[)

Let $x \in [-1;1[$ be a point at which $\Psi(T)$ is differentiable such that T is also differentiable at $g(x)$.
The following table sums up the different cases.

x is in	g at x	T at $g(x)$	g^{-1} at $T \circ g(x)$	$\Psi(T) = g^{-1} \circ T \circ g$ at x
$[-1;0[$	reverses the orientation	reverses the orientation if $T(-x,1) = (y,0)$	preserves the orientation	preserves the orientation
		preserves the orientation if $T(-x,1) = (y,1)$	reverses the orientation	
$[0;1[$	preserves the orientation	reverses the orientation if $T(x,0) = (y,1)$	preserves the orientation	
		preserves the orientation if $T(x,0) = (y,0)$	reverses the orientation	

So $\Psi(T)$ is an IET on [-1;1[.
Conversely, let $\tilde{T}$ be an IET on [-1;1[.
The transformation $g \circ \tilde{T} \circ g^{-1}$ is a composition of local isometries, thus it is a local isometry too. The transformations g, $\tilde{T}$ and g^{-1} have finitely many points of discontinuities, so has $g \circ \tilde{T} \circ g^{-1}$. We must check that it preserves and reverses the orientation according to the definition of an OFIET.
Let $(x,\epsilon) \in [0;1[\times\{0\} \cup]0;1] \times \{1\}$ be a point at which $g \circ \tilde{T} \circ g^{-1}$ is differentiable such that $\tilde{T}$ is also differentiable at $g^{-1}(x)$. The following table sums up the different cases.

ϵ	g^{-1} at (x,ϵ)	g at $\tilde{T} \circ g^{-1}((x,\epsilon))$	$g \circ \tilde{T} \circ g^{-1}$ at (x,ϵ)
1	reverses the orientation	reverses the orientation if $\tilde{T}(-x) \in [-1;0[$	preserves the orientation and $g \circ \tilde{T} \circ g^{-1}((x,\epsilon)) = (y,1)$
		preserves the orientation if $\tilde{T}(-x) \in [0;1[$	reverses the orientation and $g \circ \tilde{T} \circ g^{-1}((x,\epsilon)) = (y,0)$
0	preserves the orientation	reverses the orientation if $\tilde{T}(x) \in [-1;0[$	reverses the orientation and $g \circ \tilde{T} \circ g^{-1}((x,\epsilon)) = (y,1)$
		preserves the orientation if $\tilde{T}(x) \in [0;1[$	preserves the orientation and $g \circ \tilde{T} \circ g^{-1}((x,\epsilon)) = (y,0)$

In each case, either $g \circ \tilde{T} \circ g^{-1}((x,\epsilon)) = (y,\epsilon)$ and $g \circ \tilde{T} \circ g^{-1}$ preserves the orientation at (x,ϵ), either $g \circ \tilde{T} \circ g^{-1}((x,\epsilon)) = (y,1-\epsilon)$ and $g \circ \tilde{T} \circ g^{-1}$ reverses the orientation at (x,ϵ). So $g \circ \tilde{T} \circ g^{-1}$ is an OFIET on [0;1[.

Thus Ψ is an isomorphism between OFIET([0;1[) and IET([-1;1[). □

5.2 Affine interval exchange transformations

Definition 10 (AIET). *An* affine interval exchange transformation *(AIET) on* $[0;1[$ *is a bijection from* $[0;1[$ *onto itself which is everywhere continuous on the right, continuous except on a finite number of points and differentiable but on a finite set and with constant differential on every interval where it is defined.*
One denotes by $\Delta(T)$ *the union of discontinuities of* T *and of the finite set where its differential* T' *is not defined.*

In the following illustrations, we represent in black the intervals where the AIET is an isometry. The intervals that are expanded (resp. tightenend) by the AIET are in deep (resp. light) colors and their images in light (resp. deep) colors.

Example 6. Let $a \in]0;1[$ and $\theta > 0$ such that $0 < \theta a < 1$. Define $R_{a,\theta}$ by:

$$\forall x \in [0;1[,\ R_{a,\theta}(x) = \begin{cases} \theta a & \text{if } 0 \leqslant x < a \\ \frac{1-\theta a}{1-a}x + \frac{\theta - 1}{1-a}a & \text{if } a \leqslant x < 1 \end{cases}.$$

Then $R_{a,\theta}$ is a continuous AIET with $\Delta(R) = \{a\}$.

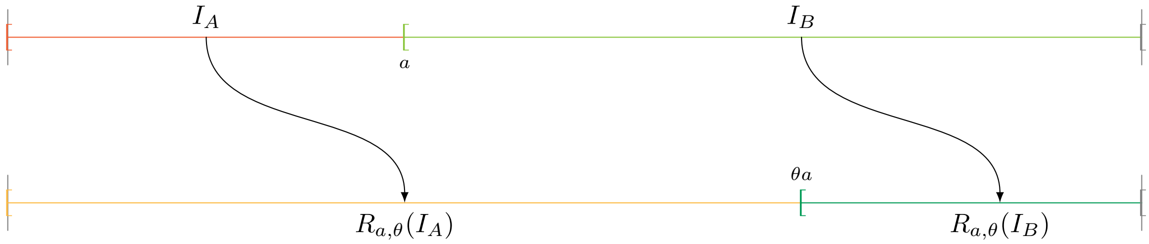


Figure 20: Example of the AIET $R_{a,\theta}$

Example 7. Let T be defined by:

$$\forall x \in [0;1[,\ T(x) = \begin{cases} 2x + \frac{3}{5} & \text{if } x \in A = \left[0; \frac{2}{10}\right[\\ x + \frac{3}{10} & \text{if } x \in B = \left[\frac{2}{10}, \frac{3}{10}\right[\\ \frac{1}{3}x + \frac{3}{10} & \text{if } x \in C = \left[\frac{3}{10}; \frac{6}{10}\right[\\ x - \frac{3}{5} & \text{if } x \in D = \left[\frac{6}{10}; 1\right[\end{cases}.$$

One easily checks that T is an AIET, and A, B, C and D are its intervals of continuity. Figure 21 illustrates T.

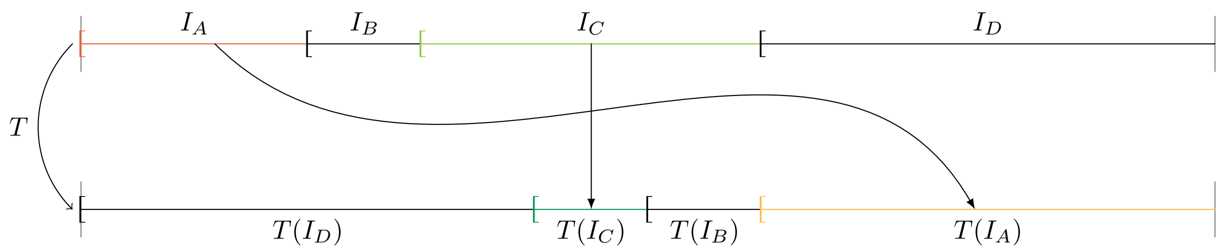


Figure 21: Example of the AIET T

An example of a subgroup

The following theorem is an adaptation of Theorem 8.1 of [DFG13].

Theorem 5. *There is a subgroup $F < AIET([0;1[)$ generated by two elements that contains an isomorphic copy of all finite groups and a free semigroup.*

Proof. Consider an AIET r, defined on $[0;1[$, which is the identity on $[\frac{2}{3};1[$ and that acts on $[0;\frac{2}{3}[$ similarly as $R_{a,\theta}$ of Example 6. More precisely, let $a \in [0;\frac{2}{3}[$ and $\theta > 1$ such that $\theta a \in [\frac{1}{3};\frac{2}{3}[$ and define:

$$\forall x \in [0;1[,\ r(x) = \begin{cases} \theta a & \text{if } 0 \leqslant x < a \\ \frac{1-\theta a}{1-a}x + \frac{\theta - 1}{1-a}a & \text{if } a \leqslant x < \frac{2}{3} \\ x & \text{if } \frac{2}{3} \leqslant x < 1 \end{cases}.$$

Consider the involution s that switches $[\frac{2}{3};1[$ and $[\frac{1}{3};\frac{2}{3}[$ (see Figure 22).

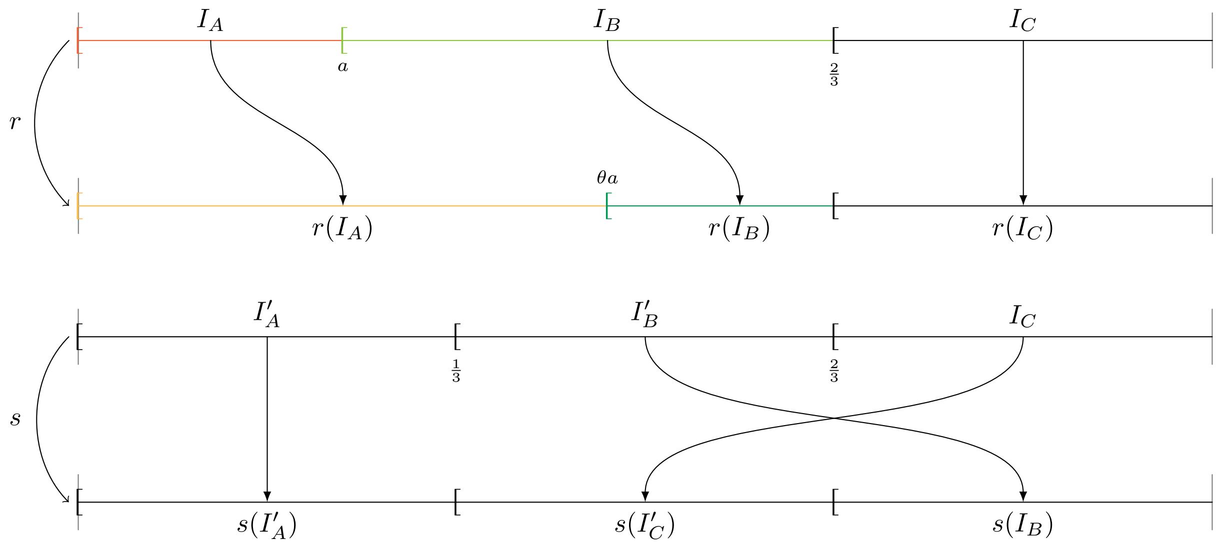


Figure 22: The AIETs r and s

We prove that the subgroup $F = \langle r, s \rangle < AEIT([0;1[)$ contains every permutation group $\mathfrak{S}_n$.
Let $x = \frac{1}{2\theta}$. For every integer $k \geqslant 0$, set $I_k = [r^k(x), r^{k+1}(x)[$.
We claim that $\langle r, s \rangle$ contains, for every $k \geqslant 0$, the involution that exchanges (with a piecewise affine map) I_k and I_{k+1} and that is the identity everywhere else. Thus $\langle r, s \rangle$ contains the symmetric group over the non-negative numbers.
Consider the elements $r' = srs$, $r'' = r^{-1}r'r$, $t = r'^{-1}r''$. Then the element $\sigma_k = r^k t r^{-k}$ exchanges (with a piecewise affine map) I_k and I_{k+1} and is the identity everywhere else. See Figure 23.

Now we prove the second part of the theorem: F contains a free semigroup, namely $G = \langle r, r' \rangle$.
Let W be a non trivial reduced word over the letters r and r'. Denote by w the associated AIET. Both r and r' are non decreasing piecewise affine maps, distinct to the identity, so there exist $x, x' \in [0;1[$ such that $r(x) > x$ and $r'(x') > x'$. Since W is not empty, we have either $w(x) > x$ (if W contains the letter r) or $w(x') > x'$ (if W contains the letter r'). Thus w is not the identity.
Let $W' \neq W$ be another non trivial reduced word over the letters r and r'. Denote by w' the associated AIET and let us show that $w \neq w'$.
Up to replacing W and W' by $V^{-1}W$ and $V^{-1}W'$ where V is the common prefix of W and W', we may assume that W and W' do not begin with the same letter. Up to exchanging the role of W and W', we assume that $W = rW_1$ begins with an r and $W' = r'W_1$ begins with an r'.
Since r and r' act the same way on $[0;\frac{1}{3\theta}[$ (by multiplication by θ), any word of length n sends $[0;\frac{1}{3\theta^n}[$ to $[0;\frac{1}{3}[$ by multiplication by θ^n. We conclude that two words over the letters r and r' with different lengths cannot lead to the same AIET. So if W and W' do not have the same length, then $w \neq w'$.

It remains the case where W and W' have the same length n. Both $w_1 = r^{-1}w$ and $w_1' = r'^{-1}w'$ send $[0; \frac{1}{3\theta^{n-1}}[$ to $[0; \frac{1}{3}[$ by multiplication by θ^{n-1}. In particular:

$$w_1\left(\frac{1}{3\theta^n}\right) = w_1'\left(\frac{1}{3\theta^n}\right) = \frac{1}{3\theta}$$

and

$$w\left(\frac{1}{3\theta^n}\right) = r\left(\frac{1}{3\theta}\right) = \frac{1}{3}$$

whereas

$$w'\left(\frac{1}{3\theta^n}\right) = r'\left(\frac{1}{3\theta}\right) = \frac{2}{3}.$$

So $w \neq w'$.

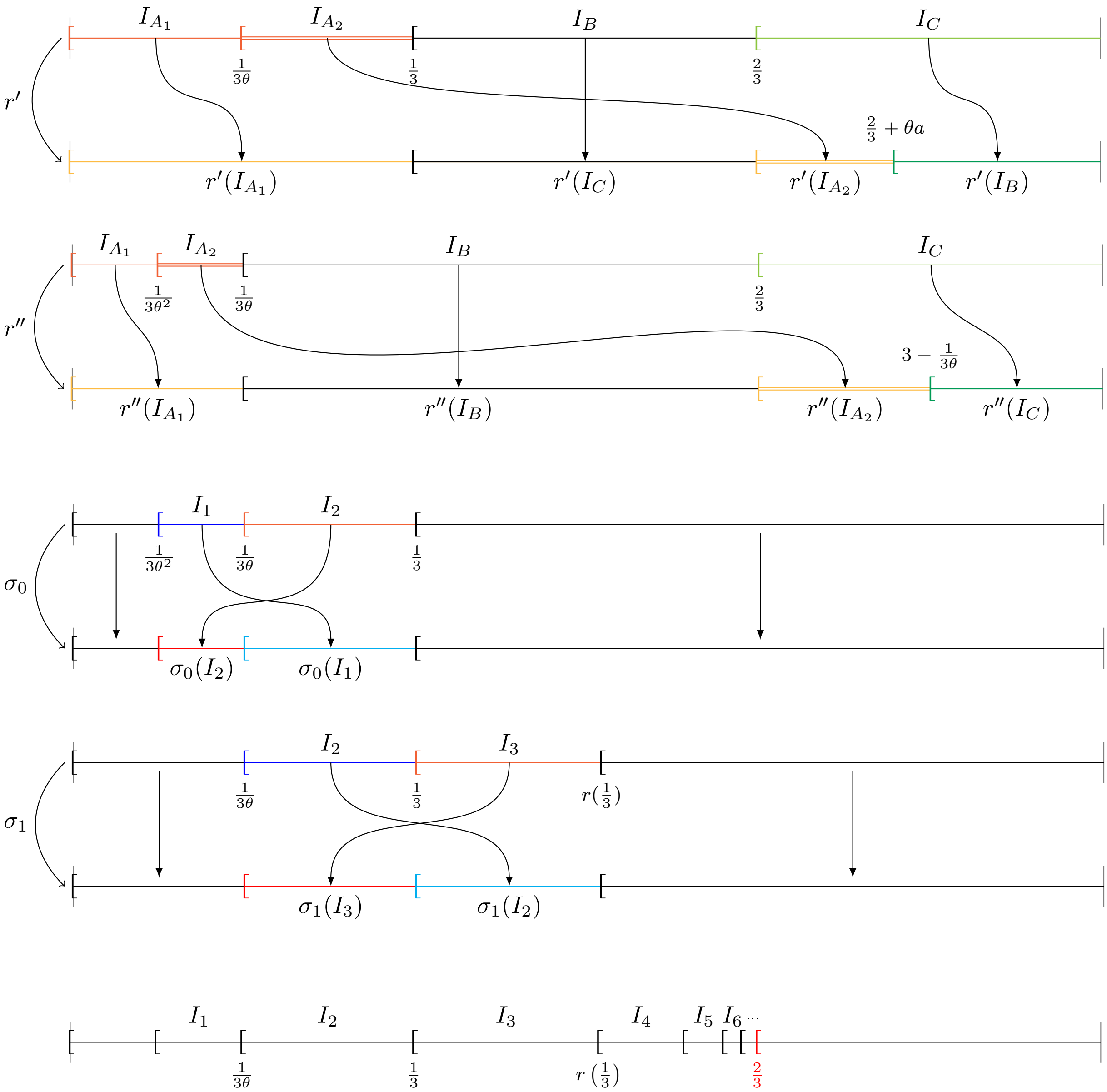


Figure 23: The AIETs r', r'', $t = \sigma_0$, σ_1 and the first few intervals I_k

□

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