ÉCOLE NORMALE SUPÉRIEURE DE RENNES

RESEARCH INTERNSHIP

Differential Geometric formulation of Maxwell's equations and beyond

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Introduction

0.1 Maxwell's equations and how they transform under change of coordinates

A lot of mathematicians have heard of Maxwell's equations during an electromagnetism course in university, mostly in second or third year of license degree. These equations are often introduced with the only variables $\mathbf{E}$ and $\mathbf{B}$, where $\mathbf{E}$ represents the electric field and $\mathbf{B}$ represents the magnetic induction field:

$$\begin{array}{llll} \text{div } \mathbf{B} = 0 & \text{(Maxwell-Thompson)} & \mathbf{curl}\ \mathbf{B} = \mu_0 \mathbf{j} + \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} & \text{(Maxwell-Ampère)} \\ \text{div } \mathbf{E} = \dfrac{\rho}{\varepsilon_0} & \text{(Maxwell-Gauss)} & \mathbf{curl}\ \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{(Maxwell-Faraday)} \end{array}$$

but it will be more convenient to work with Maxwell's equations in *duplex form*, that is introducing the electric flux density $\mathbf{D}$ and the magnetic field $\mathbf{H}$:

$$\begin{array}{ll} \text{div } \mathbf{B} = 0 & \mathbf{curl}\ \mathbf{H} = \mathbf{j} + \dfrac{\partial \mathbf{D}}{\partial t} \\ \text{div } \mathbf{D} = \rho & \mathbf{curl}\ \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}. \end{array}$$

along with the consitutive relations linking $\mathbf{B}$ with $\mathbf{H}$ and $\mathbf{E}$ with $\mathbf{D}$ which, in an isotropic material, reads:

$$\mathbf{B} = \mu_0 \mathbf{H} \qquad \mathbf{D} = \varepsilon_0 \mathbf{E}$$

The unknown variables $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$ are all vectorial functions, depending to space and time as well, namely they are elements of $\mathscr{C}^\infty\left(\mathbb{R}^4, \mathbb{R}^3\right)$. However, math major students, with their knowledge in linear algebra, know that vectors of $\mathbb{R}^3$ can represent *a lot* of mathematical objects, namely, every object in a 3-dimensional vector space. And this construction was, in fact, not the one used by James Clerk Maxwell himself ([7]) to derive these equations, who rather used quaternions, but his approach using imaginary fluid ([4]) is now preferentially stated using vectorial calculus. And the following fact will definitely confuse these students: if φ is a diffeomorphism of the space: $\varphi(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$ and J denotes the Jacobian matrix of this diffeomorphism, Maxwell's equations remain invariant if we set:

$$\begin{array}{ll} \mathbf{E'} = \left(J^{-1}\right)^T \mathbf{E} & \mathbf{H'} = \left(J^{-1}\right)^T \mathbf{H} \\ \mathbf{D'} = \frac{J}{\det J}\mathbf{D} & \mathbf{B'} = \frac{J}{\det J}\mathbf{B} \end{array}$$

Therefore, the duplex form reveals some underlying structure of the 3 dimensional object that these variables represent. While the constitutive relations seem, at first sight, quite useless as it appears that the magnetic induction field $\mathbf{B}$ and the electric displacement field $\mathbf{D}$ coincide respectively with the magnetic field $\mathbf{H}$ and the electric field $\mathbf{E}$, up to a constant, the fact that $\mathbf{B}$ and $\mathbf{D}$ do not transform in the same way as $\mathbf{H}$ and $\mathbf{E}$ under change of coordinates really tell us that they are not the same mathematical objects, even though they are represented in the form of vectors. The goal of the internship was to understand how to didactically introduce Maxwell's equations for advanced mathematicians familiar with differential forms, separating time and space and working in flat space ([6], [7], [8]), explore the possibilities of exploring lagrangian formulations for these equations ([5] [11] [10] [1] [2] [16] [18] [14] [15]) to try to apply Noether's theorem to this formulation ([3], [5], [17] [14]) and finally to try building a similar didactical discourse to introduce Maxwell's equations in more general 4-dimensional manifolds, namely foliated Minkowski space ([9], [12], [13]).

0.2 Tensor bundles and tensor fields

In this report, the notion of a *metric* on a manifold will be extremely useful, but in order to define such an object, the notions of tensor bundles and tensor fields need to be addressed. We will begin this section with a reminder of linear algebra regarding the tensor product of vector spaces:

Definition 0.2.1 (Tensor product of two vector spaces) *Let V and W be two vector spaces over the same field F. The tensor product of the vector spaces V and W is a pair $(E, \otimes)$ where E is a vector space over F and $\otimes$ is a bilinear application:*

$$\otimes \quad : \quad V \times W \quad \longrightarrow \quad E$$

satisfying the following universal property:

> *For all vector space G over F and for all bilinear application:*
>
> $$b \quad : \quad V \times W \quad \longrightarrow \quad G$$
>
> *there exists a unique linear application $\tilde{b} \in \mathscr{L}(E, G)$ such that:*
>
> $$\forall (v, w) \in V \times W, \quad b(v, w) = \tilde{b}(v \otimes w).$$

It is common to denote this tensor product by $V \otimes W$.

Remark 0.2.1 (Isomorphism between tensor products and spaces of multilinear forms) *If V and W are some vector spaces over the same field, then the universal property of the tensor product $V \otimes W$ builds an isomorphism between the space $Hom(V \times W, F)$ of bilinear forms over $V \times W$ and the space $(V \otimes W)^*$. Moreover, the application:*

$$\begin{array}{ccc} V^* \times W^* & \longrightarrow & Hom(V \times W, F) \\ (\alpha, \beta) & \longmapsto & (v, w) \mapsto \alpha(v).\beta(w) \end{array}$$

is bilinear, thus it induces the following application on $V^* \otimes W^*$:

$$\begin{array}{ccc} V^* \otimes W^* & \longrightarrow & Hom(V \times W, F) \\ \alpha \otimes \beta & \longmapsto & (v, w) \mapsto \alpha(v).\beta(w) \end{array}$$

which is an isomorphism. Therefore, we have the following isomorphisms:

$$V^* \otimes W^* \simeq (V \otimes W)^* \simeq Hom(V \times W, F).$$

The same arguments remain valid for a finite tensor product $\displaystyle\bigotimes_{i=1}^{p} V_i$: denoting by $Hom\left(\displaystyle\prod_{i=1}^{p} V_i, F\right)$ the space of p-fold linear forms over $\displaystyle\prod_{i=1}^{p} V_i$, the following isomorphisms remain true:

$$\left(\bigotimes_{i=1}^{p} V_i\right)^* \simeq \bigotimes_{i=1}^{p} V_i^* \simeq Hom\left(\prod_{i=1}^{p} V_i, F\right).$$

Now, in the context of a manifold M, the vector spaces one will obviously encounter are the tangent spaces $T_x M$ and cotangent spaces $T_x^* M$ for $x \in M$, which we will use to construct the tensor bundles:

Definition 0.2.2 (Tensor bundles and tensor fields) *Let M be a smooth manifold. For all $x \in M$ and for all $k, l \in \mathbb{N}$, define the following tensor products:*

$$T_x^{(k,l)}M = (T_x^*M)^{\otimes k} \otimes (T_xM)^{\otimes l} \simeq Hom\left((T_xM)^k \times (T_x^*M)^l, \mathbb{R}\right)$$

where, for $n \in \mathbb{N}$ and a vector space V, the space $V^{\otimes n}$ denotes the space $V \otimes \ldots \otimes V$ with n repeated tensor products. The $(k,l)-$tensor bundle of M is then defined as:

$$T^{(k,l)}M = \bigsqcup_{x \in M} T_x^{(k,l)}M$$

and is equipped with a manifold structure like the tangent bundle TM and the cotangent bundle T^*M by stating that the set $\Gamma\big(T^{(k,l)}M\big)$ of cross-sections τ of $T^{(k,l)}M$ such that the application:

$$\begin{array}{ccc} M & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \tau_x(X_1(x), \ldots, X_k(x); \omega_1(x), \ldots, \omega_l(x)) \end{array}$$

*is smooth for all $(X_i)_{1 \leq i \leq k} \in \mathscr{X}(M)^k$ and for all $(\omega_i)_{1 \leq i \leq l} \in \Omega^1(M)^l$, is the space of smooth cross-sections of $T^{(k,l)}M$. These cross-sections are called **tensor fields**.*

Example 0.2.1 (Some simple examples of tensor bundles and cross-sections)
The set $T^{(0,0)}M$ seems to not fit in our previous definition, as it is difficult to interpret an empty tensor product at first sight. A possible interpretation can arise from the dimension of the spaces $T_x^{(k,l)}M$. Indeed, these spaces have dimension n^{k+l} if M is a manifold of dimension n. Thus, if $k = l = 0$, then we have:

$$\dim\left(T_x^{(k,l)}M\right) = n^0 = 1.$$

Thus we can define this space as follows:

$$\forall x \in M, \quad T_x^{(0,0)}M = \{x\} \times \mathbb{R},$$

leading to the following definition:

$$T^{(0,0)}M = \bigsqcup_{x \in M} \{x\} \times \mathbb{R} = M \times \mathbb{R}.$$

Now, let us deal with more familiar tensor bundles.

- *If $k = 0$ and $l = 1$, we have*

  $$T_x^{(k,l)}M = T_xM.$$

  Thus:

  $$T^{(0,1)}M = TM$$

  which is the well-known tangent bundle of M.

- *For $k = 1$ and $l = 0$, we have:*

  $$T_x^{(k,l)}M = T_x^*M$$

  Thus:

  $$T^{(1,0)}M = T^*M$$

  which is the cotangent bundle of M.

Now, let us consider simple example of cross-sections of the tensor bundles $T^{(k,l)}M$:
For $k=0$ and $l=1$, we have:

$$\Gamma\left(T^{(k,l)}M\right) = \Gamma(TM) = \mathscr{X}(M),$$

which is the set of vector fields on M.
For $k=1$ and $l=0$, we have:

$$\Gamma\left(T^{(k,l)}M\right) = \Gamma(T^*M) = \Omega^1(M),$$

the set of differential 1-forms on M.
More generally, for all $k \in [\![0,n]\!]$, we have:

$$\Omega^k(M) \subset \Gamma\left(T^{(k,0)}M\right).$$

We will see that general relativity uses a lot of these constructions and, in order to make computations look more compact, physicists use a shortcut for sums which is Einstein's summation convention, which is detailed in Appendix A.

One last notion that needs to be addressed in these preliminaries is the notion of Lie group and Lie algebra:

0.3 Lie groups and Lie algebras

Definition 0.3.1 (Lie group) *A group $(G,\star)$ is called a* ***Lie group*** *if G is equipped with a manifold structure so that the applications:*

$$\star \ : \ G\times G \longrightarrow G$$

and

$$\begin{array}{rcccc} Inv & : & G & \longrightarrow & G \\ & & g & \longmapsto & g^{-1} \end{array}$$

are smooth.

Example 0.3.1 (Simple examples of Lie groups)

- $(\mathbb{R}^n,+)$ *is a Lie group for all $n \in \mathbb{N}^*$.*
- *The tores* $\left(\frac{\mathbb{R}^n}{\mathbb{Z}^n},+\right)$ *are also Lie groups.*
- *For $n \in \mathbb{N}^*$, the group $(GL_n(\mathbb{R}),\times)$ of invertible real matrices is a Lie group.*

To a Lie group, one can associate a vector space called the Lie algebra:

Definition 0.3.2 (Lie algebra of a Lie group) *Let G be a Lie group. For all $g \in G$ the following application:*

$$\begin{array}{cccc} R_g & : & G & \longrightarrow & G \\ & & h & \longmapsto & h \star g \end{array}$$

called the right multiplication map is a diffeomorphism with inverse:

$$(R_g)^{-1} = R_{g^{-1}}.$$

Denoting by $\mathscr{X}(G)$ the set of vector fields on the manifold G, the following set:

$$\mathfrak{g} = \left\{ v \in \mathscr{X}(G) \;\middle|\; \forall g, h \in G,\ d(R_g)_h \cdot (v(h)) = v(R_g(h)) = v(h \star g) \right\}$$

is called the Lie algebra of the Lie group G. The vector fields $v \in \mathfrak{g}$ are called right-invariant vector fields.

This definition does not give us more insight about the underlying structure of this Lie algebra, so let us dig into it, by defining a first operation on vector fields:

Definition 0.3.3 (Lie bracket) *Let M be a manifold. Each $v \in \mathscr{X}(M)$ can act on a smooth real-valued function $f \in \Omega^0(M)$ via this application:*

$$\begin{array}{ccc} \Omega^0(M) & \longrightarrow & \Omega^0(M) \\ f & \longmapsto & v(f) : x \mapsto df_x \cdot (v(x)). \end{array}$$

Now, the Lie bracket is a bilinear application:

$$[\cdot,\cdot] \quad : \quad \mathscr{X}(M) \times \mathscr{X}(M) \quad \longrightarrow \quad \mathscr{X}(M)$$

such that:

$$\forall f \in \Omega^0(M),\ \forall v, w \in \mathscr{X}(M), \quad [v,w](f) = v(w(f)) - w(v(f)).$$

Remark 0.3.1 (Is the Lie bracket well-defined ?) *To determine whether the Lie bracket is well-defined we must answer the following questions:*

1. *Is a vector field determined by its action on all smooth real-valued functions ? That is, if $v, w \in \mathscr{X}(M)$ are such that:*

   $$\forall f \in \Omega^0(M), \quad v(f) = w(f),$$

   does that mean that $v = w$?

2. *Is the Lie bracket between two vector fields still a vector field?*

The answers to these questions are detailed in Appendix B.

Now that we checked that the Lie bracket is well-defined, let us explore its properties:

Proposition 0.3.4 *The Lie bracket $[\cdot,\cdot]$ has the following properties:*

1. *Bilinearity:*

   $$\forall \lambda, \mu \in \mathbb{R},\ \forall v, v', w, w' \in \mathscr{X}(M), \quad [\lambda v + v', w] = \lambda [v,w] + [v', w]$$

   and

   $$[v, \mu w + w'] = \mu [v,w] + [v, w'].$$

2. *Skew-symmetry:*
$$\forall v, w \in \mathscr{X}(M), \quad [v,w] = -[w,v]$$

3. *Jacobi identity:*
$$\forall v_1, v_2, v_3 \in \mathscr{X}(M), \quad [v_1,[v_2,v_3]] + [v_2,[v_3,v_1]] + [v_3,[v_1,v_2]] = 0.$$

Refer to Appendix B for the proof.

Proposition 0.3.5 (The structure of a Lie algebra) *Let G be a Lie group and $\mathfrak{g}$ its associated Lie algebra. We have the following properties*

1. $(\mathfrak{g}, +, \cdot)$*, where $\cdot$ denotes the scalar multiplication with a real number, is a real vector space*

2. *The application:*
$$\begin{array}{rcccl} \Phi & : & \mathfrak{g} & \longrightarrow & T_eG \\ & & v & \longmapsto & v(e) \end{array}$$
is an isomorphism of vector spaces

3. *The Lie bracket is an intern composition law on $\mathfrak{g}$, that means:*
$$\forall v, w \in \mathfrak{g}, \quad [v,w] \in \mathfrak{g}.$$

The proof of these 3 statements are to be found in Appendix B.
With these properties, one can define the notion of a Lie algebra in a more abstract way:

Definition 0.3.6 (Lie algebra) *A Lie algebra is a collection $(\mathfrak{g}, +, \cdot, [\cdot,\cdot])$ where:*

- $(\mathfrak{g}, +, \cdot)$ *is a real vector space,*
- $[\cdot,\cdot]$ *is an intern composition law on $\mathfrak{g}$ satisfying the 3 properties of the Lie bracket.*

Example 0.3.2 *A simple example is $(\mathscr{M}_n(\mathbb{R}), +, \cdot, [\cdot,\cdot])$ where $\mathscr{M}_n(\mathbb{R})$ is the set of n-dimensional square matrices, and the Lie bracket $[\cdot,\cdot]$ is the commutator of two matrices, that is:*
$$\begin{array}{rcccl} [\cdot,\cdot] & : & \mathscr{M}_n(\mathbb{R})^2 & \longrightarrow & \mathscr{M}_n(\mathbb{R}) \\ & & (A,B) & \longmapsto & [A,B] = AB - BA \end{array}$$
With this construction, we have:
$$(\mathscr{M}_n(\mathbb{R}), +, \cdot, [\cdot,\cdot]) \simeq (\mathfrak{gl}_n(\mathbb{R}), +, \cdot, [\cdot,\cdot])$$
where $\mathfrak{gl}_n(\mathbb{R})$ denotes the Lie algebra associated to the Lie group $(GL_n(\mathbb{R}), \times)$.

Now that the notions of tensor fields and Lie group and Lie algebra are defined, let us begin with the content of the internhsip.

1 Maxwell's equations in the light of differential forms: the 3+1 formalism

1.1 Why is it relevant to reformulate Maxwell's equations with differential forms ?

Maxwell's equations are formulated using differential operators acting on 3 dimensional vectors in the euclidean space, or acting on scalar functions. We give their expressions in the cartesian coordinate system, that is, $\mathbb{R}^3$ with the metric given by the identity matrix.

- div: $\forall \mathbf{V} := \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$, div $\mathbf{V} \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$, with the expression:

$$\operatorname{div} \mathbf{V} = \partial_x V_x + \partial_y V_y + \partial_z V_z.$$

- **curl**: $\forall \mathbf{V} \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$, **curl** $\mathbf{V} \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$, with the expression:

$$\mathbf{curl}\, \mathbf{V} = \begin{pmatrix} \partial_y V_z - \partial_z V_y \\ \partial_z V_x - \partial_x V_z \\ \partial_x V_y - \partial_y V_x \end{pmatrix}.$$

- ∇: $\forall s \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$, $\nabla s \in \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$, with the expression:

$$\nabla s = \begin{pmatrix} \partial_x s \\ \partial_y s \\ \partial_z s \end{pmatrix}.$$

Denote by $\Omega^k(U)$ the space of differential k-forms on an open subset $U \subset \mathbb{R}^3$ with $k \in [\![0,3]\!]$, we have the following isomorphisms of $\mathscr{C}^\infty(U, \mathbb{R})$-module:

- $\Omega^0(U) := \mathscr{C}^\infty(U, \mathbb{R}) \simeq \Omega^3(U)$,
- $\Omega^1(U) \simeq \mathscr{C}^\infty(U, \mathbb{R}^3) \simeq \Omega^2(U)$,

with the following correspondences:

- $\alpha = f(x,y,z)\,\mathrm{d}x + g(x,y,z)\,\mathrm{d}y + h(x,y,z)\,\mathrm{d}z \in \Omega^1(U) \longleftrightarrow \mathbf{A} = \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{pmatrix} \in \mathscr{C}^\infty(U, \mathbb{R}^3)$,

- $\beta = F(x,y,z)\,\mathrm{d}y \wedge \mathrm{d}z + G(x,y,z)\,\mathrm{d}z \wedge \mathrm{d}x + H(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(U) \longleftrightarrow \mathbf{B} = \begin{pmatrix} F(x,y,z) \\ G(x,y,z) \\ H(x,y,z) \end{pmatrix}$,

- $\gamma = s(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \longleftrightarrow s \in \mathscr{C}^\infty(U, \mathbb{R})$.

Now, recall the formulae for the *exterior derivative* of 0-,1- and 2-forms:

- $\forall s \in \Omega^0(U), \quad \mathrm{d}s = \partial_x s\,\mathrm{d}x + \partial_y s\,\mathrm{d}y + \partial_z s\,\mathrm{d}z \longleftrightarrow \nabla s$,

- $\forall\alpha \in \Omega^1(U)$, $\quad \mathrm{d}\,\alpha = (\partial_x g - \partial_y f)\,\mathrm{d}x\wedge\mathrm{d}y + (\partial_z f - \partial_x h)\,\mathrm{d}z\wedge\mathrm{d}x + (\partial_y h - \partial_z g)\,\mathrm{d}y\wedge\mathrm{d}z \longleftrightarrow$ **curl A**,
- $\forall\beta \in \Omega^2(U)$, $\quad \mathrm{d}\,\beta = (\partial_x F + \partial_y G + \partial_z H)\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \longleftrightarrow$ div **B**.

What we can see is that the differential operators ∇, **curl** and div are all unified in this exterior derivative d as long as this operator is applied on the right object. This exterior derivative can also unify the expressions of div, **curl**, and ∇ in other systems of coordinates (cylindrical or spherical coordinates). Now we can begin to build the set of Maxwell's equations using an important tool: Poincaré's lemma.

Lemma 1.1.1 (Poincaré's lemma) *Let U be an open star-shaped subset of $\mathbb{R}^n$. If $\alpha \in \Omega^k(U)$, $k \in [\![1,n]\!]$ is such that α is closed, that is $\mathrm{d}\,\alpha = 0$, then α is exact, that is $\exists\beta \in \Omega^{k-1}(U)$, $\alpha = \mathrm{d}\,\beta$.*

First step: define the electric displacement **D** and the magnetic induction field **B** via physical densities.

In classical electrodynamics, *electric charge* is a quantifiable quantity: considering a compact 3-dimensional domain $\mathscr{V}$ (it can represent a physical object, or just any domain of the space), this domain has a total electric charge Q that can be calculated by *counting* the charges that are inside of the domain. This leads to define a differential 3-form ρ on $\mathring{\mathscr{V}}$ representing the charge density in the domain:

$$Q = \int_{\mathscr{V}} \rho.$$

Note that this 3-form depends, in general, on the time t. The form ρ, as well as the other forms introduced here, are in fact elements of $\mathscr{C}^\infty\left(\mathbb{R},\Omega\left(\mathring{\mathscr{V}}\right)\right)$.

Now, assuming that the support of ρ is included in $\mathring{\mathscr{V}}$ and that $\mathring{\mathscr{V}}$ is a star-shaped subset of $\mathbb{R}^3$, we can define a differential 2-form on $\mathring{\mathscr{V}}$ D using Poincaré's lemma such that:

$$\mathrm{d}D = \rho,$$

given that $\rho \in \Omega^3\left(\mathring{\mathscr{V}}\right)$ and thus $\mathrm{d}\rho = 0$. In the same fashion, we can introduce a 3-form ρ^m representing the the density of *magnetic charges* in $\mathscr{V}$. However, while an electric dipole exists in nature, a magnetic dipole does not, meaning that the concept of *magnetic charge* has no physical reality. We can then set $\rho^m = 0$ and define a 2-form B such that:

$$\mathrm{d}B = \rho^m = 0.$$

These two equations are nothing less than Maxwell-Thompson and Maxwell-Gauss equations.

Second step: apply Poincaré's Lemma.

We have defined a 3-form ρ, from which stem the electric displacement **D**. If we consider $\partial_t\rho$, we still have a 3-form. Thus, we can define the *electric charge current* **j** via Poincaré's lemma:

$$\exists j \in \mathscr{C}^\infty\left(\mathbb{R},\Omega^2\left(\mathring{\mathscr{V}}\right)\right), \quad \mathrm{d}(-j) = \partial_t\rho.$$

This equation is the continuity equation for the electric charge. Now, given that $\rho = \mathrm{d}D$ and using Schwarz' lemma, we have:

$$\mathrm{d}(-j) = \partial_t\,\mathrm{d}D = \mathrm{d}(\partial_t D),$$

thus

$$\mathrm{d}(\partial_t D + j) = 0.$$

Therefore, applying Poincaré's lemma once again, we can define the magnetic field **H**:

$$\exists H \in \mathscr{C}^\infty\left(\mathbb{R}, \Omega^1\left(\mathring{\mathscr{V}}\right)\right), \quad \mathrm{d}H = j + \partial_t D,$$

and we derive Maxwell-Ampère equation. We are only left with Maxwell-Faraday equation. Given that $\mathrm{d}B = 0$, we also have $\mathrm{d}(\partial_t B) = 0$ using Schwarz' lemma once again. Thus we can define the electric field **E** using Poincaré's lemma one last time:

$$\exists E \in \mathscr{C}^\infty\left(\mathbb{R}, \Omega^1\left(\mathring{\mathscr{V}}\right)\right), \quad \mathrm{d}(-E) = \partial_t B.$$

However, we are left with a difficulty: how can we derive the constitutive relations $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$ using differential forms? All of these objects are represented as 3 dimensional vecotrs, so this notation makes sense, but in the case of differential forms, B and D are 2-forms while H and E are 1-forms: they are not the same geometrical objects. Therefore, we must define the suitable geometrical objects in order to recover these relations, namely the notion of a *metric* as well as the Hodge star operator.

1.2 Pseudo-Riemannian manifolds and metrics

Definition 1.2.1 (Pseudo-Riemannian manifold, metric) *Let M be a smooth manifold and g a tensor of rank $(2,0)$, that is:*

$$g \in \Gamma\left(T^{(2,0)}M\right).$$

If g is symmetric at every point, that is:

$$\forall x \in M,\ \forall u, v \in T_x M, \quad g_x(u,v) = g_x(v,u),$$

and g is non-degenerate at every point, meaning that:

$$\forall x \in M,\ \forall u \in T_x M,\ (\forall v \in T_x M,\ g_x(u,v) = 0) \Longrightarrow (u = 0)$$

then the pair (M,g) is called a pseudo-Riemannian manifold, and g itself is called a metric, or a pseudo-Riemannian metric. When the bilinear form g_x is definite-positive for all $x \in M$, then the metric g is called a Riemannian metric and the pair (M,g) is called a Riemannian manifold.

Example 1.2.1 (Simple examples of Riemannian and pseudo-Riemannian manifolds)

1. *$(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold, where $\langle\cdot,\cdot\rangle$ denotes the canonical scalar product, or dot product of $\mathbb{R}^n$.*
2. *Considering the sphere of center 0 and radius $R > 0$ in 3 dimensions $\mathbb{S}^2(R)$ with spherical coordinates (θ, ϕ), the following application:*

$$g : \begin{array}{ccc} [-\pi, \pi[\times[0, 2\pi[& \longrightarrow & \mathscr{M}_2(\mathbb{R}) \\ (\theta, \phi) & \longmapsto & \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin(\theta)^2 \end{pmatrix} \end{array}$$

defines a Riemannian metric on $\mathbb{S}^2(R)$.

3. *Let $\mathbb{H}$ be the Poincaré upper-half plane, that is:*

$$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}.$$

Define on $\mathbb{H}$ the following application:

$$\begin{array}{rccl} g & : & \mathbb{H} & \longrightarrow & \mathbb{C}^* \otimes \mathbb{C}^* \\ & & z & \longmapsto & g_z : (v,w) \mapsto \frac{\operatorname{Re}(v\overline{w})}{\operatorname{Im}(z)^2}. \end{array}$$

The pair $(\mathbb{H}, g)$ is a well-known Riemannian manifold and g is often called the Poincaré metric.

4. *Define the following quadratic form on $\mathbb{R}^4$:*

$$\eta(t', x, y, z) = -t'^2 + x^2 + y^2 + z^2.$$

This quadratic form induces a bilinear form whose matrix representation is:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This matrix is not definite-positive, but is non-degenerate and induce on $\mathbb{R}^4$ a structure of pseudo-Riemannian manifold called Minkowski space.

This notion of a metric oon a manifold will enable to properly define the Hodge star operator, which will be the key to state the constitutive relations in the language of differential geometry.

1.3 The Hodge star operator

Denoting by $\bigwedge^k(V)$ the vector space of k-linear alternate forms on a F-vector space V, that is the set of all elements ω such that:

$$\begin{array}{rccl} \omega & : & V^k & \longrightarrow & F \\ & & (v_1, \ldots, v_k) & \longmapsto & \omega(v_1, \ldots, v_k) \end{array},$$

$$\forall \sigma \in \mathfrak{S}_k,\ \forall (v_1, \ldots, v_k) \in V^k, \quad \omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) = \varepsilon(\sigma).\omega(v_1, \ldots, v_k).$$

If V is finite-dimensional, and its dimension is denoted by $n \in \mathbb{N}^*$, then $\bigwedge^k(V)$ is also finite-dimensional, and its dimension is $\binom{n}{k}$. Therefore, for all $k \in [\![0, n]\!]$, the spaces $\bigwedge^k(V)$ and $\bigwedge^{n-k}(V)$ are of the same dimension, so one can build an isomorphism between these two spaces. Likewise, one can build an isomorphism between the spaces $\Omega^k(U)$ and $\Omega^{n-k}(U)$ and the Hodge star operator is a somewhat canonical isomorphism between these two spaces, and depends on the geometry of the ambient space as we will see.

Definition 1.3.1 (An inner product on the space of $k-$forms on a pseudo-Riemannian manifold) *Let (M, g) be a pseudo-Riemannian manifold of dimension n and denote by $\left(g^{i,j}(x)\right)_{1 \leq i,j \leq n}$ the inverse matrix of the matrix representation of the bilinear form g_x for $x \in M$. If we denote, for $x \in M$, by $\left(\mathrm{d}x^1, \ldots, \mathrm{d}x^n\right)$*

the local coordinates around x, one can define the following $\Omega^0(M)$-bilinear form:

$$\begin{array}{cccc} \langle\cdot,\cdot\rangle_x & : & \Omega^k(M)^2 & \longrightarrow & \mathbb{R} \\ & & \left(\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_k}, \mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_k}\right) & \longmapsto & \det\begin{pmatrix} g^{i_1,j_1}(x) & \cdots & g^{i_1,j_k}(x) \\ \vdots & \ddots & \vdots \\ g^{i_k,j_1}(x) & \cdots & g^{i_k,j_k}(x) \end{pmatrix} \end{array}$$

This bilinear form is non-degenerate due to the non-degeneracy of the metric g. Therefore it defines an inner product on $\Omega^k(M)$ for all $k \in [\![1,n]\!]$, with $\langle 1,1\rangle_x = 1$ defining the inner product on $\Omega^0(M)$.

Remark 1.3.1 *If g is a Riemannian metric, then, for all $x \in M$ g_x is a scalar product on T_xM. Thus, $\left(g_{i,j}(x)\right)_{1\leq i,j\leq n} \in \mathscr{S}_n^{++}(\mathbb{R})$, the space of $n\times n$ symmetric definite-positive matrices. Therefore, $\langle\cdot,\cdot\rangle_x$ is also a scalar product on $\Omega^k(M)$.*

Thanks to this inner product, we can define the Hodge star operator via this universal property:

Definition 1.3.2 (Hodge star operator) *Let (M,g) be a pseudo-Riemannian manifold. The Hodge star operator, denoted by $*$ is the $\Omega^0(M)$-linear application defined by the following property:*

$$* \;:\; \Omega^k(M) \longrightarrow \Omega^{n-k}(M),$$

$$\forall \alpha,\beta \in \Omega^k(M),\ \forall x \in M, \quad (\alpha\wedge(*\beta))_x = \langle\alpha,\beta\rangle_x\sqrt{|\det(g_x)|}\,\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^n.$$

Does such a definition guarantee the well-defined nature of this operator? You will see in the following examples that it is, and the computations always follow the same pattern.

Example 1.3.1

- $*1 = \sqrt{|\det(g_x)|}\,\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^n$ *in local coordinates. So $*1$ is equal to the canonical volume form on M induced by the metric g on orientable pseudo-Riemannian manifolds. So, considering a function $f \in \Omega^0(M)$, we can define:*

$$\int_M f := \int_M f(*1) = \int_M (*f).$$

- *Considering the Riemannian manifold $\mathbb{S}^2(R)$, we have:*

$$\langle \mathrm{d}\theta, \mathrm{d}\theta\rangle_{(\theta,\phi)} = \frac{1}{R^2},$$

$$\langle \mathrm{d}\phi, \mathrm{d}\phi\rangle_{(\theta,\phi)} = \frac{1}{R^2\sin(\theta)^2},$$

and

$$\sqrt{\left|\det\left(g_{(\theta,\phi)}\right)\right|} = R^2\sin(\theta).$$

So:

$$*\mathrm{d}\theta = \sin(\theta)\,\mathrm{d}\phi,$$

and

$$*\mathrm{d}\phi = -\frac{1}{\sin(\theta)}\,\mathrm{d}\theta.$$

Indeed:

$$\mathrm{d}\theta\wedge(\sin(\theta)\,\mathrm{d}\phi)=\sin(\theta)\,\mathrm{d}\theta\wedge\mathrm{d}\phi=\frac{1}{R^2}R^2\sin(\theta)\,\mathrm{d}\theta\wedge\mathrm{d}\phi$$

and:

$$\mathrm{d}\phi\wedge\left(-\frac{1}{\sin(\theta)}\,\mathrm{d}\theta\right)=\frac{1}{\sin(\theta)}\,\mathrm{d}\theta\wedge\mathrm{d}\phi=\frac{1}{R^2\sin(\theta)^2}R^2\sin(\theta)\,\mathrm{d}\theta\wedge\mathrm{d}\phi.$$

- *Let us consider now the simple example $(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$. The inner product on $\Omega^k(\mathbb{R}^n)$ has then a simple expression:*

$$\forall I=(i_1,\ldots,i_k)\in[\![1,n]\!]^k, i_1<\ldots<i_k, \forall J\in[\![1,n]\!]^k,\quad \langle\mathrm{d}x^I,\mathrm{d}x^J\rangle=\delta_{I,J}.$$

that is that the canonical basis of $\Omega^k(\mathbb{R}^n)$ is an orthonormal basis for the inner product $\langle\cdot,\cdot\rangle$ on $\Omega^k(\mathbb{R}^n)$. Thus, we have:

$$\forall I=(i_1\ldots,i_k), i_1<\ldots<i_k,\quad *\mathrm{d}x^I=\varepsilon(\sigma_{I,\bar{I}})\,\mathrm{d}x^{\bar{I}}$$

where $\bar{I}=(i'_1,\ldots,i'_{n-k}), i'_1<\ldots<i'_{n-k}$ is such that $\{i_1,\ldots,i_k,i'_1,\ldots,i'_{n-k}\}=[\![1,n]\!]$ and $\sigma_{I,\bar{I}}$ is the permutation defined by $(I,\bar{I}):=(i_1,\ldots,i_k,i'_1,\ldots,i'_{n-k})$.

The constitutive relations in Maxwell's equations with the Hodge star operator:

We know that:

$$E\in\Omega^1\left(\mathring{\mathscr{V}}\right),\quad H\in\Omega^1\left(\mathring{\mathscr{V}}\right)$$

and

$$D\in\Omega^2\left(\mathring{\mathscr{V}}\right),\quad B\in\Omega^2\left(\mathring{\mathscr{V}}\right).$$

Thus, we can relate **E** and **D** on the one hand and **H** and **B** on the other hand via the Hodge star operator:

$$D=\varepsilon_0*E,\quad B=\mu_0*H.$$

Indeed, the vectorial expressions

$$\mathbf{D}=\varepsilon_0\mathbf{E},\quad \mathbf{B}=\mu_0\mathbf{H}$$

lead to the following expressions for D and B in the language of differential forms:

$$D=D_x\,\mathrm{d}y\wedge\mathrm{d}z+D_y\,\mathrm{d}z\wedge\mathrm{d}x+D_z\,\mathrm{d}x\wedge\mathrm{d}y=\varepsilon_0(E_x\,\mathrm{d}y\wedge\mathrm{d}z+E_y\,\mathrm{d}z\wedge\mathrm{d}x+E_z\,\mathrm{d}x\wedge\mathrm{d}y),$$

and

$$B=B_x\,\mathrm{d}y\wedge\mathrm{d}z+B_y\,\mathrm{d}z\wedge\mathrm{d}x+B_z\,\mathrm{d}x\wedge\mathrm{d}y=\mu_0(H_x\,\mathrm{d}y\wedge\mathrm{d}z+H_y\,\mathrm{d}z\wedge\mathrm{d}x+H_z\,\mathrm{d}x\wedge\mathrm{d}y),$$

Given that, in $\mathbb{R}^3$ with cartesian coordinates, the metric is simply the identity, we have:

$$*\mathrm{d}x=\mathrm{d}y\wedge\mathrm{d}z,\quad *\mathrm{d}y=\mathrm{d}z\wedge\mathrm{d}x,\quad *\mathrm{d}z=\mathrm{d}x\wedge\mathrm{d}y,$$

we have:

$$D=\varepsilon_0*(E_x\,\mathrm{d}x+E_y\,\mathrm{d}y+E_z\,\mathrm{d}z)=\varepsilon_0*E,$$

and, similarly,

$$B=\mu_0*H.$$

These relations remain true in other linear materials with relative permittivity and permeability being equal to 1, but, if the material is no longer isotropic, the metric used to derive the constitutive relations will have to change, meaning that in vectorial anlysis, the relations will change a bit.

Remark 1.3.2 *We could define the correspondences between the $\Omega^0(U)$-modules $\Omega^1(U)$ (resp. $\Omega^2(U)$) and $\mathscr{C}^\infty(\mathbb{R}^3,\mathbb{R}^3)$ in a more abstract way by saying that for all $\alpha \in \Omega^1(U)$, α and $*\alpha$ should have the same image in $\mathscr{C}^\infty(\mathbb{R}^3,\mathbb{R}^3)$.*

1.4 Pulling back Maxwell's equations to justify how E, B, D and H transform under the action of a diffeomorphism

Let φ be a diffeomorphism of the space $\mathbb{R}^3$ changing the coordinates (x,y,z) into the coordinates (u,v,w):

$$\begin{array}{rcrcl}\varphi & : & \left(\mathbb{R}^3,(\mathrm{d}x,\mathrm{d}y,\mathrm{d}z)\right) & \longrightarrow & \left(\mathbb{R}^3,(\mathrm{d}u,\mathrm{d}v,\mathrm{d}w)\right) \\ & & (x,y,z) & \longmapsto & (u(x,y,z),v(x,y,z),w(x,y,z))\end{array}.$$

For simplicity, we will denote the inverse mapping of φ as:

$$\begin{array}{rcrcl}\varphi^{-1} & : & \left(\mathbb{R}^3,(\mathrm{d}u,\mathrm{d}v,\mathrm{d}w)\right) & \longrightarrow & \left(\mathbb{R}^3,(\mathrm{d}x,\mathrm{d}y,\mathrm{d}z)\right) \\ & & (u,v,w) & \longmapsto & (x(u,v,w),y(u,v,w),z(u,v,w))\end{array}.$$

In the set of coordinates $(\mathrm{d}x,\mathrm{d}y,\mathrm{d}z)$ we have:

$$E(x,y,z) = E_x(x,y,z)\,\mathrm{d}x + E_y(x,y,z)\,\mathrm{d}y + E_z(x,y,z)\,\mathrm{d}z,$$

$$H(x,y,z) = H_x(x,y,z)\,\mathrm{d}x + H_y(x,y,z)\,\mathrm{d}y + H_z(x,y,z)\,\mathrm{d}z,$$

$$D(x,y,z) = D_x(x,y,z)\,\mathrm{d}y\wedge\mathrm{d}z + D_y(x,y,z)\,\mathrm{d}z\wedge\mathrm{d}x + D_z(x,y,z)\,\mathrm{d}x\wedge\mathrm{d}y,$$

$$B(x,y,z) = B_x(x,y,z)\,\mathrm{d}y\wedge\mathrm{d}z + B_y(x,y,z)\,\mathrm{d}z\wedge\mathrm{d}x + B_z(x,y,z)\,\mathrm{d}x\wedge\mathrm{d}y.$$

Given that in the languages of differential forms, only the operators d and ∂_t appear, and, given that the pullback operation $\left(\varphi^{-1}\right)^*$ commutes with these two operators, the pulled back forms $\left(\varphi^{-1}\right)^* E$, $\left(\varphi^{-1}\right)^* B$, $\left(\varphi^{-1}\right)^* H$ and $\left(\varphi^{-1}\right)^* D$ also satisfy Maxwell's equations. Let us compute these forms:

$$\begin{array}{rcl}\left(\varphi^{-1}\right)^* E & = & \left(E_x(x,y,z)\frac{\partial x}{\partial u} + E_y(x,y,z)\frac{\partial y}{\partial u} + E_z(x,y,z)\frac{\partial z}{\partial u}\right)\mathrm{d}u \\ & & +\left(E_x(x,y,z)\frac{\partial x}{\partial v} + E_y(x,y,z)\frac{\partial y}{\partial v} + E_z(x,y,z)\frac{\partial z}{\partial v}\right)\mathrm{d}v \\ & & +\left(E_x(x,y,z)\frac{\partial x}{\partial w} + E_y(x,y,z)\frac{\partial y}{\partial w} + E_z(x,y,z)\frac{\partial z}{\partial w}\right)\mathrm{d}w.\end{array}$$

Denoting by **E'** the image in $\mathscr{C}^\infty\left(\mathbb{R}^3,\mathbb{R}^3\right)$ of this form, we have:

$$\begin{array}{rcl}\mathbf{E'}(u,v,w) & = & \begin{pmatrix} E_x\left(\varphi^{-1}(u,v,w)\right)\frac{\partial x}{\partial u} + E_y\left(\varphi^{-1}(u,v,w)\right)\frac{\partial y}{\partial u} + E_z\left(\varphi^{-1}(u,v,w)\right)\frac{\partial z}{\partial u} \\ E_x\left(\varphi^{-1}(u,v,w)\right)\frac{\partial x}{\partial v} + E_y\left(\varphi^{-1}(u,v,w)\right)\frac{\partial y}{\partial v} + E_z\left(\varphi^{-1}(u,v,w)\right)\frac{\partial z}{\partial v} \\ E_x\left(\varphi^{-1}(u,v,w)\right)\frac{\partial x}{\partial w} + E_y\left(\varphi^{-1}(u,v,w)\right)\frac{\partial y}{\partial w} + E_z\left(\varphi^{-1}(u,v,w)\right)\frac{\partial z}{\partial w}\end{pmatrix} \\ & = & \left(J^{-1}(u,v,w)\right)^T \mathbf{E}\left(\varphi^{-1}(u,v,w)\right)\end{array}$$

where:

$$J(x,y,z) = \begin{pmatrix} \frac{\partial u}{\partial x}(x,y,z) & \frac{\partial u}{\partial y}(x,y,z) & \frac{\partial u}{\partial z}(x,y,z) \\ \frac{\partial v}{\partial x}(x,y,z) & \frac{\partial v}{\partial y}(x,y,z) & \frac{\partial v}{\partial z}(x,y,z) \\ \frac{\partial w}{\partial x}(x,y,z) & \frac{\partial w}{\partial y}(x,y,z) & \frac{\partial w}{\partial z}(x,y,z)\end{pmatrix}$$

denotes the Jacobian matrix of the diffeomorphism φ. The same calculations hold for all 1-forms in $\mathbb{R}^3$, thus, denoting by **H'** the image in $\mathscr{C}^\infty(\mathbb{R}^3,\mathbb{R}^3)$ of the form $\left(\varphi^{-1}\right)^* H$ we have:

$$\mathbf{H'}(u,v,w) = \left(J^{-1}(u,v,w)\right)^T \mathbf{H}\left(\varphi^{-1}(u,v,w)\right).$$

For the 2-forms B and D we have:

$$
\begin{aligned}
\left(\varphi^{-1}\right)^* B \;=\; & \Big(B_x(x,y,z)\left(\tfrac{\partial y}{\partial v}\tfrac{\partial z}{\partial w} - \tfrac{\partial y}{\partial w}\tfrac{\partial z}{\partial v}\right) \\
& + B_y(x,y,z)\left(\tfrac{\partial z}{\partial v}\tfrac{\partial x}{\partial w} - \tfrac{\partial z}{\partial w}\tfrac{\partial x}{\partial v}\right) \\
& + B_z(x,y,z)\left(\tfrac{\partial x}{\partial v}\tfrac{\partial y}{\partial w} - \tfrac{\partial x}{\partial w}\tfrac{\partial y}{\partial v}\right)\Big)\,\mathrm{d}v\wedge\mathrm{d}w \\
& + \Big(B_x(x,y,z)\left(\tfrac{\partial y}{\partial w}\tfrac{\partial z}{\partial u} - \tfrac{\partial y}{\partial u}\tfrac{\partial z}{\partial w}\right) \\
& + B_y(x,y,z)\left(\tfrac{\partial z}{\partial w}\tfrac{\partial x}{\partial u} - \tfrac{\partial z}{\partial u}\tfrac{\partial x}{\partial w}\right) \\
& + B_z(x,y,z)\left(\tfrac{\partial x}{\partial w}\tfrac{\partial y}{\partial u} - \tfrac{\partial x}{\partial u}\tfrac{\partial y}{\partial w}\right)\Big)\,\mathrm{d}w\wedge\mathrm{d}u \\
& + \Big(B_x(x,y,z)\left(\tfrac{\partial y}{\partial u}\tfrac{\partial z}{\partial v} - \tfrac{\partial y}{\partial v}\tfrac{\partial z}{\partial u}\right) \\
& + B_y(x,y,z)\left(\tfrac{\partial z}{\partial u}\tfrac{\partial x}{\partial v} - \tfrac{\partial z}{\partial v}\tfrac{\partial x}{\partial u}\right) \\
& + B_z(x,y,z)\left(\tfrac{\partial x}{\partial u}\tfrac{\partial y}{\partial v} - \tfrac{\partial x}{\partial v}\tfrac{\partial y}{\partial u}\right)\Big)\,\mathrm{d}u\wedge\mathrm{d}v
\end{aligned}
$$

Thus, denoting by **B'** the image of this form in $\mathscr{C}^\infty(\mathbb{R}^3,\mathbb{R}^3)$, we have:

$$
\begin{aligned}
\mathbf{B'}(u,v,w) &= \begin{pmatrix}
\frac{\partial y}{\partial v}\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\frac{\partial z}{\partial v} & \frac{\partial z}{\partial v}\frac{\partial x}{\partial w} - \frac{\partial z}{\partial w}\frac{\partial x}{\partial v} & \frac{\partial x}{\partial v}\frac{\partial y}{\partial w} - \frac{\partial x}{\partial w}\frac{\partial y}{\partial v} \\
\frac{\partial y}{\partial w}\frac{\partial z}{\partial u} - \frac{\partial y}{\partial u}\frac{\partial z}{\partial w} & \frac{\partial z}{\partial w}\frac{\partial x}{\partial u} - \frac{\partial z}{\partial u}\frac{\partial x}{\partial w} & \frac{\partial x}{\partial w}\frac{\partial y}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial y}{\partial w} \\
\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u} & \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}
\end{pmatrix}\mathbf{B}\left(\varphi^{-1}(u,v,w)\right) \\
&= \mathrm{Com}\left(J^{-1}(u,v,w)\right)^T \mathbf{B}\left(\varphi^{-1}(u,v,w)\right) \\
&= \tfrac{J(u,v,w)}{\det(J(u,v,w))}\mathbf{B}\left(\varphi^{-1}(u,v,w)\right)
\end{aligned}
$$

where $\mathrm{Com}(A)$ denotes the *cofactor matrix* of the matrix A. Similarly, it follows:

$$
\mathbf{D'} = \frac{J(u,v,w)}{\det(J(u,v,w))}\mathbf{D}\left(\varphi^{-1}(u,v,w)\right).
$$

2 Maxwell's equations in Minkowski space: the bridge between 3+1 formalism and 4 dimensions

This section focus on the researches made during the last two weeks of the internship in order to draw a suitable framework in order to generalize the 3+1 formalism introduced in section 1 to specific 4-dimensional manifolds.

2.1 Maxwell's equations in flat Minkowski space: introducing the four-potential, the four-current and the Faraday form

In this section, we will work in the flat Minkowski space $\mathbb{R}^4$ equipped with the metric η introduced in Example 1.2.1 and the coordinates (t',x,y,z) with $t'=ct$ where c is the speed of light. In this framework, the *time dependent* forms on $\mathbb{R}^3$ which were used in the last section can be identified as elements of $\Omega(\mathbb{R}^4)$, with 0 component attached to the basis elements containing a $\mathrm{d}t'$. Moreover, we have the following relations:

$$
\mathrm{d}t' = c\,\mathrm{d}t, \quad \text{as well as} \quad \partial_{t'} = \frac{1}{c}\partial_t.
$$

The four-current: The four-current $\bar{j}$ unifies in 4 dimensions the electric charge density ρ and the electric current j with this formula:

$$\bar{j} = c\rho + j \wedge \mathrm{d}t' \in \Omega^3(\mathbb{R}^4).$$

With this definition, we have:

$$\mathrm{d}\,\bar{j} = (\partial_{t'} c\rho + \operatorname{div}\mathbf{j})\,\mathrm{d}t' \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = (\partial_t \rho + \operatorname{div}\mathbf{j})\,\mathrm{d}t' \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Therefore, in this context, the continuity equation becomes:

$$\mathrm{d}\,\bar{j} = 0.$$

And thus, Poincaré lemma can be applied on the form $\bar{j}$:

$$\exists G \in \Omega^2(\mathbb{R}^4), \quad \mathrm{d}G = \bar{j}.$$

We will see that this equation will, in fact, unify Maxwell-Ampère and Maxwell-Gauss equations.

The four-potential: An astute reader would have certainly noticed that, in the last section, we did not apply Poincaré lemma directly on the form B in 3 dimensions, but rather on $\partial_t B$. So let's do it now:

$$\exists A \in \mathscr{C}^\infty\left(\mathbb{R}, \Omega^1\left(\mathring{\mathscr{V}}\right)\right), \quad \mathrm{d}A = B.$$

This form is the 1-form analogue of a vectorial quantity called the *vector-potential* of the magnetic induction field $\mathbf{B}$. In vectorial analysis, this last equation reads:

$$\mathbf{curl}\,\mathbf{A} = \mathbf{B}.$$

And now, coupling this last equation with Maxwell-Faraday equation in 3 dimensions, it reads, thanks to Schwarz' lemma:

$$\mathrm{d}(\partial_t A + E) = 0.$$

Thus, using Poincaré lemma one last time:

$$\exists \phi \in \mathscr{C}^\infty\left(\mathbb{R}, \Omega^0\left(\mathring{\mathscr{V}}\right)\right), \quad \mathrm{d}\phi = \partial_t A + E.$$

Note that ϕ is a scalar quantity, corresponding to the *electric potential*. In vectorial analysis, this equation reads:

$$\nabla\phi = \partial_t \mathbf{A} + \mathbf{E}.$$

In four dimensions, the four-potential $\bar{A}$ unifies both quantities ϕ and A as follows:

$$\bar{A} = \frac{\phi}{c}\,\mathrm{d}t' + A.$$

The faraday form: The Faraday form F is simply defined as:

$$F = \mathrm{d}\bar{A}.$$

Let us compute this exterior derivative:

$$F = \frac{1}{c}\,\mathrm{d}\phi \wedge \mathrm{d}t' + \mathrm{d}t' \wedge \partial_{t'} A + \mathrm{d}A$$

where, in this case, $\mathrm{d}\phi$ and $\mathrm{d}A$ are computed in 3 dimensions. Therefore, it reads:

$$F = \frac{1}{c}\left(\mathrm{d}\phi - \partial_t A\right) \wedge \mathrm{d}t' + B$$

i.e.

$$F = \frac{E}{c} \wedge \mathrm{d}t' + B.$$

Given that $F = \mathrm{d}\bar{A}$, we necessarily have:

$$\mathrm{d}F = 0.$$

Computing $\mathrm{d}F$, the last equation reads:

$$-\frac{1}{c}\,\mathrm{d}E \wedge \mathrm{d}t' + \mathrm{d}t' \wedge \partial_{t'}B + \mathrm{d}B = 0,$$

i.e.

$$-\frac{1}{c}\left(\mathrm{d}E + \partial_t B\right) \wedge \mathrm{d}t' + \mathrm{d}B = 0.$$

Here again, $\mathrm{d}E$ and $\mathrm{d}B$ are computed in 3 dimensions.
This equation, in 3 dimensions is in fact equivalent to the following set of equations:

$$\left\{\begin{array}{rcl} \mathrm{d}E & = & -\partial_t B, \\ \mathrm{d}B & = & 0. \end{array}\right.$$

which are exactly Maxwell-Thompson and Maxwell-Faraday equations unified!

Now, how can we recover the last two equations, which are Maxwell-Gauss and Maxwell-Ampère equations? In this 4-dimensional discussion, we still did not use the Hodge star operator. This operator is now depending on the metric η. Let us compute the hodge star of some basic forms:

$$\left\{\begin{array}{rclrcl} *\mathrm{d}t' & = & -\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z, & *\mathrm{d}x & = & -\mathrm{d}t' \wedge \mathrm{d}y \wedge \mathrm{d}z, \\ *\mathrm{d}y & = & \mathrm{d}t' \wedge \mathrm{d}x \wedge \mathrm{d}z, & *\mathrm{d}z & = & -\mathrm{d}t' \wedge \mathrm{d}x \wedge \mathrm{d}y, \\ *(\mathrm{d}t' \wedge \mathrm{d}x) & = & -\mathrm{d}y \wedge \mathrm{d}z, & *(\mathrm{d}t' \wedge \mathrm{d}y) & = & \mathrm{d}x \wedge \mathrm{d}z, \\ *(\mathrm{d}t' \wedge \mathrm{d}z) & = & -\mathrm{d}x \wedge \mathrm{d}y, & *(\mathrm{d}x \wedge \mathrm{d}y) & = & \mathrm{d}t' \wedge \mathrm{d}z, \\ *(\mathrm{d}x \wedge \mathrm{d}z) & = & -\mathrm{d}t' \wedge \mathrm{d}y, & *(\mathrm{d}y \wedge \mathrm{d}z) & = & \mathrm{d}t' \wedge \mathrm{d}x. \end{array}\right.$$

These relations in mind, we can now compute $*F$:

$$*F = \frac{1}{c\varepsilon_0}D + \mu_0\,\mathrm{d}t' \wedge H = \mu_0\left(cD + \mathrm{d}t' \wedge H\right).$$

And now, computing $\mathrm{d}(*F)$ reads:

$$\mathrm{d}(*F) = \mu_0\left(c\,\mathrm{d}t' \wedge \partial_{t'}D + c\,\mathrm{d}D - \mathrm{d}t' \wedge \mathrm{d}H\right)$$

i.e.

$$\mathrm{d}(*F) = \mu_0\left(\mathrm{d}t' \wedge \left(\partial_t D - \mathrm{d}H\right) + c\,\mathrm{d}D\right).$$

Therefore, the equation:

$$\mathrm{d}(*F) = \mu_0 \bar{j}$$

is equivalent to the set of equations in 3 dimensions:

$$\left\{\begin{array}{rcl} \mathrm{d}H & = & j + \partial_t D, \\ \mathrm{d}D & = & \rho. \end{array}\right.$$

which unifies Maxwell-Ampère and Maxwell-Gauss equations!

In the end, if one would aim to didactically introduce Maxwell's equations in flat Minkowski space using Poincaré lemma, it would require 4 steps:

1. Introduce the four-current $\bar{j}$ via the physical quantities ρ and j, linked together via the continuity equation and argue about the exactness of $\bar{j}$ thanks to this continuity equation.

2. Use Poincaré lemma to $\bar{j}$ to recover $G = \frac{1}{\mu_0} * F$.

3. Define, thanks to the form G, the vector field corresponding to the time variation:

$$\nabla \hat{t} \equiv (-1,0,0,0)$$

and the following application, depending on $t \in \mathbb{R}$:

$$\begin{array}{cccc} f_t & : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^4 \\ & & (x,y,z) & \longmapsto & (ct,x,y,z) \end{array}$$

the time-dependent forms D and H in 3 dimensions as follows:

$$H \quad = \quad -f_t^* \left(\iota_{\nabla \hat{t}} G\right) \quad \text{and} \quad D \quad = \quad \frac{1}{c} f_t^* G$$

Indeed, if we have:

$$G = cD + \mathrm{d}t' \wedge H,$$

then, given that D has no component attached to a $\mathrm{d}t'$, we have:

$$\iota_{\nabla \hat{t}} D = 0$$

and thus:

$$\iota_{\nabla \hat{t}} G = -H.$$

In order to make H a time-dependent form in 3 dimensions, one must finally consider the pulled-back form $f_t^* H$, which is simply in fact the form $H_{(ct,\cdot,\cdot,\cdot)}$.
Moreover, given the expression of f_t, we have, for all $(x,y,z) \in \mathbb{R}^3$:

$$\forall h \in \mathbb{R}^3, \quad (df_t)_{(x,y,z)} \cdot h = (0,h).$$

Thus, it reads:

$$f_t^* \left(\mathrm{d}t' \wedge H\right) = 0$$

and therefore:

$$f_t^* G = cD_{(ct,\cdot,\cdot,\cdot)},$$

so that the equation given by Poincaré lemma gives Maxwell-Ampère and Maxwell-Gauss equations.

4. Argue that $*G = -\frac{1}{\mu_0} F$ is an exact form and define the forms E and B in the same way as D and H:

$$E \quad = \quad c f_t^* \left(\iota_{\nabla \hat{t}} F\right) \quad \text{and} \quad B \quad = \quad f_t^* F$$

so that the equation $\mathrm{d}F = 0$ is equivalent to Maxwell-Thompson and Maxwell-Faraday equations, and argue that the relation:

$$*F = \mu_0 G$$

gives the constitutive relations between E, D, B and H.

5. (Bonus) Apply Poincaré lemma to $*G$ or F and recover the four-potential $\bar{A}$.

2.2 Foliated manifolds, a way to generalize the last discussion?

In this subsection, we discuss the ways to loosen the framework of the flat Minkowski space for Maxwell's equations in order to attempt on generalizing the discussion of the last subsection. Our attempt here was based on the fact that the forms introduced in 4-dimensions have a "time-component" and a "space-component" (they are expressed as a sum of the type "time-dependent form on $\mathbb{R}^3$+ time-dependent form on $\mathbb{R}^3 \wedge \mathrm{d}t'$) so our goal was to define a suitable framework that would be as general as possible in order to define what is "time", and how to single-out "time" from "space" in a 4-dimensional manifold.

2.2.1 The manifold structure

Definition 2.2.1 (Foliation of a manifold) *Let N be an n-dimensional manifold. A p-dimensional foliation of N is a decomposition of N as a disjoint union of connected subsets $(\mathscr{L}_a)_{a\in A}$ which are called the leaves of the foliation such that for all $x \in N$, there exists a local chart (U, φ) around x, with $\varphi = \left(\varphi^1, \dots, \varphi^n\right)$ such that:*

$$\forall a \in A,\ \exists \left(c_a^{p+1}, \dots, c_a^n\right) \in \mathbb{R}^{n-p}, \quad U \cap \mathscr{L}_a = \left\{x \in U \ \middle|\ \varphi^i(x) = c_a^i,\ i \in [\![p+1, n]\!]\right\}.$$

It is common to refer to the codimension of the foliation rather than to the dimension of this foliation.

Remark 2.2.1 *With this definition one can readily see that every leaf $\mathscr{L}_a$ of the foliation is a p-dimensional embedded submanifold of N, with the embedding being the identity for all $a \in A$:*

$$\begin{array}{rccc} f_a & : & \mathscr{L}_a & \longrightarrow & N \\ & & x & \longmapsto & x \end{array}$$

For our case, we will denote by M a 4-dimensional manifold. Let also $\hat{t}$ be a regular scalar field, that is a function:

$$\hat{t} \ : \ M \longrightarrow \mathbb{R}$$

such that for all $x \in M$, its differential $\mathrm{d}\hat{t} \in \Omega^1(M)$ is never 0. In this case, the family $(\Sigma_t)_{t\in\mathbb{R}}$ such that:

$$\forall t \in \mathbb{R}, \quad \Sigma_t = \{x \in M \mid \hat{t}(x) = ct\}.$$

forms a 1-codimensional foliation of M. Moreover, the form $\mathrm{d}\hat{t}$ maps every point $p \in M$ to a basis element of T_p^*M and the scalar field $\hat{t}$ in itself can be used as the first coordinate of every local chart on M.

Our 4-dimensional manifold M will also be supposed to be equipped with a metric η that is Lorentzian, that is it has signature $(-,+,+,+)$ at every point of the manifold. Such a pseudo-riemannian manifold induce isomorphisms between the bundles TM and T^*M, namely the musical isomorphisms:

$$\begin{array}{rcccl} \flat & : & TM & \longrightarrow & T^*M \\ & & X & \longmapsto & X^\flat : Y \mapsto \eta(X, Y). \end{array}$$

and

$$\begin{array}{rcccl} \sharp & : & T^*M & \longrightarrow & TM \\ & & \omega & \longmapsto & \omega^\sharp \end{array}$$

such that for all $x \in M$, for all $v \in T_xM$ and for all $\omega \in T_x^*M$, we have:

$$\eta_x\left(\omega^\sharp, v\right) = \omega(v).$$

Therefore, thanks to these isomorphisms, we can define a vector field on M using our regular scalar field $\hat{t}$ called the gradient of $\hat{t}$ and denoted by:

$$\nabla\hat{t} = (\mathrm{d}\hat{t})^{\sharp}.$$

And thanks to this vector field, we have the following *orthogonal decomposition* of the tangent spaces:

$$\forall p \in M, \quad T_pM = T_p\Sigma_{\frac{1}{c}\hat{t}(p)} \oplus \operatorname{span}\left(\nabla\hat{t}(p)\right)$$

where the orthogonality is linked to the metric. Indeed, given that for all $t \in \mathbb{R}$, Σ_t is a level set of the function $\hat{t}$ we have:

$$\forall p \in \Sigma_t,\ \forall v \in T_p\Sigma_t, \quad \mathrm{d}\hat{t}_p(v) = \eta_p\left(\nabla\hat{t}(p), v\right) = 0.$$

One can then define a projection on the tangent space of the leaves:

$$\begin{array}{rcccl} \pi & : & TM & \longrightarrow & \bigsqcup_{t\in\mathbb{R}} T\Sigma_t. \\ & & v \in T_pM & \longmapsto & \pi(v) \in T_p\Sigma_{\frac{1}{c}\hat{t}(p)} \end{array}$$

In the last days of the internship, this projection appeared quite natural as it is a suitable way to "restrict" space-time vectors of four dimensions to simply space vectors in three dimensions. This projections appeared, in the example of flat Minkowski space, to have a certain notion of *smoothness*. So, we conjectured the following fact: this projection is smooth, in the sense that $\bigsqcup_{t\in\mathbb{R}} T\Sigma_t$ is equipped with a manifold structure by stating that the cross-sections X such that:

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & f_t^*\omega \cdot X_t \end{array}$$

is smooth for all $\omega \in T^*M$ form the set of smooth cross-sections of this space.

Definition 2.2.2 (Induced metric on the leaves of a foliation) *Let $\left(N, (\mathscr{L}_a)_{a\in A}, g\right)$ be a foliated pseudo-Riemannian manifold. For all $a \in A$, the metric $\gamma_a := f_a^* g$ defined as follows:*

$$\forall p \in \mathscr{L}_a,\ \forall v_1, v_2 \in T_p\mathscr{L}_a \quad (\gamma_a)_p(v_1, v_2) = g_p\left((df_a)_p \cdot v_1, (df_a)_p \cdot v_2\right)$$

is called the induced metric on the leaf $\mathscr{L}_a$. Therefore, the pair $(\mathscr{L}_a, \gamma_a)$ is a pseudo-riemannian manifold.

With this operation in mind, we can define the notions of *time-like*, *space-like* and *null* vector fields and submanifolds that appear in general relativity and which will motivate how to single-out "time" from "space" in our four-dimensional manifold.

Definition 2.2.3 (time-like, space-like, null vector field) *Let (M, η) be a 4-dimensional pseudo-riemannian manifold equipped with a lorentzian metric η. A vector field $X \in \mathscr{X}(M)$ is called:*

- *time-like if:*
$$\forall p \in M, \quad \eta_p(X(p), X(p)) < 0,$$
- *space-like if:*
$$\forall p \in M, \quad \eta_p(X(p), X(p)) > 0,$$
- *null if:*
$$\forall p \in M, \quad \eta_p(X(p), X(p)) = 0.$$

Definition 2.2.4 (Time-like, space-like, null embedded manifolds) *Let (N,g) be a pseudo-riemannian metric equipped with a lorentzian metric. An embedded manifold Σ in N via an application Φ is said to be:*

- *time-like if the induced metric on Σ, Φ^*g is lorentzian.*
- *space-like if the induced metric on Σ, Φ^*g is riemannian.*
- *null if the induced metric on Σ, Φ^*g is degenerate.*

These definitions allow us to draw the following parallel. In our situation of a foliated manifold $\left(M, (\Sigma_t)_{t\in\mathbb{R}}, \eta\right)$, we have:

- For all $t \in \mathbb{R}$, Σ_t is space-like if and only if $\nabla\hat{t}$ is time-like,
- for all $t \in \mathbb{R}$, Σ_t is time-like if and only if $\nabla\hat{t}$ is space-like, and
- for all $t \in \mathbb{R}$, Σ_t is null if and only if $\nabla\hat{t}$ is null.

Therefore, we will consider from now on a regular scalar field that is time-like, so that the leaves of the foliation induced with this scalar field are space-like manifolds. Moreover, up to dividing the scalar field by the quantity $\sqrt{-\eta_p\left(\nabla\hat{t}(p), \nabla\hat{t}(p)\right)}$ (which is never zero) at every point p of the manifold, we can consider that the vector field $\nabla\hat{t}$ is *normalized*, that is:

$$\forall p \in M, \quad \eta_p\left(\nabla\hat{t}(p), \nabla\hat{t}(p)\right) = -1.$$

2.2.2 Induced forms on a leaf of a foliated manifold, forms on a leaf that can be identified as foms on the manifold

On a foliated manifold $\left(N, (\mathscr{L}_a)_{a\in A}\right)$, each differential k-form on N induce a k-form on the leaves $\mathscr{L}_a$ with the pullback operation:

$$\begin{array}{cccl} f_a^* & : & \Omega^k(N) & \longrightarrow & \Omega^k(\mathscr{L}_a) \\ & & \omega & \longmapsto & f_a^*\omega_p : (v_1, \dots, v_k) \mapsto \omega_{f_a(p)} \cdot \left((df_a)_p \cdot v_1, \dots, (df_a)_p \cdot v_k\right). \end{array}$$

For our situation, we will be interested in viewing well-behaving family of forms on the leaves of a foliation as a global form on the manifold. Indeed, if $(\omega_t)_{t\in\mathbb{R}}$ is a family of differential forms such that for all $t \in \mathbb{R}$, $\omega_t \in \Omega^k(\Sigma_t)$ where k is an integer independent from the parameter t such that the application:

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \displaystyle\bigsqcup_{t\in\mathbb{R}} \Omega^k\left(\Sigma_t\right) \\ t & \longmapsto & \omega_t \end{array}$$

enables to define a k-form on M via the following application:

$$\begin{array}{ccl} M & \longrightarrow & \bigwedge^k(TM) \\ p & \longmapsto & \omega_p : (v_1, \dots, v_k) \mapsto \left(\omega_{\frac{1}{c}\hat{t}(p)}\right)_p \cdot \left(\pi(v_1), \dots, \pi(v_k)\right). \end{array}$$

Such a definition was motivated by the example in flat Minkowski space where it was easy to define smooth forms in 3 dimensions with respect to time as well as their time derivative. So, I attempted to define the time-derivative of these forms using the *flow* of the vector field $\nabla\hat{t}$.

Definition 2.2.5 (Flow generated by a vector field on a manifold) *Let $v \in \mathscr{X}(M)$. The flow generated by v through a point $x \in M$ is a smooth curve in M parameterized by the function:*

$$\Phi_v^x \quad : \quad \tilde{I} \quad \longrightarrow \quad M$$

satisfying the following differential equation:

$$\begin{cases} (\Phi_v^x)'(\varepsilon) & = & v(\Phi_v^x(\varepsilon)) \quad \forall \varepsilon \in \tilde{I}, \\ \Phi_v^x(0) & = & x. \end{cases}$$

where $\tilde{I}$ denotes the maximal interval of definition of the function Φ_v^x.

In this definition, the flow of a certain vector field v appears to be a function of the real variable, but, fixing ε small enough, we can define a smooth function Φ_ε defined on an open subset of M:

$$\Phi_\varepsilon \quad : \quad x \quad \longmapsto \quad \Phi_v^x(\varepsilon).$$

Therefore, the definition for the time derivative of a form α defined via a family of forms $(\alpha_t)_{t\in\mathbb{R}}$ I have come up with is the following:

$$(\partial_t \alpha)_p = c \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\left(\alpha_{\frac{1}{c}(\hat{t}(p)+\varepsilon)} \right)_{\Phi_{-\varepsilon}(p)} - \left(\alpha_{\frac{1}{c}\hat{t}(p)} \right)_p \right),$$

where Φ_ε denotes the flow of the vector field $\nabla \hat{t}$ for ε fixed. This definition is motivated by the following fact:

Proposition 2.2.6 *Using the same notations as before, if $p \in \Sigma_t$, then for all ε such that $\Phi_\varepsilon(p)$ is defined, we have:*

$$\Phi_\varepsilon(p) \in \Sigma_{t-\frac{\varepsilon}{c}}.$$

PROOF Let $p \in M$, $t \in \mathbb{R}$ so that $p \in \Sigma_t$ and let $I(p)$ be the maximal open interval of definition of the flow $\Phi_{\nabla \hat{t}}^p$. Then the function:

$$\begin{array}{cccc} T & : & I(p) & \longrightarrow & \mathbb{R} \\ & & \varepsilon & \longmapsto & \hat{t}(\Phi_\varepsilon(p)) \end{array}$$

is smooth and its derivative can be computed:

$$\forall \varepsilon \in I(p), \quad T'(\varepsilon) = \mathrm{d}\hat{t}_{\Phi_\varepsilon(p)} \cdot (\nabla \hat{t}(\Phi_\varepsilon(p))) = \eta_{\Phi_\varepsilon(p)}(\nabla \hat{t}(\Phi_\varepsilon(p)), \nabla \hat{t}(\Phi_\varepsilon(p)))$$

given that $\nabla \hat{t} = (\mathrm{d}\hat{t})^\sharp$. Finally, given that $\nabla \hat{t}$ is normalized everywhere, it reads:

$$\forall \varepsilon \in I(p), \quad T'(\varepsilon) = -1.$$

Therefore, we have:

$$\forall \varepsilon \in I(p), \quad \hat{t}(\Phi_\varepsilon(p)) = \hat{t}(p) - \varepsilon = ct - \varepsilon.$$

This definition seems to be satisfying given that it agrees with the notion of *Lie derivative* in the direction of the vector field $\nabla \hat{t}$.

Definition 2.2.7 (Lie derivative) *Let $v \in \mathscr{X}(M)$ and $\omega \in \Omega(M)$. The Lie derivative of ω in the direction of v, denoted as $\mathscr{L}_v \omega$ is defined as:*

$$(\mathscr{L}_v \omega)_p = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left((\Phi_\varepsilon^* \omega)_{\Phi_\varepsilon(p)} - \omega_p \right).$$

Therefore, with this definition it seems that we have, for all forms α defined with a family of forms on the leaves $(\alpha_t)_{t\in\mathbb{R}}$:

$$\mathscr{L}_{\nabla\hat{t}}\alpha = -\frac{1}{c}\partial_t\alpha,$$

but we could not justify this construction before the end of the internship.

Example 2.2.1 (The construction in flat Minkowski space) *In flat Minkowski space, that is* $\mathbb{R}^4$ *equipped with the metric:*

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we consider the regular scalar field:

$$\begin{array}{cccc} \hat{t} & : & \mathbb{R}^4 & \longrightarrow & \mathbb{R} \\ & & (t',x,y,z) & \longmapsto & t' \end{array}$$

and its gradient is then:

$$\nabla\hat{t} \equiv (-1,0,0,0).$$

Indeed:

$$\forall (t',x,y,z),(\tau,h_1,h_2,h_3)\in\mathbb{R}^4, \quad \mathrm{d}\hat{t}_{(t',x,y,z)}\cdot(\tau,h_1,h_2,h_3) = \tau.$$

and:

$$\forall (\tau,h_1,h_2,h_3)\in\mathbb{R}^4, \quad \eta((-1,0,0,0),(\tau,h_1,h_2,h_3)) = \tau.$$

Moreover, this vector field is normalized everywhere:

$$\eta((-1,0,0,0),(-1,0,0,0)) = (-1)(-1)(-1) = -1.$$

This manifold can be equipped with the following foliation $(\Sigma_t)_{t\in\mathbb{R}}$*:*

$$\forall t\in\mathbb{R}, \quad \Sigma_t = \left\{(ct,x,y,z) \;\middle|\; (x,y,z)\in\mathbb{R}^3\right\} = \{ct\}\times\mathbb{R}^3,$$

such that the space $\mathscr{C}^\infty\left(\mathbb{R},\Omega^k\left(\mathbb{R}^3\right)\right)$ *composed of "time-dependent" k-forms on* $\mathbb{R}^3$ *can be easily seen as global forms on* $\mathbb{R}^4$*, indeed, taking* $k=1$ *for example, in coordinate representation, we have that an element* $\alpha\in\mathscr{C}^\infty\left(\mathbb{R},\Omega^k\left(\mathbb{R}^3\right)\right)$ *can be expressed this way:*

$$\forall t\in\mathbb{R},\ \forall (ct,x,y,z)\in\Sigma_t,$$

$$\alpha(t) = \alpha_1(ct,x,y,z)\,\mathrm{d}x + \alpha_2(ct,x,y,z)\,\mathrm{d}y + \alpha_3(ct,x,y,z)\,\mathrm{d}z$$

and such an expression defines directly a form on $\mathbb{R}^4$*, whose time derivative is simply:*

$$\forall p=(t',x,y,z)\in\mathbb{R}^4, \quad (\partial_t\alpha)_p = c\left(\lim_{\varepsilon\to 0}\frac{\alpha_1(t'+\varepsilon,x,y,z)-\alpha_1(t'x,y,z)}{\varepsilon}\,\mathrm{d}x+\cdots\right),$$

that is:

$$\partial_t\alpha = c\left(\partial_{t'}\alpha_1\,\mathrm{d}x + \partial_{t'}\alpha_2\,\mathrm{d}y + \partial_{t'}\alpha_3\,\mathrm{d}z\right)$$

which agrees with the "natural" way to compute the time-derivative in this context.
Now, note that every leaf of this foliation is a 3-dimensional embedded submanifold of $\mathbb{R}^4$ with the embedding being:

$$\begin{array}{cccc} f_t & : & \Sigma_t & \longrightarrow & \mathbb{R}^4 \\ & & (ct,x,y,z) & \longmapsto & (ct,x,y,z), \end{array}$$

whose differential can be represented by the following matrix:

$$df_t = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

at every point. Thus, the metric induced by η on a leaf Σ_t is:

$$\gamma_t = f_t^* \eta = df_t^T \eta \, df_t = I_3,$$

meaning that each leaf of the foliation is naturally equipped with the euclidean metric, which is Riemannian: these leaves are space-like submanifolds, which is consistent with the fact that $\eta(\nabla\hat{t}, \nabla\hat{t}) = -1$, meaning that $\nabla\hat{t}$ is time-like.
In order to finish the study of the flat Minkowski space, let's compute the Lie derivative of a time-dependent form in the direction of $\nabla\hat{t}$ in order to compare it with the time-derivative. In order to compute this Lie derivative, we use ***Cartan's formula****:*

$$\mathscr{L}_{\nabla\hat{t}}\omega = \mathrm{d}(\iota_{\nabla\hat{t}}\omega) + \iota_{\nabla\hat{t}}(\mathrm{d}\,\omega).$$

For example, applying this formula to the 1-form α, we end up with:

$$\mathscr{L}_{\nabla\hat{t}}\alpha = -\partial_{t'}\alpha_1 \,\mathrm{d}x - \partial_{t'}\alpha_2 \,\mathrm{d}y - \partial_{t'}\alpha_3 \,\mathrm{d}z$$

and so:

$$\mathscr{L}_{\nabla\hat{t}}\alpha = -\frac{1}{c}\partial_t \alpha.$$

3 Noether's theorem and Lagrangian formalism of partial differential equations

3.1 Quick overview of Langrangian mechanics

Lagrangian mechanics is part of classical mechanics. So we focus on a particle or a system of particles whose position in time and space are well-defined. We will denote the *generalized coordinates* of the particle by the letter q. In most of the cases, we will denote by a dot above a variable its time derivative.

Lagrangian formalism

Let $n \in \mathbb{N}$ and set that our particles belong to $\mathbb{R}^n$ during the motion. We define a smooth function L, called the *Lagrangian function*:

$$\begin{array}{rcrcl} L & : & \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ & & (q, \dot{q}, t) & \longmapsto & L(q, \dot{q}, t) \end{array}$$

This function *encodes* the motion of the particle. In most of the cases, it is defined to be $L = T - U$ where T denotes the kinetic energy and U denotes the potential energy of the system.

Definition 3.1.1 (Lagrangian action of the system) *Let C be a set of possible trajectories of the system between two points q_1, $q_2 \in \mathbb{R}^n$ reached at times t_1 and $t_2 > t_1$,*
i.e. $C = \{q \in \mathscr{C}^\infty([t_1, t_2], \mathbb{R}^n) \mid q(t_1) = q_1,\ q(t_2) = q_2\}$. The functional

$$\begin{array}{rcrcl} S & : & C & \longrightarrow & \mathbb{R} \\ & & q & \longmapsto & S[q] = \displaystyle\int_{t_1}^{t_2} L\,(q(t), \dot{q}(t), t)\,\mathrm{d}t. \end{array}$$

is called the Lagrangian action of the system.

In classical mechanics, **a particle in motion on which are exerted conservative forces travels along the trajectory that minimizes the Lagrangian action.** This principle is called the *least action principle*. Therefore, in order to find the equations of motion, we must find necessary conditions on L along the minimizing trajectory q.

Proposition 3.1.2 *If the system travels along the trajectory q, then L satisfies the Euler-Lagrange equation :*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)(q(t), \dot{q}(t), t) - \frac{\partial L}{\partial q}(q(t), \dot{q}(t), t) = 0. \quad \forall t \in [t_1, t_2].$$

PROOF Let $q, y \in C$ where q is supposed to minimize the funcional S, and define $\delta q = y - q$ such that $y = q + \delta q$. We can define, for $s \in (-1, 1)$, the trajectory $q + s\delta q$ which is still in C. We can then define the function:

$$\begin{array}{rcrcl} \Sigma & : & (-1, 1) & \longrightarrow & \mathbb{R} \\ & & s & \longmapsto & S[q + s\delta q] \end{array}$$

Given that the Lagrangian function is smooth, our function Σ is differentiable and, given also that q minimze the functional S, it also minimizes Σ. Therefore :

$$\Sigma'(0) = 0.$$

Let us compute this derivative and derive the Euler-Lagrange equations.
For $s \in [-1, 1]$, we have:

$$S[q + s\delta q] - S[q] = \int_{t_1}^{t_2} \Big(L\Big(q(t) + \delta q(t), \dot{q}(t) + \dot{\delta q}(t), t\Big) - L\,(q(t), \dot{q}(t), t)\Big)\,\mathrm{d}t.$$

We can then expand the right hand side to the first order in s thanks to the smoothness of L, and we get:

$$S[q + s\delta q] - S[q] = \int_{t_1}^{t_2} s\left(\frac{\partial L}{\partial q}(q(t), \dot{q}(t), t)\,\delta q(t) + \frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t), t)\,\dot{\delta q}(t) + \underset{s \to 0}{O}(s)\right)\mathrm{d}t.$$

Therefore:

$$\Sigma'(0) = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}(q(t), \dot{q}(t), t)\,\delta q(t) + \frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t), t)\,\dot{\delta q}(t)\right)\mathrm{d}t.$$

Using integration by parts on the second term of the integral, we get:

$$\Sigma'(0) = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}(q(t),\dot{q}(t),t) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)(q(t),\dot{q}(t),t) \right) \delta q(t)\mathrm{d}t + \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2}.$$

But, given that y and q are in C, $y(t_1) = q(t_1)$ and $y(t_2) = q(t_2)$, so $\delta q(t_1) = \delta q(t_2) = 0$. Thus:

$$\Sigma'(0) = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}(q(t),\dot{q}(t),t) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)(q(t),\dot{q}(t),t) \right) \delta q(t)\mathrm{d}t = 0.$$

Given that the choice of y was arbitrary, the last equality holds true for all differentiable functions δq vanishing at t_1 and t_2. Therefore, the integrand must be identically 0 along the trajectory, that is:

$$\forall t \in [t_1,t_2], \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)(q(t),\dot{q}(t),t) - \frac{\partial L}{\partial q}(q(t),\dot{q}(t),t) = 0.$$

Example 3.1.1 (The harmonic oscillator) *We consider a particle of mass m attached to a weightless unstretchable spring of rigidity k. The spring is set to be placed on the horizontal, so we only focus on the coordinate x of the particle. Up to a translation, we can set the equilibrium of the particle to be at $x = 0$. The spring gives the particle elastic potential energy, which is defined as follows:*

$$U(x) = \frac{1}{2}kx^2.$$

We can then define the Lagrangian of the system to be:

$$L(x,\dot{x}) = \frac{1}{2}\left(m\dot{x}^2 - kx^2\right).$$

So if we calculate $\frac{\partial L}{\partial x}$ and $\frac{\partial L}{\partial \dot{x}}$, we get:

$$\frac{\partial L}{\partial x} = -kx$$

and

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}.$$

Therefore, Euler - Lagrange equation gives us the equation of motion of the harmonic oscillator:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\iff m\ddot{x} + kx = 0.$$

3.2 Lagrangian formalism for field theory

In the last section, we were intersted in following the motion of a particle in the space, therefore, the only *independent* variable appearing in the Lagrangian formalism was time. However, in Maxwell's equations, the fields **E**, **D**, **B** and **H** depend not only on time but also on space. Therefore, we must find a way to generalize this formalism for more than one independent variable and more than two dependent variables corresponding to position and velocity. The Lagrangian one can consider could depend on a quantity $u(x_1,\ldots,x_n)$ but also on its derivatives up to order s.

Definition 3.2.1 (independent variables, dependent variables, jet-spaces) *Let $X \simeq \mathbb{R}^p$ be a set of independent variables, whose coordinates are denoted by $(x^1, \ldots, x^p)$, and $U \simeq \mathbb{R}^q$ be a set of dependent variables whose coordinates are denoted by $(u^1, \ldots, u^q)$. If we consider a smooth function*

$$f \;:\; X \;\longrightarrow\; \mathbb{R}$$

there are exactly $p_k := \binom{p+k-1}{k}$ different partial derivatives of order k of f. Indeed they are determined by an unordered k-uple $J = (j_1, \ldots, j_k) \in [\![1, p]\!]^k$ with the formula:

$$\partial^J f = \frac{\partial^k f}{\partial x^{j_1} \ldots \partial x^{j_k}}.$$

We will denote by $\mathscr{J}_k$ the set of such unordered k-tupkes. Therefore, if $f = (f^1, \ldots, f^q)$ is now a vectorial function whose image is included in U, there are $q \times p_k$ different partial derivatives of order k of f. Define a $q \times p_k$-dimensional space $U_k \simeq \mathbb{R}^{qp_k}$ whose coordinates will be denoted as $(u^\alpha_J)_{1 \leq \alpha \leq q, J \in \mathscr{J}_k}$ or, alternatively, by $(u^{(k)})$. The product space:

$$U^{(n)} = U \times \prod_{k=1}^{n} U_k$$

is called the n-th jet space of U. If $M \subset X \times U$ is an open subset of $X \times U$, then the product space:

$$M^{(n)} = M \times \prod_{k=1}^{n} U_k$$

is called the n-th jet-space of M

Remark 3.2.1 (The dimension of $U^{(n)}$) *We have:*

$$\dim\left(U^{(n)}\right) = \sum_{k=0}^{n} q \binom{p+k-1}{k} = q \binom{p+n}{n}.$$

Definition 3.2.2 (Prolongation of a function) *Let $f : X \longrightarrow U$ be a smooth function. The function:*

$$\begin{array}{rccl} pr^{(n)} f & : & X & \longrightarrow & U^{(n)} \\ & & x & \longmapsto & \left(f(x), \left(\partial^{J_1} f(x)\right)_{J_1 \in \mathscr{J}_1}, \ldots, \left(\partial^{J_n} f(x)\right)_{J_n \in \mathscr{J}_n}\right) \end{array}$$

is called the n-th prolongation of the function f, as it prolongs f to the space $U^{(n)}$.

Example 3.2.1 *Consider the space $X \simeq \mathbb{R}^3$ and $U \simeq \mathbb{R}^3$ and let f correspond to the transformation between spherical and cartesian coordinates:*

$$\begin{array}{rcrcl} f & : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & & (r, \theta, \phi) & \longmapsto & (r\cos(\phi)\sin(\theta), r\sin(\phi)\sin(\theta), r\cos(\theta)) \end{array}$$

We have:

$$\begin{array}{rcl} pr^{(1)} f(r, \theta, \phi) & = & \big(r\cos(\phi)\sin(\theta), r\sin(\phi)\sin(\theta), r\cos(\theta); \\ & & \cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta), r\cos(\phi)\cos(\theta), r\sin(\phi)\cos(\theta), \\ & & -r\sin(\phi)\sin(\theta), r\cos(\phi)\sin(\theta), 0\big) \end{array}$$

To introduce the notion of Lagrangian and Euler-Lagrange equations, we need to introduce the notion of partial derivative equations, total derivative and Euler-Lagrange operators.

Definition 3.2.3 (System of partial derivative equations) *Let $X \simeq \mathbb{R}^p$ be a space of independent variables and $U \simeq \mathbb{R}^q$ a space of dependent variables. Let also n and m be some fixed natural numbers and $M \subset X \times U$ be an open subset. A system of equations*

$$\Delta^\alpha \left(x, pr^{(n)} f(x)\right) = 0, \quad \forall \alpha \in [\![1, m]\!]$$

where:

$$\Delta \ : \ M^{(n)} \longrightarrow \mathbb{R}^m$$

is a smooth function, is called a system of m partial derivative equations of order n on the open subset M. The dimension of U is called the number of unknowns.

Example 3.2.2

- *Maxwell's equations introduced in the beginning of this report are a set of 14 partial derivative equations of order 1 depending on the variables (t,x,y,z). Therefore, the relevant spaces to consider are:*

$$X = \mathbb{R}^4, \text{ with coordinates } (t,x,y,z) = (t,\boldsymbol{r}),$$
$$U = \mathbb{R}^{12}, \text{ with coordinates } (E^1,E^2,E^3,H^1,H^2,H^3,B^1,B^2,B^3,D^1,D^2,D^3),$$

and

$$U_1 = \mathbb{R}^{48}, \text{ with coordinates } (E^1_t,E^1_x,E^1_y,E^1_z,\ldots,D^3_t,D^3_x,D^3_y,D^3_z),$$

with U_1 encoding the partial derivatives of order 1 of the unknowns $\boldsymbol{E}$,$\boldsymbol{B}$,$\boldsymbol{D}$,$\boldsymbol{H}$. Here we can define the function Δ to be:

$$\begin{aligned}\Delta(t,\boldsymbol{r},\boldsymbol{E},\boldsymbol{H},\boldsymbol{B},\boldsymbol{D},\ldots) \ = \ \big(&B^1_x + B^2_y + B^3_z; D^1_x + D^2_y + D^3_z - \rho(t,\boldsymbol{r});\\ &H^3_y - H^2_z - j^1(t,\boldsymbol{r}) - D^1_t, H^1_z - H^3_x - j^2(t,\boldsymbol{r}) - D^2_t,\\ &H^2_x - H^1_y - j^3(t,\boldsymbol{r}) - D^3_t; E^3_y - E^2_z + B^1_t, E^1_z - E^3_x + B^2_t, E^2_x - E^1_y + B^3_t;\\ &B^1 - \mu_0 H^1, B^2 - \mu_0 H^2, B^3 - \mu_0 H^3; D^1 - \varepsilon_0 E^1, D^2 - \varepsilon_0 E^2, D^3 - \varepsilon_0 E^3\big)\end{aligned}$$

- *A simpler example is the heat equation:*

$$\partial_t u = \Delta u.$$

It is a system of 1 partial derivative equation of order 2. Thus, we introduce:

$$X = \mathbb{R}^4, \text{ with coordinates } (t,x,y,z),$$
$$U = \mathbb{R}, \text{ with coordinate } (u),$$
$$U_1 = \mathbb{R}^4 \text{ with coordinates } (u_t,u_x,u_y,u_z),$$

and

$$U_2 = \mathbb{R}^{10} \text{ with coordinates } (u_{tt},u_{tx},u_{ty},u_{tz},u_{xx},u_{xy},u_{xz},u_{yy},u_{yz},u_{zz}).$$

This equation is defined by the following function:

$$\begin{array}{cccc} \Delta \ : & X \times U^{(2)} & \longrightarrow & \mathbb{R} \\ & (t,\boldsymbol{r},u,u^{(1)},u^{(2)}) & \longmapsto & u_t - u_{xx} - u_{yy} - u_{zz} \end{array}$$

*Here, the function Δ does not depend on the independent variables $(t,\boldsymbol{r})$. The heat equation is then classified as an **autonomous** partial derivative equation.*

Definition 3.2.4 (Total derivative) *Using the notations introduced before, the differential operator:*

$$\begin{array}{rccl} D_{x^i} & : & \bigcup_{n=0}^{\infty} \mathscr{C}^\infty\left(M^{(n)},\mathbb{R}\right) & \longrightarrow & \bigcup_{n=0}^{\infty} \mathscr{C}^\infty\left(M^{(n)},\mathbb{R}\right) \\ & & F & \longmapsto & \dfrac{\partial F}{\partial x^i} + \sum_{j=0}^{+\infty} \sum_{\alpha=1}^{q} \sum_{J \in \mathscr{J}_j} u^\alpha_{(i,J)} \dfrac{\partial F}{\partial u^\alpha_J} \end{array}$$

is called the total derivative with respect to the variable x^i. Here, the notation (i,J) with $J \in \mathscr{J}_j$ denotes the unordered $j+1$-tuple containing i and all elements of the tuple J.

Remark 3.2.2 (Why is it called the total derivative ?) *Recall that the space U was a space of dependent variables, implying that they somewhat depend on the variables of X. Thus the total derivative is defined to be the derivative with respect to the variable x^i using the chain rule, as if the variables u^α_J depended on it. More rigorously, if we have a function $f = (f^1,\dots,f^q)$ depending on $x \in X$ taking its value in U, then, we have:*

$$\frac{\partial}{\partial x^i}\left(F\left(x, pr^{(n)} f(x)\right)\right) = D_{x^i} F\left(x, pr^{(n+1)} f(x)\right)$$

Taking, for the sake of illustration, $n = 0$, we have:

$$\frac{\partial}{\partial x^i}\left(F\left(x, f(x)\right)\right) = \frac{\partial F}{\partial x^i}\left(x, f(x)\right) + \frac{\partial f^\alpha}{\partial x^i}(x) \frac{\partial F}{\partial u^\alpha}\left(x, f(x)\right)$$

via the chain rule (note that Einstein's summation convention is used here). So the statement holds.

Remark 3.2.3 (About this "infinite" sum) *The total derivative is here defined with an infinite sum. But, given that it takes in argument a certain function in some $\mathscr{C}^\infty\left(M^{(n)},\mathbb{R}\right)$, all the partial derivatives with respect to the variables u^k_J will vanish when $J \in \mathscr{J}_j$ with $j > n$. Therefore, the infinite sum really stops to $j = n$. Also, if $F \in \mathscr{C}^\infty\left(M^{(n)},\mathbb{R}\right)$, then $D_{x^i} F \in \mathscr{C}^\infty\left(M^{(n+1)},\mathbb{R}\right)$.*

To better understand this sum, let us develop it a little bit. We have:

$$D_{x^i} F = \frac{\partial F}{\partial x^i} + \sum_{\alpha=1}^{q}\left(u^\alpha_i \frac{\partial F}{\partial u^\alpha} + \sum_{j=1}^{p} u^\alpha_{ij} \frac{\partial F}{\partial u^\alpha_j} + \sum_{j=1}^{p} \sum_{l=j}^{p} u^\alpha_{ijl} \frac{\partial F}{\partial u^\alpha_{jl}} + \dots\right).$$

Example 3.2.3 *Let us compute some simple examples: For all $i \in [\![1,p]\!]$, we have:*

$$\forall \alpha \in [\![1,q]\!], \quad D_{x^i}\left(u^\alpha\right) = u^\alpha_i$$

and more generally,

$$\forall k \in \mathbb{N},\ \forall J \in \mathscr{J}_k,\ \forall \alpha \in [\![1,q]\!], \quad D_{x^i}\left(u^k_J\right) = u^k_{(i,J)}.$$

Therefore, we have, if we denote by:

$$D_J = D_{x^{j_1}} \dots D_{x^{j_k}}, \quad \forall J \in \mathscr{J}_k$$

we have:

$$\forall k \in \mathbb{N},\ \forall J \in \mathscr{J}_k,\ \forall \alpha \in [\![1,q]\!], \quad D_J\left(u^\alpha\right) = u^\alpha_J.$$

Now that the total derivative is defined, we can define the Euler-Lagrange operators, or variational derivatives.

Definition 3.2.5 (Variational derivative) *Let $\alpha \in [\![1,q]\!]$. The application:*

$$\begin{array}{rccl} \dfrac{\delta}{\delta u^\alpha} & : & \displaystyle\bigcup_{n=0}^{\infty} \mathscr{C}^\infty\left(M^{(n)},\mathbb{R}\right) & \longrightarrow & \displaystyle\bigcup_{n=0}^{\infty} \mathscr{C}^\infty\left(M^{(n)},\mathbb{R}\right) \\ & & F & \longmapsto & \displaystyle\sum_{j=0}^{+\infty}\sum_{J\in\mathscr{J}_j}(-1)^j D_J\left(\frac{\partial F}{\partial u^\alpha_J}\right) \end{array}$$

is called the variational derivative with respect to the variable u^α, or the Euler-Lagrange operator with respect to the variable u^α.

Remark 3.2.4 (On which space is defined the variational derivative of a function.) *We know that if $F \in \mathscr{C}^\infty\left(M^{(n)},\mathbb{R}\right)$, then for all $i \in [\![1,p]\!]$, $D_{x^i}F \in \mathscr{C}^\infty\left(M^{(n+1)},\mathbb{R}\right)$. So, given that the sum in the definition of the variational derivative stops at $j=n$, we have a term of highest order of derivative that is:*

$$\sum_{J\in\mathscr{J}_n}(-1)^n D_J\left(\frac{\partial F}{\partial u^\alpha_J}\right).$$

So, given that $\frac{\partial F}{\partial u^\alpha_J} \in \mathscr{C}^\infty\left(M^{(n)},\mathbb{R}\right)$ too, we have:

$$D_J\left(\frac{\partial F}{\partial u^\alpha_J}\right) \in \mathscr{C}^\infty\left(M^{(2n)},\mathbb{R}\right).$$

Therefore:

$$\frac{\delta F}{\delta u^\alpha} \in \mathscr{C}^\infty\left(M^{(2n)},\mathbb{R}\right).$$

Example 3.2.4 *Of course, this is not a coincidence if it is named Euler-Lagrange operator. Indeed, taking $p=1$, that is $X \simeq \mathbb{R}$ with coordinate (t), we have, if we set:*

$$U \simeq \mathbb{R}^n, \quad \textit{with coordinates } (q^1,\ldots,q^n) = \boldsymbol{q}$$

and

$$U_1 \simeq \mathbb{R}^n, \quad \textit{with coordinates } (q^1_t,\ldots,q^n_t) = \dot{\boldsymbol{q}}$$

we have, for all functions $L \in \mathscr{C}^\infty\left(X\times U^{(1)},\mathbb{R}\right)$:

$$\frac{\delta L}{\delta \boldsymbol{q}}(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) := \begin{pmatrix} \frac{\delta L}{\delta q^1}(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) \\ \vdots \\ \frac{\delta L}{\delta q^n}(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) \end{pmatrix} = \frac{\partial L}{\partial \boldsymbol{q}}(t,\boldsymbol{q},\dot{\boldsymbol{q}}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}).$$

Therefore, the equation:

$$\frac{\delta L}{\delta \boldsymbol{q}} = 0$$

is nothing but Euler-Lagrange equation for the general coordinate $\boldsymbol{q}$ as introduced before.

Proposition 3.2.6 (Variational principle for Euler-Lagrange equations) *Let* $L \in \bigcup_{n=0}^{\infty} \mathscr{C}^{\infty}\left(M^{(n)}, \mathbb{R}\right)$ *and* $\Omega \subset X$ *be an open subset. If* $f \in \mathscr{C}^{\infty}(X, U)$ *minimizes the functional:*

$$I[u] = \int_{\Omega} L\left(x, pr^{(n)} u(x)\right) \mathrm{d}x,$$

then f *satisfy the following system of partial derivative equations:*

$$\forall \alpha \in [\![1, q]\!], \quad \frac{\delta L}{\delta u^{\alpha}}\left(x, pr^{(2n)} f(x)\right) = 0$$

called Euler-Lagrange equations.

PROOF Let $n \in \mathbb{N}$ be such that $f \in \mathscr{C}^{\infty}\left(M^{(n)}, \mathbb{R}\right)$. Now, let $\delta f \in \mathscr{C}^{\infty}\left(M^{(n)}, \mathbb{R}\right)$ with $\operatorname{supp}(\delta f) \subset \Omega$ and consider the following function:

$$\begin{array}{rcccl} \Upsilon & : & (-1,1) & \longrightarrow & \mathbb{R} \\ & & \varepsilon & \longmapsto & I[f + \varepsilon \delta f] \end{array}$$

Given that f is a minimizing curve for the functional I, we must have:

$$\Upsilon'(0) = 0.$$

We already computed the same kind of derivative, we will then compute this derivative quicker:

$$\begin{aligned} \Upsilon(\varepsilon) - \Upsilon(0) &= \int_{\Omega} \left(L\left(x, \mathrm{pr}^{(n)}(f + \varepsilon \delta f)\right) - L\left(x, \mathrm{pr}^{(n)}(f)\right)\right) \mathrm{d}x \\ &= \int_{\Omega} \varepsilon \left(\sum_{\alpha=1}^{q} \sum_{j=0}^{n} \sum_{J \in \mathscr{J}_j} \frac{\partial L}{\partial u_J^{\alpha}}\left(x, \mathrm{pr}^{(n)} f(x)\right) \times \partial^J (\delta f)^{\alpha}(x) + \underset{\varepsilon \to 0}{O}(\varepsilon) \right) \mathrm{d}x. \end{aligned}$$

Thus:

$$\Upsilon'(0) = \int_{\Omega} \left(\sum_{\alpha=1}^{q} \sum_{j=0}^{n} \sum_{J \in \mathscr{J}_j} \frac{\partial L}{\partial u_J^{\alpha}}\left(x, \mathrm{pr}^{(n)} f(x)\right) \times \partial^J (\delta f)^{\alpha}(x) \right) \mathrm{d}x.$$

Then, using Green's formula, we can integrate by parts the terms in the sum, with the boundary term on $\partial\Omega$ disappearing given that δf is supposed to be of compact support in Ω. Thus, it reads:

$$\Upsilon'(0) = \int_{\Omega} \left(\sum_{\alpha=1}^{q} \sum_{j=0}^{n} \sum_{J \in \mathscr{J}_j} (-1)^j \partial^J \left(\frac{\partial L}{\partial u_J^{\alpha}}\left(x, \mathrm{pr}^{(n)} f(x)\right)\right) \times (\delta f)^{\alpha} \right) \mathrm{d}x.$$

Then, using the remark on the total derivative, it reads:

$$\Upsilon'(0) = \int_{\Omega} \left(\sum_{\alpha=1}^{q} \sum_{j=0}^{n} \sum_{J \in \mathscr{J}_j} (-1)^j D_J \left(\frac{\partial L}{\partial u_J^{\alpha}} \right)\left(x, \mathrm{pr}^{(2n)} f(x)\right) \times (\delta f)^{\alpha} \right) \mathrm{d}x.$$

Thus:

$$\int_{\Omega} \left(\sum_{\alpha=1}^{q} \sum_{j=0}^{n} \sum_{J \in \mathscr{J}_j} (-1)^j D_J \left(\frac{\partial L}{\partial u_J^{\alpha}} \right)\left(x, \mathrm{pr}^{(2n)} f(x)\right) \times (\delta f)^{\alpha} \right) \mathrm{d}x. = 0$$

and this holds true for all functions δf with compact support in Ω. Therefore we must have:

$$\forall \alpha \in [\![1,q]\!],\ \forall x \in \Omega, \quad \sum_{j=1}^{n} \sum_{j \in \mathscr{J}_j} (-1)^j D_J \left(\frac{\partial L}{\partial u_J^\alpha} \right) \left(x, \mathrm{pr}^{(2n)} f(x) \right) = 0.$$

That is exactly:

$$\forall \alpha \in [\![1,q]\!],\ \forall x \in \Omega \quad \frac{\delta L}{\delta u^\alpha} = 0.$$

Therefore, this notion of Euler-Lagrange equations motivates the following definition:

Definition 3.2.7 (Lagrangian function associated to a system of partial differential equations) *Using the same notations as earlier, let $\Delta \in \bigcup_{n \in \mathbb{N}} \mathscr{C}^\infty \left(M^{(n)}, \mathbb{R}^q \right)$ define a system of partial differential equations with q equations. A smooth function $L \in \bigcup_{n \in \mathbb{N}} \mathscr{C}^\infty \left(M^{(n)}, \mathbb{R} \right)$ is called a **Lagrangian function associated to the system of partial differential equations** Δ if:*

$$\forall \alpha \in [\![1,q]\!], \quad \Delta^\alpha = \frac{\delta L}{\delta u^\alpha}.$$

Remark 3.2.5 *In this definition, it is important to notice that a system of partial differential equations must have the same number of equations than the number of unknowns in order to have an associated Lagrangian function!*

Theorem 3.2.8 (Every system of partial differential equations has a Lagrangian) *Let $M \subset X \times U$ be an open subset, with $U \simeq \mathbb{R}^q$ and let $\Delta \in \mathscr{C}^\infty \left(M^{(n)}, \mathbb{R}^q \right)$ define a system of partial differential equations of order n with q equations. We say that the function:*

$$\begin{array}{rccl} \Delta^* & : & M^{(n)} \times U^{(n)} & \longrightarrow & \mathbb{R}^q \\ & & (x, u^{(n)}, v^{(n)}) & \longmapsto & \dfrac{\delta}{\delta u^\alpha} \left(\displaystyle\sum_{\beta=1}^{q} v^\beta \Delta^\beta \right) (x, u^{(n)}, v^{(n)}) \end{array}$$

defines the adjoint equation to Δ. The system of $2q$ partial differential equations with $2q$ unknowns defined by the couple (Δ, Δ^*) admits the function:

$$\begin{array}{rccl} L & : & M^{(n)} \times U^{(n)} & \longrightarrow & \mathbb{R} \\ & & (x, u^{(n)}, v^{(n)}) & \longmapsto & \displaystyle\sum_{\beta=1}^{q} v^\beta \Delta^\beta (x, u^{(n)}) \end{array}$$

as a Lagrangian.

PROOF For $\alpha \in [\![1,q]\!]$ we have:

$$\frac{\delta L}{\delta u^\alpha} = (\Delta^*)^\alpha$$

by definition, and:

$$\frac{\delta L}{\delta v^\alpha} = \Delta^\alpha.$$

Now, if one would aim to find a Lagrangian formulation for Maxwell's equations in duplex form with unknowns **E**, **B**, **D** and **H**, there would be a problem: the number of unknowns and the number of equations don't match. So, we first wanted to build a Lagrangian of the partial set of Maxwell's equations with only the constitutive relations and Maxwell-Ampère and Maxwell-Faraday equations:

$$\left\{\begin{array}{rclc} \textbf{curl H} & = & \mathbf{j}+\dfrac{\partial \mathbf{D}}{\partial t}, & \mathbf{B}=\mu_0\mathbf{H}, \\ \textbf{curl E} & = & -\dfrac{\partial \mathbf{B}}{\partial t}, & \mathbf{D}=\varepsilon_0\mathbf{E}. \end{array}\right.$$

This leads to the following Lagrangian:

$$\begin{aligned} L\left(\mathbf{E}^{(1)},\mathbf{B}^{(1)},\mathbf{D}^{(1)},\mathbf{H}^{(1)},\mathbf{V},\mathbf{W},\mathbf{X},\mathbf{Y}\right) &= \mathbf{V}\cdot\left(\textbf{curl H}-\mathbf{j}-\frac{\partial \mathbf{D}}{\partial t}\right)+\mathbf{W}\cdot(\mathbf{B}-\mu_0\mathbf{H})+\mathbf{X}\cdot\left(\textbf{curl E}+\frac{\partial \mathbf{B}}{\partial t}\right) \\ &\quad +\mathbf{Y}\cdot(\mathbf{D}-\varepsilon_0\mathbf{E}). \end{aligned}$$

with the adjoint equation being:

$$\left\{\begin{array}{rclcrcl} \dfrac{\delta L}{\delta \mathbf{E}}=\textbf{curl X}-\varepsilon_0\mathbf{Y} & = & 0, & & \dfrac{\delta L}{\delta \mathbf{B}}=-\dfrac{\partial \mathbf{X}}{\partial t}+\mathbf{W} & = & 0, \\ \dfrac{\delta L}{\delta \mathbf{D}}=\dfrac{\partial \mathbf{V}}{\partial t}+\mathbf{Y} & = & 0, & & \dfrac{\delta L}{\delta \mathbf{H}}=\textbf{curl V}-\mu_0\mathbf{W} & = & 0. \end{array}\right.$$

However this Lagrangian is not satisfying, given that Maxwell-Gauss and Maxwell-Thompson equations are not included in these equations. In the end, we aimed for a Lagrangian formulation for the differential geometric formulation of Maxwell-Ampère and Maxwell-Gauss equations:

$$\mathrm{d}(*F)=\mu_0\bar{j}$$

given that the relation:

$$\mathrm{d}\bar{A}=F$$

gives already Maxwell-Faraday and Maxwell-Thompson equations. Our goal was then to build a 4-form in $\mathbb{R}^4$, having in mind the technique and formalism used by Mitsou in [15] and the well-known Lagrangians for Maxwell's equations in Gaussian units ([14]), leading to the following Lagrangian:

$$\Lambda(\bar{A},F)=\frac{1}{2}F\wedge(*F)+\mu_0\bar{j}\wedge\bar{A}.$$

Indeed, in the context of a Lagrangian $\mathscr{L}$ being a form of maximal degree D being a polynomial in some p-form ϕ and its differential $\mathrm{d}\phi$, Euler-Lagrange equations become:

$$\frac{\partial \mathscr{L}}{\partial \phi}-(-1)^p\,\mathrm{d}\left(\frac{\partial \mathscr{L}}{\partial(\mathrm{d}\phi)}\right)=0$$

where the quantities $\frac{\partial \mathscr{L}}{\partial \phi}$ and $\frac{\partial \mathscr{L}}{\partial(\mathrm{d}\phi)}$ are being computed by sort of expanding $\mathscr{L}$ to the first order in ϕ and $\mathrm{d}\phi$. More precisely, the operators $\frac{\partial}{\partial \phi}$ and $\frac{\partial}{\partial(\mathrm{d}\phi)}$ are anti-derivations of degree $-p$ and $-p-1$ respectively satisfying the following additional property:

$$\frac{\partial}{\partial(\mathrm{d}\phi)}(\mathrm{d}\phi)=\frac{\partial}{\partial \phi}\phi=1.$$

In a graded algebra, like the algebra of differential forms on a manifold $\Omega(M)$, an anti-derivation D of degree p is a linear map from $\Omega(M)$ to itself satisfying the following properties:

$$\forall \alpha, \beta \in \Omega(M), \quad D(\alpha \wedge \beta) = D(\alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge D(\beta),$$

and:

$$\forall \alpha \in \Omega(M), \quad \deg D(\alpha) = \deg \alpha + p.$$

Therefore, these two operators are well-defined on all $\Omega(M)$. In the case of our Lagrangian Λ, we have:

$$\frac{\partial \Lambda}{\partial \bar{A}} = -\mu_0 \bar{j}$$

and

$$\frac{\partial \Lambda}{\partial F} = *F,$$

leading to the equation:

$$-\mu_0 \bar{j} + \mathrm{d}(*F) = 0,$$

which is what we expected.

3.3 Action of a Lie group and symmetries

One of the goals of the internship was to investigate Noether's theorem and try to apply it for the duplex formulation of Maxwell's equations. Noether's theorem states that if a Lie group somewhat leaves a Lagrangian action invariant, then this leads to some conservation law for the associated Euler-Lagrange equations. So let's investigate what it means for a Lie group to leave a Lagrangian action invariant.

Definition 3.3.1 (Local group action) *Let M be a manifold and G a Lie group. A local group action is given by an open subset $\mathscr{U}$ such that:*

$$\{e\} \times M \subset \mathscr{U} \subset G \times M$$

and a smooth map

$$\Psi \quad : \quad \mathscr{U} \quad \longrightarrow \quad M$$

satisfying the following properties:

1. *If $(g,x) \in \mathscr{U}$, $(g, \Psi(h,x)) \in \mathscr{U}$ and $(g \star h, x) \in \mathscr{U}$, then:*

$$\Psi(g, \Psi(h,x)) = \Psi(g \star h, x)$$

2. *For all $x \in M$:*

$$\Psi(e,x) = x$$

3. *If $(g,x) \in \mathscr{U}$, then $\left(g^{-1}, \Psi(g,x)\right) \in \mathscr{U}$ and:*

$$\Psi\left(g^{-1}, \Psi(g,x)\right) = x.$$

It is common to use the following notations:

$$\forall x \in M, \quad G_x = \{g \in G \mid (g,x) \in \mathscr{U}\} \subset G$$

$$\forall g \in G, \quad M_g = \{x \in M \mid (g,x) \in \mathscr{U}\} \subset M$$

Definition 3.3.2 (Infinitesimal group action) *If G is a Lie group locally acting on a manifold M via a smooth function Ψ defined on $\mathscr{U} \subset G \times M$, then there is a subsequent action of the Lie algebra $\mathfrak{g}$ on M via the following function:*

$$\begin{array}{rcccl} \psi & : & \mathfrak{g} & \longrightarrow & \mathscr{X}(M) \\ & & v & \longmapsto & \psi(v) : x \mapsto (d\Psi_x)_e \cdot (v(e)) \end{array}$$

where, for $x \in M$, Ψ_x denotes the function $\Psi(\cdot, x)$.

This group action builds in fact a Lie algebra homomorphism between $\mathfrak{g}$ and $(\mathscr{X}(M), +, \cdot, [\cdot,\cdot])$ that is:

$$\forall v, w \in \mathfrak{g}, \quad [\psi(v), \psi(w)] = \psi([v,w])$$

so that the set $\psi(\mathfrak{g})$ is a finite-dimensional Lie subalgebra of $(\mathscr{X}(M), +, \cdot, [\cdot,\cdot])$. An even more powerful result arises thanks to this application: if $\mathfrak{a} \subset \mathscr{X}(M)$ is a finite-dimensional Lie algebra, then there exists a lie group G locally acting on M such that $\mathfrak{a} = \psi(\mathfrak{g})$:

Proposition 3.3.3 *Let $w_1, \ldots, w_r$ be r linearly independant vector fields on a manifold M spanning a Lie algebra, that is, there exists r^3 constants $(c_{ij}^k)_{1 \leq i,j,k \leq r} \in \mathbb{R}^{r^3}$ such that:*

$$\forall i, j \in [\![1, r]\!], \quad [w_i, w_j] = \sum_{k=1}^{r} c_{ij}^k w_k.$$

Then, there exists a Lie group G such that its Lie algebra $\mathfrak{g}$ is spanned by r vector fields $(v_1, \ldots, v_r)$ satisfying:

$$\forall i, j \in [\![1, r]\!], \quad [v_i, v_j] = \sum_{k=1}^{r} c_{ij}^k v_k,$$

and:

$$\forall i \in [\![1, r]\!], \quad \psi(v_i) = w_j.$$

That means that one can always identify the Lie algebra of our locally acting Lie group with its image $\psi(\mathfrak{g}) \subset \mathscr{X}(M)$ without loss of generality, in the sense that, if G is a given Lie group locally acting on a manifold M via a function Ψ, and $\mathfrak{g}$ denotes its associated Lie algebra, then, if $v_1, \ldots, v_r$ are r linearly independent elements of $\mathfrak{g}$ such that $\psi(v_1), \ldots, \psi(v_r)$ form a basis of $\psi(\mathfrak{g})$, then the vector space spanned by $v_1, \ldots, v_r$ form a Lie algebra of a certain Lie group acting on M with the same infinitesimal action.

Proposition 3.3.4 (Group action on a function) *Let $X \simeq \mathbb{R}^p$ be a space of independent variables, and $U \simeq \mathbb{R}^q$ be a space of dependent variables. Let $M \subset X \times U$ be an open subset and G a Lie group locally acting on M. Let also $f \in \mathscr{C}^\infty(\Omega, U)$, where $\Omega \subset X$ is the domain of definition of f. If, for $g \in G$, the graph Γ_f of f is such that:*

$$\Gamma_f \subset M_g,$$

then g can act on the graph of f as follows:

$$g \cdot \Gamma_f = \{g \cdot (x, f(x)),\ x \in \Omega\}.$$

Given that:

$$e \cdot \Gamma_f = \Gamma_f$$

and G acts smoothly on M, then, there exists $V \in \mathscr{N}_G(e)$ such that:

$$\forall g \in V,\ \exists \tilde{\Omega} \subset X,\ \exists \tilde{f} \in \mathscr{C}^\infty(\tilde{\Omega}, U), \quad \big(g \cdot \Gamma_f\big) \cap \big(\tilde{\Omega} \times U\big) = \Gamma_{\tilde{f}}.$$

Therefore, we say that G can locally act on a function f by stating that:

$$g \cdot f = \tilde{f}.$$

PROOF Denote, for $g \in G$, by $\Xi_g \in \mathscr{C}^\infty(M,X)$ and $\Phi_g \in \mathscr{C}^\infty(M,U)$ two smooth functions such that:

$$\forall g \in G,\ \forall (x,u) \in M, \quad g \cdot (x,u) = (\Xi_g(x,u), \Phi_g(x,u)).$$

Given that G acts smoothly on M, the applications:

$$\begin{array}{rcl} G & \longrightarrow & \mathscr{C}^\infty(M,X) \\ g & \longmapsto & \Xi_g \end{array}$$

and

$$\begin{array}{rcl} G & \longrightarrow & \mathscr{C}^\infty(M,U) \\ g & \longmapsto & \Phi_g \end{array}$$

are smooth. So, denoting by $\mathrm{Id} \times f \in \mathscr{C}^\infty(\Omega, M)$ the function such that:

$$\forall x \in \Omega, \quad \mathrm{Id} \times f(x) = (\mathrm{Id}(x), f(x)) = (x, f(x)).$$

We have:

$$g \cdot \Gamma_f = \left\{ (\tilde{x}, \tilde{u}) = (\Xi_g \circ (\mathrm{Id} \times f)(x), \Phi_g \circ (\mathrm{Id} \times f)(x)),\ x \in \Omega \right\}.$$

Given that e acts as the identity, we have:

$$\Xi_e \circ (\mathrm{Id} \times f) = \mathrm{Id}.$$

Therefore, due to the smoothness of the action of G, there exists a certain neighborhood of the identity $V \in \mathscr{N}_G(e)$ such that:

$$\forall g \in V,\ \forall x \in \Omega, \quad \det\left(d\left(\Xi_g \circ (\mathrm{Id} \times f) \right)_x \right) \neq 0.$$

Thus, by the local inversion theorem, we have:

$$\forall g \in V,\ \exists \Omega' \subset \Omega, \quad \Xi_g \circ (\mathrm{Id} \times f) \text{ is invertible on } \Omega'.$$

And so, we have:

$$\forall g \in V, \exists \tilde{\Omega} \subset X, \quad \left(g \cdot \Gamma_f \right) \cap \left(\tilde{\Omega} \times U \right) = \Gamma_{\tilde{f}}$$

where:

$$\tilde{\Omega} = \Xi_g \circ (\mathrm{Id} \times f)(\Omega')$$

and

$$\tilde{f} = (\Phi_g \circ (\mathrm{Id} \times f)) \circ (\Xi_g \circ (\mathrm{Id} \times f))^{-1}.$$

Proposition 3.3.5 (The flow induces a local one-parameter group action on the manifold) *Let M be a smooth manifold and $v \in \mathscr{X}(M)$. For $x \in M$, denote by $I(x)$ the maximal interval of definition of Φ_v^x, and, for $\varepsilon \in \mathbb{R}$, denote by Φ_ε the application:*

$$\begin{array}{rcl} \{x \in M \mid \varepsilon \in I(x)\} & \longrightarrow & M \\ x & \longmapsto & \Phi_v^x(\varepsilon), \end{array}$$

the following property holds true:

$$\forall x \in M,\ \forall \varepsilon, \delta \in I(x), \quad (\varepsilon + \delta \in I(x)) \Longrightarrow (\varepsilon \in I(\Phi_\delta(x))) \wedge (\Phi_\varepsilon \circ \Phi_\delta(x) = \Phi_{\varepsilon+\delta}(x)).$$

This means that every vector field on M generates a local group action of $\mathbb{R}$ on M with:

$$\{0\} \times M \subset \mathscr{U} = \{(\varepsilon, x) \in \mathbb{R} \times M \mid \varepsilon \in I(x)\}$$

and:

$$\begin{array}{rcccl} \Psi & : & \mathscr{U} & \longrightarrow & M \\ & & (\varepsilon, x) & \longmapsto & \Phi_v^x(\varepsilon). \end{array}$$

You will find in Appendix B a quick discussion about one-parameter subgroups of a Lie group and the exponential map.

Definition 3.3.6 (prolongation of a local group action) *Let G be a Lie group locally acting on an open subset $M \subset X \times U$. We define the n^{th} prolongation of the action of G on M as the action on $M^{(n)}$:*

$$\begin{array}{rcccl} \Psi^{(n)} & : & G \times M^{(n)} & \longrightarrow & M^{(n)} \\ & & \left(g, x, u^{(n)}\right) & \longmapsto & pr^{(n)} g \cdot (x, u) \end{array}$$

such that for all $\left(x_0, u_0^{(n)}\right) \in M^{(n)}$, $pr^{(n)} g \cdot \left(x_0, u_0^{(n)}\right)$ is defined as follows:

- *Take any smooth function $f \in \mathscr{C}^\infty$ such that $pr^{(n)} f(x_0) = u_0^{(n)}$ and $g \cdot f$ is well defined on a small neighborhood of $\tilde{x}_0$, where we denote:*

  $$g \cdot (x_0, u_0) = (\tilde{x}_0, \tilde{u}_0)$$

  with u_0 being the component on U of $u^{(n)}$.

- *Compute $g \cdot f$ and $pr^{(n)}(g \cdot f)$.*

- *We have:*

  $$pr^{(n)} g \cdot \left(x_0, u_0^{(n)}\right) = \left(\tilde{x}_0, pr^{(n)}(g \cdot f)(\tilde{x}_0)\right),$$

  That is:

  $$\Gamma_{pr^{(n)}(g \cdot f)} = pr^{(n)} g \cdot \Gamma_{pr^{(n)} f}.$$

Definition 3.3.7 (Prolongation of vector fields) *Let $v \in \mathscr{X}(M)$ be a vector field. This vector field, as we saw, induce a local group action of $\mathbb{R}$ on M via the flow $\Phi_v^{\cdot}$. The n^{th}-prolongation of the vector field v is defined to be a vector field on $M^{(n)}$ corresponding to an infinitesimal generator of the n^{th}-prolongation of this group action, that is a basis element of the Lie algebra associated to this group action. More precisely:*

$$pr^{(n)} v \left(x, u^{(n)}\right) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left(\Phi_v^{\left(x, u^{(n)}\right)}(\varepsilon)\right)_{\big|\varepsilon=0}.$$

We can compute an explicit expression for $\mathrm{pr}^{(n)} v$ in local coordinates in terms of the local coordinates of v:

$$v = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha \frac{\partial}{\partial u^\alpha}$$

and end up with the following expression:

$$\mathrm{pr}^{(n)} v = \sum_{\alpha=1}^{q} \sum_{j=1}^{+\infty} \sum_{J \in \mathscr{J}_j} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha} + \sum_{i=1}^{p} \xi^i \left(\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{j=1}^{+\infty} \sum_{J \in \mathscr{J}_j} u_{(i,J)}^\alpha \frac{\partial}{\partial u_J^\alpha}\right),$$

where Q_α is the *characteristic* of the vector field v, which is defined as follows:

$$Q_\alpha\left(x,u^{(1)}\right)=\phi_\alpha(x,u)-\sum_{i=1}^{p}\xi^i(x,u)u_i^\alpha.$$

For simplicity, given that the expression in brackets corresponds to the expression of the total derivative operator, we write:

$$\mathrm{pr}^{(n)}v=\mathrm{pr}^{(n)}v_Q+\sum_{i=1}^{p}\xi^iD_{x^i},$$

where we defined:

$$v_Q=\sum_{\alpha=1}^{q}Q_\alpha\frac{\partial}{\partial u^\alpha}$$

and thus:

$$\mathrm{pr}^{(n)}v_Q=\sum_{\alpha=1}^{q}\sum_{j=1}^{+\infty}\sum_{J\in\mathscr{J}_j}D_JQ_\alpha\frac{\partial}{\partial u_J^\alpha}.$$

Definition 3.3.8 (Variational symmetry) *Let $\Omega_0\subset X$ be an open subset. A Lie group G locally acting on an open subset $M\subset\Omega_0\times U$ is a variational symmetry group of the functional:*

$$I[u]=\int_{\Omega_0}L\left(x,pr^{(n)}u(x)\right)\mathrm{d}x$$

if, for all open subset Ω such that $\overline{\Omega}\subset\Omega_0$, for all $f\in\mathscr{C}^\infty(\Omega,U)$ satisfying:

$$\Gamma_f\subset M$$

and for all $g\in G$ such that $\tilde{f}=g\cdot f$ is well-defined over $\tilde{\Omega}$, we have:

$$\int_{\tilde{\Omega}}L\left(\tilde{x},pr^{(n)}\tilde{f}(\tilde{x})\right)\mathrm{d}\tilde{x}=\int_{\Omega}L\left(x,pr^{(n)}f(x)\right)\mathrm{d}x.$$

Theorem 3.3.9 (Infinitesimal criterion of symmetry) *Let G be a connected Lie group locally acting on $M\subset\Omega_0\times U$. Then G is a variational symmetry of the functional I if and only if:*

$$pr^{(n)}v(L)+LDiv\ \xi=0$$

for all $v=\sum_{i=1}^{p}\xi^i\frac{\partial}{\partial x^i}+\sum_{\alpha=1}^{q}\phi_\alpha\frac{\partial}{\partial u^\alpha}\in\mathfrak{g}$.

Remark 3.3.1 *Note that proposition 3.3.3 is used in this definition as we identified the Lie algebra $\mathfrak{g}$ with its image via the infinitesimal group action homomorphism.*

PROOF Let $g\in G$ and let Ξ_g and Φ_g be smooth functions so that:

$$\forall(x,u)\in\mathscr{U},\quad g\cdot(x,u)=(\Xi_g(x,u),\Phi_g(x,u)):=(\tilde{x},\tilde{u}).$$

This transformation is invertible (it's a group action) and thus it can be seen as a change of variables. Therefore, the symmetry condition can be rewritten as:

$$\int_\Omega L\left(x,\mathrm{pr}^{(n)}(g\cdot f)(x)\right)\det\left(J_g\left(x,\mathrm{pr}^{(1)}f(x)\right)\right)\mathrm{d}x=\int_\Omega L\left(x,\mathrm{pr}^{(n)}f(x)\right)\mathrm{d}x,$$

wher J_g denotes the Jacobian matrix of the transformation, whose entries are:

$$J_g^{ij} = D_{x^i} \Xi_g^j.$$

The last equality must hold true for all open subset Ω with closure inside of Ω_0, and for all smooth functions. Therefore we must have:

$$\forall \left(x, u^{(n)}\right) \in M^{(n)}, \quad L\left(\mathrm{pr}^{(n)} g \cdot \left(x, u^{(n)}\right)\right) \det\left(J_g\left(x, u^{(1)}\right)\right) = L\left(x, u^{(n)}\right).$$

In particular, this equation must hold true for all group elements of the form $g_\varepsilon = \exp(\varepsilon v)$ (see Appendix section B.2) where $v \in \mathfrak{g}$ and $\varepsilon \in \mathbb{R}$. Therefore, differentiating the expression

$$L\left(\mathrm{pr}^{(n)} g_\varepsilon \cdot \left(x, u^{(n)}\right)\right) \det\left(J_{g_\varepsilon}\left(x, u^{(1)}\right)\right) = L\left(x, u^{(n)}\right)$$

with respect to ε, it reads:

$$\left(\mathrm{pr}^{(n)} v(L) + L \mathrm{Div}\ \xi\right) \det\left(J_{g_\varepsilon}\right) = 0,$$

given that:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\det\left(J_{g_\varepsilon}\left(x, u^{(1)}\right)\right)\right) = \mathrm{Div}\ \xi\left(\mathrm{pr}^{(1)} g_\varepsilon \cdot \left(x, u^{(1)}\right)\right) \det\left(J_{g_\varepsilon}\left(x, u^{(1)}\right)\right).$$

Therefore, when $\varepsilon = 0$, J_{g_ε} is the identity matrix, leading to:

$$\mathrm{pr}^{(n)} v(L) + L \mathrm{Div}\ \xi = 0.$$

The converse will not be proved here and can be found in [17].

Theorem 3.3.10 (Noether's theorem) *Let G be a Lie group locally acting on $M \subset \Omega_0 \times U$. Let $v \in \mathfrak{g}$ and consider the 1-parameter subgroup of G formed by the flow of v emerging form e. Then, if this 1-parameter subgroup is a variational symmetry gorup for the above functional, then, if v is expressed in terms of local coordinates:*

$$v = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

and denoting by Q its characteristic:

$$Q_\alpha\left(x, u, u^{(1)}\right) = \phi_\alpha(x, u) - \sum_{i=1}^{q} \xi^i(x, u) u_i^\alpha$$

then $(Q_1, \ldots, Q_q)$ is such that:

$$\exists P \in \mathscr{C}^\infty\left(M^{(n)}, \mathbb{R}^p\right) = (P_1, \ldots, P_p), \quad Div\ P := \sum_{i=1}^{p} D_{x^i} P_i = \sum_{\nu=1}^{q} Q_\nu \frac{\delta L}{\delta u^\nu}.$$

Therefore, P is a solution of the conservation law of Euler-Lagrange equations:

$$Div\ P = 0$$

for all solutions of Euler-Lagrange equations.

PROOF By the infinitesimal criterion of invariance, we have:

$$\begin{aligned}
0 &= \mathrm{pr}^{(n)} v(L) + L\mathrm{Div}\ \xi \\
&= \mathrm{pr}^{(n)} v_Q(L) + \sum_{i=1}^{p} \xi^i D_{x^i} L + L\mathrm{Div}\ \xi \\
&= \mathrm{pr}^{(n)} v_Q(L) + \mathrm{Div}\ (L\xi) \\
&= \sum_{\alpha,J} D_J Q_\alpha \frac{\partial L}{\partial u_J^\alpha} + \mathrm{Div}\ (L\xi) \\
&= \sum_{\alpha,J} Q_\alpha (-1)^j D_j \left(\frac{\partial L}{\partial u_J^\alpha}\right) + \mathrm{Div}\ (A + L\xi) \quad \text{using the product rule for the total derivatives} \\
&= \sum_{\alpha=1}^{q} Q_\alpha \frac{\delta L}{\delta u^\alpha} + \mathrm{Div}\ (A + L\xi).
\end{aligned}$$

where A is a p-tuple whose components depend on Q, L and their derivatives. Therefore, the result follows with $P = -(A + L\xi)$.

Of course, the interest of this theorem is to know a closed form for P, which is possible given that it follows just from a product rule for derivatives. However, its expression in the most general case is not really interesting, given that for physical problems such as Maxwell's equations, the Lagrangian only depends on the first derivatives of the dependent variables. In this case, we have the following expression for P:

$$P_i = \sum_{\alpha=1}^{q} \phi_\alpha \frac{\partial L}{\partial u_i^\alpha} + \xi^i L - \sum_{\alpha=1}^{q} \sum_{j=1}^{p} \xi^j u_j^\alpha \frac{\partial L}{\partial u_i^\alpha}.$$

It would have been satisfying to explore some simple example of applications of Noether's theorem for our Lagrangian Λ, however, we did not have time exploring that.

Conclusion and perspectives

During this internship, our main focus was on finding literature to understand how to didactically introduce Maxwell's equations for advanced mathematicians that might work with theoretical physicists, both being familiar with differential forms. So, after understanding how to construct Maxwell's equations in the light of differential forms using Poincaré lemma on time-dependent forms on $\mathbb{R}^3$ ([6], [7]), we wanted to build a similar discourse on more general 4-dimensional manifolds, and we found that the notion of foliated 4-dimensional Lorentzian manifolds ([13], [9], [8], [12]) appeared to be natural for our consturcion. However, it was hard to correctly define, in this more general framework, the easy constructions appearing in flat space-time. The bridge between the 3+1 formalism and the 4-dimensional one is far from trivial to build, and it surely is worth diving into it, though we were short on time to do so. However, the work we did in this direction will probably be found in a future article.

We also attempted to build a specific Lagrangian for the duplex form of Maxwell's equations and apply Noether's thoerem to whatever Lagrangian we would have found. It required me to find literature for both Lagrangian mechanics and Noether's theorem ([5] [17] [14] [10]) and so it required me to learn about Lie groups and Lie algebra ([16], [17], [2]). However, when we were trying to find a Lagrangian for the duplex formulation of Maxwell's equations, we were left with a problem: the number of unknowns didn't match the number of equations, and our attempts on decreasing the number of equations to 12 appeared fruitless, so our first intuition on applying Noether's theorem to a "new" Lagrangian for Maxwell's equations went dry and it seems that the Noether machinery was already well studied for Maxwell's equations anyway ([3] [5] [17], [18] [10]). An interesting perspective could be to reformulate Noether's theorem in the case where the Lagrangian L is a form of maximal degree on a manifold.

Appendices

A Einstein's summation convention

This convention is stated in a really simple manner: *"Repeated indices are summed over"*. But what does that mean exactly?

This summation convention is used, in its simplest formulation, in the frame of vector spaces and dual vector spaces.

Example A.0.1 (Einstein's summation convention in vector spaces and their dual) *Let V be a vector space over a field F, and $(e_i)_{i\in I}$ be a basis of this vector space. Each vector $v\in V$ can be decomposed as:*

$$\exists (v^i)_{i\in I}\in F^{(I)}, \quad v=\sum_{i\in I} v^i e_i,$$

where $F^{(I)}$ denotes the set of collections of elements of F indexed by I with finite support: only finitely many elements of this family are non-zero.

In this case, Einstein's summation convention is written as follows:

$$v=v^i e_i.$$

That is why, in general relativity textbooks or papers, it is common to denote vectors by the notation v^i instead of just v.

But this convention is also used to express values of the form $l(v)$ where l is element of V^*. In this case, it is common to write the dual basis of the basis $(e_i)_{i\in I}$ as $(e^i)_{i\in I}$, and for $l\in V^*$, it is common to denote the coordinates of this covector as $(l_i)_{i\in I}$ such that, with this convention, it reads:

$$l=l_i e^i$$

and

$$l(v)=l_i v^i.$$

Note that in Einstein's summation convention, only "subscripts and superscripts" are summed together. A sum such as:

$$\sum_{i\in I} v^i w^i = v^i w^i$$

for $v,w\in V$ is never written this way.

These examples will be useful to formalize this construction in the frame of tensor bundles, and more specifically, in the frame of cross-sections of these bundles.

Example A.0.2 (Einstein's summation convention for tensor fields) *For $x\in M$, consider the spaces T_xM and T_x^*M. These spaces are equipped with the usual bases $\left(\frac{\partial}{\partial x^i}\right)_{1\leq i\leq n}$ and $\left(\mathrm{d}x^i\right)_{1\leq i\leq n}$, so that the space $T_x^{(k,l)}M$ is equipped with the basis:*

$$\left(\mathrm{d}x^{i_1}\otimes\ldots\otimes\mathrm{d}x^{i_k}\otimes\frac{\partial}{\partial x^{j_1}}\otimes\cdots\frac{\partial}{\partial x^{j_l}}\right)_{1\leq i_1,\ldots,i_k,j_1,\ldots,j_l\leq n}.$$

Given these notations, we can see that every cross-section $\tau \in \Gamma\left(T^{(k,l)}M\right)$ *can be expressed this way:*

$$\forall x \in M, \quad \tau_x = \tau^{j_1,\ldots,j_l}_{i_1,\ldots,i_k}(x)\,\mathrm{d}x^{i_1} \otimes \ldots \otimes \mathrm{d}x^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \ldots \frac{\partial}{\partial x^{j_l}}.$$

In particular, every vector field $v \in \mathscr{X}(M)$ *can be expressed as:*

$$\forall x \in M, \quad v_x = v^i(x)\frac{\partial}{\partial x^i}$$

and every differential 1-form $\omega \in \Omega^1(M)$ *can be written as:*

$$\forall x \in M, \quad \omega_x = \omega_i(x)\,\mathrm{d}x^i.$$

That means that the expression:

$$\tau_x(X_1(x),\ldots,X_k(x);\omega_1(x),\ldots,\omega_l(x))$$

for $\tau \in \Gamma\left(T^{(k,l)}M\right)$, $X_1,\ldots,X_k \in \mathscr{X}(M)$ *and* $\omega_1,\ldots,\omega_l \in \Omega^1(M)$ *can be written as:*

$$\tau_x(X_1(x),\ldots,X_k(x);\omega_1(x),\ldots,\omega_l(x)) = \tau^{j_1,\ldots,j_l}_{i_1,\ldots,i_k}(x)X_1^{i_1}(x)\ldots X_k^{i_k}(x)\omega_{1_{j_1}}(x)\ldots\omega_{l_{j_l}}(x)$$

B Further discussion on Lie groups and Lie algebra

B.1 Detailed proofs of subsection 0.3

Let us detail the proof of the well-defined nature of the Lie bracket:

1. Given that for all vector fields v, w and for all smooth real-valued functions f: 4

$$v(f) - w(f) = (v-w)(f)$$

by linearity of the differential, the statement:

$$\left(\forall f \in \Omega^0(M), \quad v(f) = w(f)\right) \Longrightarrow (v = w)$$

is equivalent to:

$$\left(\forall f \in \Omega^0(M), \quad v(f) = 0\right) \Longrightarrow (v = 0)\,.$$

So let us prove the latter. The condition $v(f) = 0$ means that:

$$\forall x \in M, \quad df_x \cdot v(x) = 0.$$

So, if

$$\forall f \in \Omega^0(M),\ \forall x \in M, \quad df_x \cdot v(x) = 0,$$

then, that means:

$$\forall x \in M, \quad v(x) \in \bigcap_{l \in (T_xM)^*} \ker(l),$$

and the only possibility for this to occur is if:

$$\forall x \in M, \quad v(x) = 0.$$

Thus:

$$v = 0.$$

2. To answer this question, recall that a vector field v on a manifold M of dimension n, equipped with an atlas $(U_i, \varphi_i)_{i \in I}$ is determined by a collection of applications $(v_i)_{i \in I}$:

$$v_i \quad : \quad \varphi_i(U_i) \quad \longrightarrow \quad \mathbb{R}^n$$

satisfying the compatibility condition:

$$\forall i, j \in I, \ \left(U_i \cap U_j \neq \emptyset\right) \Longrightarrow \left(\forall x \in U_i \cap U_j, \quad v_j\left(\varphi_j(x)\right) = d\left(\varphi_j \circ \varphi_i^{-1}\right)_{\varphi_i(x)} \cdot \left(v_i\left(\varphi_i(x)\right)\right)\right).$$

Given a vector field v, these applications are given by the formula:

$$\begin{array}{rcrcl} v_i & : & \varphi_i(U_i) & \longrightarrow & \mathbb{R}^n \\ & & \varphi_i(x) & \longmapsto & d\left(\varphi_i\right)_x \cdot v(x). \end{array}$$

So let us define the Lie bracket $[v, w]$ via this characterization. For all $i \in I$, define the following application:

$$\begin{array}{rcrcl} X_i & : & \varphi_i(U_i) & \longrightarrow & \mathbb{R}^n \\ & & y & \longmapsto & d\left(w_i\right)_y \cdot \left(v_i(y)\right) - d\left(v_i\right)_y \cdot \left(w_i(y)\right). \end{array}$$

Now, let $i, j \in I$ be such that $U_i \cap U_j \neq \emptyset$. Let us prove that:

$$\forall x \in U_i \cap U_j, \quad X_j\left(\varphi_j(x)\right) = d\left(\varphi_j \circ \varphi_i^{-1}\right)_{\varphi_i(x)} \cdot \left(X_i\left(\varphi_i(x)\right)\right).$$

We have:

$$\begin{array}{rcl} X_j\left(\varphi_j(x)\right) & = & d\left(w_j\right)_{\varphi_j(x)} \cdot \left(v_j\left(\varphi_j(x)\right)\right) - d\left(v_j\right)_{\varphi_j(x)} \cdot \left(w_j\left(\varphi_j(x)\right)\right) \\ & = & d\left(d\left(\varphi_j \circ \varphi_i^{-1}\right)_{\varphi_i \circ \varphi_j^{-1}(y)} \cdot \left(w_i\left(\varphi_i \circ \varphi_j^{-1}(y)\right)\right)\right)_{\varphi_j(x)} \cdot \left(v_j\left(\varphi_j(x)\right)\right) \\ & & -d\left(d\left(\varphi_j \circ \varphi_i^{-1}\right)_{\varphi_i \circ \varphi_j^{-1}(y)} \cdot \left(v_i\left(\varphi_i \circ \varphi_j^{-1}(y)\right)\right)\right)_{\varphi_j(x)} \cdot \left(w_j\left(\varphi_j(x)\right)\right). \end{array}$$

Let us compute the differential of the application:

$$z \quad \longmapsto \quad d\left(\varphi_j \circ \varphi_i^{-1}\right)_z \cdot \left(w_i(z)\right).$$

The application:

$$\begin{array}{rcrcl} \pi & : & \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m) \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ & & (l, v) & \longmapsto & l(v) \end{array}$$

is bilinear, thus:

$$\forall h \in \mathbb{R}^n, \quad d\left(\pi(d\left(\varphi_j \circ \varphi_i^{-1}\right), w_i\right)_z \cdot h = d^2\left(\varphi_j \circ \varphi_i^{-1}\right)_z \cdot \left(w_i(z), h\right) + d\left(\varphi_j \circ \varphi_i^{-1}\right)_z \cdot \left(d\left(w_i\right)_z \cdot h\right).$$

Thus:

$$\begin{array}{rcl} X_j\left(\varphi_j(x)\right) & = & d^2\left(\varphi_j\circ\varphi_i^{-1}\right)_{\varphi_i(x)}\cdot\left(w_i(\varphi_i(x)), d\left(\varphi_i\circ\varphi_j^{-1}\right)_{\varphi_j(x)}\cdot\left(v_j\left(\varphi_j(x)\right)\right)\right) \\ & & +d\left(\varphi_j\circ\varphi_i^{-1}\right)_{\varphi_i(x)}\cdot\left(d\left(w_i\right)_{\varphi_i(x)}\cdot\left(d\left(\varphi_i\circ\varphi_j^{-1}\right)_{\varphi_j(x)}\cdot\left(v_j\left(\varphi_j(x)\right)\right)\right)\right) \\ & & -d^2\left(\varphi_j\circ\varphi_i^{-1}\right)_{\varphi_i(x)}\cdot\left(v_i(\varphi_i(x)), d\left(\varphi_i\circ\varphi_j^{-1}\right)_{\varphi_j(x)}\cdot\left(w_j\left(\varphi_j(x)\right)\right)\right) \\ & & -d\left(\varphi_j\circ\varphi_i^{-1}\right)_{\varphi_i(x)}\cdot\left(d\left(v_i\right)_{\varphi_i(x)}\cdot\left(d\left(\varphi_i\circ\varphi_j^{-1}\right)_{\varphi_j(x)}\cdot\left(w_j\left(\varphi_j(x)\right)\right)\right)\right). \end{array}$$

And, given that:

$$\forall x\in U_i\cap U_j,\quad d\left(\varphi_i\circ\varphi_j^{-1}\right)_{\varphi_j(x)}\cdot\left(w_j\left(\varphi_j(x)\right)\right)=w_i\left(\varphi_i(x)\right),$$

we have:

$$\begin{array}{rcl} X_j\left(\varphi_j(x)\right) & = & d^2\left(\varphi_j\circ\varphi_i^{-1}\right)_{\varphi_i(x)}\cdot\left(w_i(\varphi_i(x)), v_i\left(\varphi_i(x)\right)\right)+d\left(\varphi_j\circ\varphi_i^{-1}\right)_{\varphi_i(x)}\cdot\left(d\left(w_i\right)_{\varphi_i(x)}\cdot v_i\left(\varphi_i(x)\right)\right) \\ & & -d^2\left(\varphi_j\circ\varphi_i^{-1}\right)_{\varphi_i(x)}\cdot\left(v_i(\varphi_i(x)), w_i\left(\varphi_i(x)\right)\right)-d\left(\varphi_j\circ\varphi_i^{-1}\right)_{\varphi_i(x)}\cdot\left(d\left(v_i\right)_{\varphi_i(x)}\cdot w_i\left(\varphi_i(x)\right)\right) \\ & = & d\left(\varphi_j\circ\varphi_i^{-1}\right)_{\varphi_i(x)}\cdot\left(d\left(w_i\right)_{\varphi_i(x)}\cdot v_i\left(\varphi_i(x)\right)-d\left(v_i\right)_{\varphi_i(x)}\cdot w_i\left(\varphi_i(x)\right)\right). \\ & = & d\left(\varphi_j\circ\varphi_i^{-1}\right)_{\varphi_i(x)}\cdot X_i\left(\varphi_i(x)\right). \end{array}$$

So this family of application $(X_i)_{i\in I}$ defines a vector field X on M. Finally let us prove that:

$$\forall f\in\Omega^0(M),\quad X(f)=v(w(f))-w(v(f)).$$

To prove this statement, it is sufficient to prove that:

$$\forall i\in I,\ \forall y\in\varphi_i(U_i),\quad X(f)\left(\varphi_i^{-1}(y)\right)=\left(v(w(f))-w(v(f))\right)\left(\varphi_i^{-1}(y)\right).$$

Given a vector field v, using the same notations as before, we have:

$$\begin{array}{rcl} \forall y\in\varphi_i(U_i),\quad v(f)\left(\varphi_i^{-1}(y)\right) & = & df_{\varphi_i^{-1}(y)}\cdot v\left(\varphi_i^{-1}(y)\right) \\ & = & df_{\varphi_i^{-1}(y)}\cdot\left(\left(d\varphi_i\right)^{-1}_{\varphi_i^{-1}(y)}\cdot v_i(y)\right) \\ & = & d\left(f\circ\varphi_i^{-1}\right)_y\cdot v_i(y). \end{array}$$

For simplicity, we will denote by $f_i\in\mathscr{C}^\infty(\varphi_i(U_i),\mathbb{R})$ the application $f_{|U_i}\circ\varphi_i^{-1}$, so that the last equality becomes:

$$v(f)\left(\varphi_i^{-1}(y)\right)=(df_i)_y\cdot v_i(y).$$

Therefore:

$$\begin{array}{rcl} \forall i\in I,\ \forall y\in\varphi_i(U_i),\quad X(f)\left(\varphi_i^{-1}(y)\right) & = & (df_i)_y\cdot X_i(y) \\ & = & (df_i)_y\cdot\left((dw_i)_y\cdot(v_i(y))-(dv_i)_y\cdot(w_i(y))\right) \\ & = & d\left((df_i)_z\cdot w_i(z)\right)_y\cdot v_i(y)-d\left((df_i)_z\cdot v_i(z)\right)_y\cdot w_i(y) \\ & = & d\left(w(f)\circ\varphi_i^{-1}\right)_y\cdot v_i(y)-d\left(v(f)\circ\varphi_i^{-1}\right)_y\cdot w_i(y) \\ & = & v(w(f))\left(\varphi_i^{-1}(y)\right)-w(v(f))\left(\varphi_i^{-1}(y)\right). \end{array}$$

Thus:

$$\forall f\in\Omega^0(M),\quad X(f)=[v,w](f)$$

and therefore:

$$X=[v,w].$$

Proposition B.1.1 (The structure of a Lie algebra) *Let G be a Lie group and $\mathfrak{g}$ its associated Lie algebra. We have the following properties*

1. *$(\mathfrak{g},+,\cdot)$, where $\cdot$ denotes the scalar multiplication with a real number, is a real vector space*
2. *The application:*
$$\begin{array}{rcccl} \Phi & : & \mathfrak{g} & \longrightarrow & T_eG \\ & & v & \longmapsto & v(e) \end{array}$$
is an isomorphism of vector spaces
3. *The Lie bracket is an intern composition law on $\mathfrak{g}$, that means:*
$$\forall v,w\in\mathfrak{g},\quad [v,w]\in\mathfrak{g}.$$

PROOF 1. It is simple to prove that $(\mathfrak{g},+,\cdot)$ is a sub vector space of $(\mathscr{X}(G),+,\cdot)$.

2. Let $v\in\mathfrak{g}$. Given that v is right-invariant, we have:
$$\forall h\in G,\quad v(h)=v\left(R_h(e)\right)=(dR_h)_e\cdot v(e).$$
So, define the following application:
$$\begin{array}{rcccl} \Psi & : & T_eG & \longrightarrow & \mathfrak{g} \\ & & x & \longmapsto & \Psi(x): h\mapsto (dR_h)_e\cdot x. \end{array}$$
Let us verify that Ψ is well-defined. Let $x\in T_eG$. First, $\Psi(x)$ is a vector field given that for all $h\in G$:
$$(dR_h)_e \;:\; T_eG \;\longrightarrow\; T_{R_h(e)}G=T_hG.$$
Now, let us verify that $\Psi(x)$ is right-invariant:
$$\begin{array}{rcl} \forall g,h\in G,\quad \Psi(x)(h\star g) & = & \left(dR_{h\star g}\right)_e\cdot x \\ & = & d\left(R_g\circ R_h\right)_e\cdot x \\ & = & (dR_g)_h\cdot\left((dR_h)_e\cdot x\right) \\ & = & (dR_g)_h\cdot\left(\Psi(x)(h)\right). \end{array}$$
So $\Psi(x)\in\mathfrak{g}$. Now, it is easy to see that:
$$\Psi\circ\Phi=\mathrm{Id}_{\mathfrak{g}}$$
as it is what we wrote at the beginning of the proof. Also, a really simple computation leads to:
$$\Phi\circ\Psi=\mathrm{Id}_{T_eG}.$$
Therefore, Φ is indeed an isomorphism and:
$$\Psi=\Phi^{-1}.$$

3. Finally, let us prove that:
$$\forall v,w\in\mathfrak{g},\ \forall g,h\in G,\quad (dR_g)_h\cdot[v,w](h)=[v,w](h\star g).$$

In order to derive this result, we will prove the following statement:

$$\forall f \in \Omega^0(G),\ \forall v,w \in \mathfrak{g},\ \forall g,h \in G, \quad df_{h\star g} \cdot ((dR_g)_h \cdot [v,w](h)) = df_{h\star g} \cdot ([v,w](h \star g)).$$

So let $f \in \Omega^0(G)$, $v,w \in \mathfrak{g}$ and $g,h \in G$ be fixed. By the definition of the Lie bracket, it reads:

$$df_{h\star g} \cdot ([v,w](h\star g)) \quad = \quad v(w(f))(h\star g) - w(v(f))(h\star g),$$

and:

$$\begin{array}{rcl} df_{h\star g} \cdot ((dR_g)_h \cdot [v,w](h)) & = & d\left(f \circ R_g\right)_h \cdot [v,w](h) \\ & = & v\left(w\left(f \circ R_g\right)\right)(h) - w(v(f\circ R_g))(h) \\ & = & d\left(d\left(f \circ R_g\right)_k \cdot w(k)\right)_h \cdot v(h) - d\left(d\left(f \circ R_g\right)_k \cdot v(k)\right)_h \cdot w(h) \\ & = & d\left(df_{R_g(k)} \cdot ((dR_g)_k \cdot w(k))\right)_h \cdot v(h) \\ & & -d\left(df_{R_g(k)} \cdot ((dR_g)_k \cdot v(k))\right)_h \cdot w(h) \end{array}$$

the right-invariance of v and w gives:

$$\begin{array}{rcl} df_{h\star g} \cdot ((dR_g)_h \cdot [v,w](h)) & = & d\left(df_{R_g(k)} \cdot w(R_g(k))\right)_h \cdot v(h) - d\left(df_{R_g(k)} \cdot v(R_g(k))\right)_h \cdot w(h) \\ & = & \left(d^2 f\right)_{R_g(h)} (w(R_g(h)), (dR_g)_h \cdot v(h)) \\ & & +df_{R_g(h)} \cdot \left(dw_{R_g(h)} \cdot ((dR_g)_h \cdot v(h))\right) \\ & & -\left(d^2 f\right)_{R_g(h)} (v(R_g(h)), (dR_g)_h \cdot w(h)) \\ & & -df_{R_g(h)} \cdot \left(dv_{R_g(h)} \cdot ((dR_g)_h \cdot w(h))\right) \\ & = & df_{R_g(h)} \cdot \left(dw_{R_g(h)} \cdot v(R_g(h)) - dv_{R_g(h)} \cdot w(R_g(h))\right). \\ & = & v(w(f))(R_g(h)) - w(v(f))(R_g(h)). \end{array}$$

Therefore $[v,w] \in \mathfrak{g}$.

Proposition B.1.2 *The Lie bracket $[\cdot,\cdot]$ has the following properties:*

1. *Bilinearity:*

$$\forall \lambda, \mu \in \mathbb{R},\ \forall v, v', w, w' \in \mathscr{X}(M), \quad [\lambda v + v', w] = \lambda [v,w] + [v',w]$$

and

$$[v, \mu w + w'] = \mu [v,w] + [v,w'].$$

2. *Skew-symmetry:*

$$\forall v,w \in \mathscr{X}(M), \quad [v,w] = -[w,v]$$

3. *Jacobi identity:*

$$\forall v_1, v_2, v_3 \in \mathscr{X}(M), \quad [v_1,[v_2,v_3]] + [v_2,[v_3,v_1]] + [v_3,[v_1,v_2]] = 0.$$

PROOF

1. follows directly from the definition.

2. as well.

3. For $i \in I$, $y \in \varphi_i(U_i)$ and for $l,m,n \in \{1,2,3\}$, we have:

$$\begin{array}{rcl} ([v_l,[v_m,v_n]])_i(y) & = & d\left([v_m,v_n]_i\right)_y \cdot v_{l_i}(y) - (dv_{l_i})_y \cdot ([v_m,v_n]_i(y)) \\ & = & d\left((dv_{n_i})_z \cdot v_{m_i}(z) - (dv_{m_i})_z \cdot v_{n_i}(z)\right)_y \cdot v_{l_i}(y) \\ & & -(dv_{l_i})_y \cdot ((dv_{n_i})_y \cdot v_{m_i}(y) - (dv_{m_i})_y \cdot v_{n_i}(y)) \\ & = & \left(d^2 v_{n_i}\right) \cdot (v_{m_i}(y), v_{l_i}(y)) + (dv_{n_i})_y \cdot ((dv_{m_i})_y \cdot v_{l_i}(y)) \\ & & -\left(d^2 v_{m_i}\right) \cdot (v_{n_i}(y), v_{l_i}(y)) - (dv_{m_i})_y \cdot ((dv_{n_i})_y \cdot v_{l_i}(y)) \\ & & -(dv_{l_i})_y \cdot ((dv_{n_i})_y \cdot v_{m_i}(y)) + (dv_{l_i})_y \cdot ((dv_{m_i})_y \cdot v_{n_i}(y)) \end{array}$$

the structure of the last formula makes it easy to see that:

$$([v_1,[v_2,v_3]])_i(y) + ([v_2,[v_3,v_1]])_i(y) + ([v_3,[v_1,v_2]])_i(y) = 0.$$

B.2 One-parameter subgroups of a Lie group and the exponential map

Let $(G,\star)$ be a Lie group. A one-parameter subgroup of G is a Lie group homomorphism:

$$\begin{array}{rcccl} c & : & \mathbb{R} & \longrightarrow & G \\ & & t & \longmapsto & c(t) \end{array}$$

that is, c is differentiable and satisfy:

$$\forall s,t \in \mathbb{R}, \quad c(s+t) = c(s) \star c(t).$$

To any vector field $v \in \mathfrak{g}$, one can associate the one-parameter subgroup Φ_v^e, generally denoted as $\exp(\cdot v)$, that is:

$$\forall \varepsilon \in \mathbb{R}, \quad \exp(\varepsilon v) := \Phi_v^e(\varepsilon).$$

This is an important result, as it means that every non-zero element of the Lie algebra $\mathfrak{g}$ generates a flow from the identity that is **globally** defined and forms a one-parameter subgroup (note that the case when $v = 0$ is trivial as its flow is just the constant function e). In fact, the right-invariance of the vector fields of the Lie algebra implies the global definition of their flow from the identity and also the equality:

$$\forall \varepsilon, \delta \in \mathbb{R}, \quad \exp[(\varepsilon+\delta)v] = \exp(\varepsilon v)\exp(\delta v).$$

Conversely, if c forms a one-parameter subgroup of G, then $c'(0)$ is well-defined and is an element of the tangent space T_eG, which is canonically isomorphic to the Lie algebra $\mathfrak{g}$. Denoting by v_c its image in $\mathfrak{g}$, we readily have:

$$\forall \varepsilon \in \mathbb{R}, \quad c(\varepsilon) = \exp(\varepsilon v_c).$$

Therefore, each one-parameter subgroup of G is the flow of some vector field belonging to the Lie algebra $\mathfrak{g}$.
Moreover, we can determine the isomorphism classes of these one-parameter subgroups.

- If $v = 0$, then this one-parameter subgroup is simply the trivial group $\{e\}$. The other cases will focus on $v \neq 0$.

- If $\exp(\cdot v)$ is injective, then this one-parameter subgroup is isomorphic to $\mathbb{R}$.

- If $\exp(\cdot v)$ is *not* injective, then, in this case, the quantity:

$$\delta_0 = \inf\{\varepsilon > 0 \mid \exp(\varepsilon v) = e\}$$

  is well-defined and is strictly bigger than 0. Indeed, $\exp(\cdot v)$ is smooth and has a non-zero derivative at $\varepsilon = 0$ given that $v \neq 0$. Therefore $\exp(\cdot v)$ does not reach back the identity e on a small neighborhood of $\varepsilon = 0$ and so, this homomorphism has kernel $\delta_0\mathbb{Z}$ and therefore, this one-parameter subgroup is isomorphic to $\dfrac{\mathbb{R}}{\mathbb{Z}}$.

In this discussion, we have in fact manipulated what is called the **exponential** map of the Lie algebra:

$$\begin{array}{rccl} \exp & : \mathfrak{g} & \longrightarrow & G \\ & v & \longmapsto & \exp(1.v) = \Phi_v^e(1). \end{array}$$

which is a very important tool in geometry.

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