A BIJECTION BETWEEN EDGES OF THE TURÁN GRAPH AND IRREDUCIBLE ELEMENTS IN THE DOMINANCE ORDER LATTICE

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ABSTRACT. In this paper we build a bijection between the meet-irreducible elements of the lattice of the compositions of n with parts in $[1, p]$ equipped with the dominance order, and the edges of the (n, p)-Turán graph. Using this bijection, we then compute asymptotically the average value of some statistics on those meet-irreducible compositions.

1. INTRODUCTION

For two integers $n \geq 0$ and $p \geq 1$, we denote by $n \bmod p$ the remainder of the euclidean division of n by p. If a and b are two integers, we write $a \equiv b \bmod p$ if $a \bmod p = b \bmod p$. The (n, p)-*Turán graph* $\mathcal{T}_n^p$ is the complete p-partite graph on n vertices, i.e. the graph with vertex set $V(\mathcal{T}_n^p) = \{1, \ldots, n\}$, and $\{a, b\} \in E(\mathcal{T}_n^p)$ if and only if $a \not\equiv b \bmod p$, see Figure 1 for an example with $(n, p) = (8, 3)$. This graph is known for being the only one having the maximum number of edges on n vertices and being K_{p+1}-free, i.e. not having a complete induced subgraph on $p+1$ vertices, see e.g. [1]. Due to the structure of $\mathcal{T}_n^p$, one can see that its number of edges is given by

$$a_p(n) = \left(1 - \frac{1}{p}\right)\frac{n^2}{2} - \frac{(n \bmod p)(p - (n \bmod p))}{2p}. \tag{1.1}$$

We refer the reader to [2, 4] for some standard references about the Turán graph.

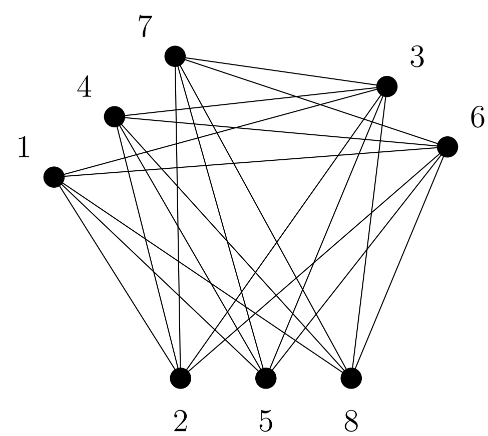


FIGURE 1. The $(8, 3)$-Turán graph $\mathcal{T}_8^3$.

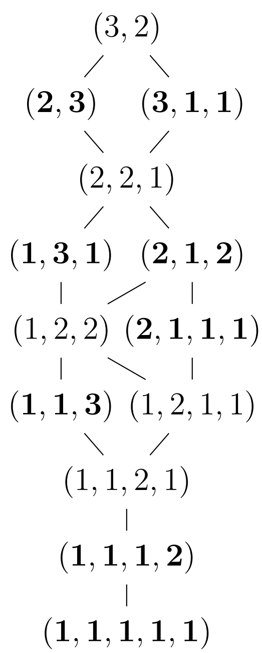
A *composition* of $n \geq 1$ is a tuple $(x_1, \ldots, x_m)$ of positive integers such that $m \geq 1$ and $x_1 + \ldots + x_m = n$. We denote by $\mathbb{F}_n^p$ the set of compositions $(x_1, \ldots, x_m)$ of n such that for all i, $1 \leq x_i \leq p$. $\mathbb{F}_n^p$ is enumerated by the *p-generalized Fibonacci sequence* $(F_n^p)_{n\geq 0}$, defined for every $p \geq 2$ by

$$F_n^p = F_{n-1}^p + F_{n-2}^p + \ldots + F_{n-p}^p$$

with initial conditions $F_n^p = 0$ if $n < 0$ and $F_0^p = 1$ (see [5]). The *dominance order* on the set of compositions of n is defined for two compositions $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_\ell)$ by

$$x \leq y \iff \text{ for all } 1 \leq k \leq \min(m, \ell), \quad \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i.$$

This order has been first introduced and studied on integer partitions, see for instance [3]. The poset $\mathbb{F}_n^p$ equipped with the dominance order forms a lattice, i.e. each pair of compositions $x, y \in \mathbb{F}_n^p$ admits a *meet* (greatest lower bound) $x \wedge y$, and a *join* (lowest upper bound) $x \vee y$. Moreover, this lattice is *distributive*, meaning that the operations meet and join are distributive relatively to the other. If $x, y \in \mathbb{F}_n^p$, we say that y *covers* x if $x < y$ and for all $z \in \mathbb{F}_n^p$, $x \leq z \leq y \Rightarrow z \in \{x, y\}$. We say equivalently that y is an *upper cover* of x, or x is a *lower cover* of y. An element $x \in \mathbb{F}_n^p$ is *meet-irreducible* if for all $y, z \in \mathbb{F}_n^p$, $x = y \wedge z \Rightarrow y = x$ or $z = x$. Similarly, an element $x \in \mathbb{F}_n^p$ is *join-irreducible* if for all $y, z \in \mathbb{F}_n^p$, $x = y \vee z \Rightarrow y = x$ or $z = x$. Since $\mathbb{F}_n^p$ is a finite lattice, x is meet-irreducible (resp. join-irreducible) if and only if x has exactly one upper cover (resp. lower cover). See for instance [7] for the standard definitions of lattice theory. Let $\textsc{mi}_n^p$ (resp. $\textsc{ji}_n^p$) be the set of meet-irreducible (resp. join-irreducible) elements in $\mathbb{F}_n^p$. See Figure 2 for an example of $\mathbb{F}_n^p$ and $\textsc{mi}_n^p$ with $(n, p) = (5, 3)$.



$\textsc{mi}_5^3$
(2, 3)
(3, 1, 1)
(1, 3, 1)
(2, 1, 2)
(2, 1, 1, 1)
(1, 1, 3)
(1, 1, 1, 2)
(1, 1, 1, 1, 1)

FIGURE 2. The lattice $\mathbb{F}_5^3$ and the set $\textsc{mi}_5^3$ of its meet-irreducible elements.

In [6] the authors gave the enumeration of many characteristic elements in $\mathbb{F}_n^p$, such as covering relations or intervals. In particular they proved that $\textsc{mi}_n^p$ is enumerated by $a_p(n)$. As a consequence of Birkhoff's representation theorem, since $\mathbb{F}_n^p$ is a finite distributive lattice,

$|\text{JI}_n^p| = |\text{MI}_n^p| = a_p(n)$. In Section 2, we build an explicit bijection between $E(\mathcal{T}_n^p)$ and MI_n^p, and then we show how this bijection can be adapted to JI_n^p. In Section 3 we use this bijection to compute asymptotically the average value of three statistics on MI_n^p.

2. The bijection

We start with the following lemma that counts the number of upper covers of a composition, and then characterizes the elements of MI_n^p. For $a, b \in [1, p]$, we say that a composition $x = (x_1, \ldots, x_m)$ has a *consecutive pattern* ab if there exists $1 \le i < m$ such that $x_i = a$ and $x_{i+1} = b$.

Lemma 2.1. *The number of upper covers of a composition $x \in \mathbb{F}_n^p$ is the number of consecutive patterns ab in x with $1 \le a \le p-1$ and $2 \le b \le p$, plus one if x has the suffix $i1$ for $1 \le i \le p-1$.*

Proof. Suppose that $x \in \mathbb{F}_n^p$ satisfies $x = (x_1, \ldots, x_j, a, b, x_{j+1}, \ldots, x_m)$ with $1 \le a \le p-1$ and $2 \le b \le p$. Then $z := (x_1, \ldots, x_j, a+1, b-1, x_{j+1}, \ldots, x_m)$ is an upper cover of x, and we say that z has the form $(\star)$. Similarly, if $x = (x_1, \ldots, x_m, i, 1)$ with $1 \le i \le p-1$, then $z := (x_1 \ldots, x_m, i+1)$ is an upper cover of x, and we say that z has the form $(\star\star)$. Conversely, suppose $x = (x_1, \ldots, x_m) \in \mathbb{F}_n^p$, and let $y = (y_1, \ldots, y_\ell) \in \mathbb{F}_n^p$ be such that $x < y$. To end the proof it suffices to prove that there exists $z \in \mathbb{F}_n^p$ with the form $(\star)$ or $(\star\star)$ such that $x < z \le y$. Let j be the smallest integer such that $x_j < y_j$. Then we have $x_j < p$.

Case 1: Suppose that $x_{j+1} > 1$. Then we set $z_j = x_j + 1$ and $z_{j+1} = x_{j+1} - 1$, and $z_i = x_i$ for $i \notin \{j, j+1\}$. Then z has the form $(\star)$, and $x < z \le y$.

Case 2: Suppose that $x_{j+1} = x_{j+2} = \ldots = x_m = 1$. Then we set $z_i = x_i$ if $i \le m-2$, and $z_{m-1} = x_{m-1} + 1$. Note that we may have $m = j+1$. Then z has the form $(\star\star)$, and $x < z \le y$.

Case 3: Suppose that $x_{j+1} = 1$ and there exists $j+1 < k \le m$ such that $x_k > 1$. We consider the smallest such k, and we set $z_{k-1} = x_{k-1} + 1$ and $z_k = x_k - 1$, and $z_i = x_i$ for $i \notin \{k-1, k\}$. Then z has the form $(\star)$, and $x < z \le y$.

Considering those three cases, the converse holds and the lemma too. □

Remark 2.2. As a direct porism of Lemma 2.1, the number of lower covers of $x \in \mathbb{F}_n^p$ is the number of consecutive patterns ab in x with $2 \le a \le p$ and $1 \le b \le p-1$, plus one if x does not end by 1.

Now we give a recursive decomposition of $E(\mathcal{T}_n^p)$, and then we will provide a similar decomposition for MI_n^p. We have

$$E(\mathcal{T}_n^p) = E(\mathcal{T}_{n-1}^p) \cup \{\{a, n\} \subseteq [1, n] \mid a \ne n \bmod p\}, \tag{2.1}$$

and

$$|\{1 \le a \le n \mid a \ne n \bmod p\}| = \left\lfloor \left(1 - \frac{1}{p}\right) n \right\rfloor.$$

In particular, the sequence $a_p(n)$ satisfies the recurrence relation

$$a_p(n) = a_p(n-1) + \left\lfloor \left(1 - \frac{1}{p}\right) n \right\rfloor.$$

In order to build our bijection naturally, we now give a similar decomposition of MI_n^p. Let $f : \mathrm{MI}_{n-1}^p \longrightarrow \mathrm{MI}_n^p$ defined for a composition $x = (x_1, \ldots, x_m)$ by

$$f(x) = \begin{cases} (x_1, \ldots, x_m + 1) & \text{if } 1 \leq x_m < p, \\ (x_1, \ldots, x_m, 1) & \text{if } x_m = p. \end{cases}$$

Then it is not hard to check that f takes indeed its values in MI_n^p, and that f is injective. Moreover, if $f(x)$ ends by 1, then it ends by $p1$. Conversely, if $y = (y_1, \ldots, y_\ell) \in \mathrm{MI}_n^p$ does not end by $i1$ with $1 \leq i \leq p-1$, then $y = f(y_1, \ldots, y_\ell - 1)$ if $y_\ell > 1$, and $y = f(y_1, \ldots, y_{\ell-1})$ if $(y_{\ell-1}, y_\ell) = (p, 1)$. Consequently, we have

$$\begin{aligned} \mathrm{MI}_n^p &= f(\mathrm{MI}_{n-1}^p) \cup \{x \in \mathrm{MI}_n^p \mid x \text{ ends by } i1 \text{ with } 1 \leq i \leq p-1\} \\ &= f(\mathrm{MI}_{n-1}^p) \cup \{(\underbrace{p, \ldots, p}_{k}, i, \underbrace{1, \ldots, 1}_{m}) \in \mathbb{F}_n^p \mid 1 \leq i \leq p-1,\ m \geq 1,\ k \geq 0\}. \end{aligned} \tag{2.2}$$

The last equality holds since by Lemma 2.1, if a composition in MI_n^p ends with $i1$ for some $1 \leq i \leq p-1$, this suffix produces one upper cover; thus such a composition avoids consecutive patterns ab with $1 \leq a \leq p-1$ and $2 \leq b \leq p$. By the injectivity of f, we have

$$|\mathrm{MI}_n^p| = |\mathrm{MI}_{n-1}^p| + |A_n^p|,$$

with $A_n^p = \{(\underbrace{p, \ldots, p}_{k}, i, \underbrace{1, \ldots, 1}_{m}) \in \mathbb{F}_n^p \mid 1 \leq i \leq p-1,\ m \geq 1,\ k \geq 0\}$.

Lemma 2.3. *Given $n, p \geq 2$, we have $|A_n^p| = \left\lfloor \left(1 - \frac{1}{p}\right) n \right\rfloor$.*

Proof. For $x = (\underbrace{p, \ldots, p}_{k}, i, \underbrace{1, \ldots, 1}_{m}) \in A_n^p$, let $g(x) = m$. Since x is a composition of n we have $n = kp + i + m$, so $g(x) \equiv (n - i) \bmod p$, and $g(x) \not\equiv n \bmod p$ since $i \not\equiv 0 \bmod p$. Conversely, if $1 \leq a \leq n$ with $a \not\equiv n \bmod p$, then $g(\underbrace{p, \ldots, p}_{\lfloor \frac{n-a}{p} \rfloor}, (n-a) \bmod p, \underbrace{1, \ldots, 1}_{a}) = a$. This proves that g is a bijection between A_n^p and $\{1 \leq a \leq n \mid a \not\equiv n \bmod p\}$. Therefore,

$$|A_n^p| = |\{1 \leq a \leq n \mid a \not\equiv n \bmod p\}| = \left\lfloor \left(1 - \frac{1}{p}\right) n \right\rfloor.$$

□

Now for $n, p \geq 2$ we give a recursive bijection $\Psi_n^p : E(\mathcal{T}_n^p) \longrightarrow \mathrm{MI}_n^p$ based on the decompositions from Eq. (2.1) and Eq. (2.2), and the bijection from Lemma 2.3. For $\{a, b\} \in E(\mathcal{T}_n^p)$ with $a < b$ we set

$$\Psi_n^p(a, b) = \begin{cases} f(\Psi_{n-1}^p(a, b)) & \text{if } b < n, \\ (\underbrace{p, \ldots, p}_{\lfloor \frac{n-a}{p} \rfloor}, (n-a) \bmod p, \underbrace{1, \ldots, 1}_{a}) & \text{if } b = n. \end{cases}$$

It follows from a direct induction that

$$\Psi_n^p(a, b) = (\underbrace{p, \ldots, p}_{\lfloor \frac{b-a}{p} \rfloor}, (b-a) \bmod p, \underbrace{1, \ldots, 1}_{a-1}, \underbrace{p, \ldots, p}_{\lfloor \frac{n-b+1}{p} \rfloor}, (n-b+1) \bmod p). \tag{2.3}$$

For $x \in \textsc{mi}_n^p$, let $k_x \geq 0$, $1 \leq i_x \leq p-1$ and $m_x \geq 0$ maximum such that x starts with $\underbrace{p,\dots,p}_{k_x}, i_x, \underbrace{1\dots,1}_{m_x}$. Then we define $\Phi_n^p : \textsc{mi}_n^p \longrightarrow E(\mathcal{T}_n^p)$ by

$$\Phi_n^p(x) = \{m_x + \delta_x, pk_x + i_x + m_x + \delta_x\},$$

where

$$\delta_x = \begin{cases} 1 & \text{if } (x \text{ ends by } 1 \Rightarrow x \text{ ends by } p1), \\ 0 & \text{otherwise.} \end{cases}$$

Observe that Φ_n^p is well defined since k_x, i_x and m_x always exist for the elements of $\textsc{mi}_n^p$. Moreover, Φ_n^p takes its values in $E(\mathcal{T}_n^p)$ because $i_x \not\equiv 0 \bmod p$. With a direct verification, we deduce the following theorem.

Theorem 2.4. *For $n, p \geq 2$, Ψ_n^p and Φ_n^p are reciprocal bijections.*

$\{a,b\} \in E(\mathcal{T}_7^2)$	$\Psi_7^2(a,b) \in \textsc{mi}_7^2$
$\{1,2\}$	$(1,2,2,2)$
$\{1,4\}$	$(2,1,2,2)$
$\{1,6\}$	$(2,2,1,2)$
$\{2,3\}$	$(1,1,2,2,1)$
$\{2,5\}$	$(2,1,1,2,1)$
$\{2,7\}$	$(2,2,1,1,1)$
$\{3,4\}$	$(1,1,1,2,2)$
$\{3,6\}$	$(2,1,1,1,2)$
$\{4,5\}$	$(1,1,1,1,2,1)$
$\{4,7\}$	$(2,1,1,1,1,1)$
$\{5,6\}$	$(1,1,1,1,1,2)$
$\{6,7\}$	$(1,1,1,1,1,1,1)$

$\{a,b\} \in E(\mathcal{T}_6^3)$	$\Psi_6^3(a,b) \in \textsc{mi}_6^3$
$\{1,2\}$	$(1,3,2)$
$\{1,3\}$	$(2,3,1)$
$\{1,5\}$	$(3,1,2)$
$\{1,6\}$	$(3,2,1)$
$\{2,3\}$	$(1,1,3,1)$
$\{2,4\}$	$(2,1,3)$
$\{2,6\}$	$(3,1,1,1)$
$\{3,4\}$	$(1,1,1,3)$
$\{3,5\}$	$(2,1,1,2)$
$\{4,5\}$	$(1,1,1,1,2)$
$\{4,6\}$	$(2,1,1,1,1)$
$\{5,6\}$	$(1,1,1,1,1,1)$

TABLE 1. Two examples of the bijection Ψ_n^p for $(n,p) = (7,2)$ and $(6,3)$.

Remark 2.5. By doing the same investigation for $\textsc{ji}_n^p$, we obtain that the following map is a bijection from $E(\mathcal{T}_n^p)$ to $\textsc{ji}_n^p$:

$$\tilde{\Psi}_n^p(a,b) = (\underbrace{1,\dots,1}_{a-1}, (b-a) \bmod p + 1, \underbrace{p,\dots,p}_{\lfloor \frac{b-a}{p} \rfloor}, \underbrace{1,\dots,1}_{n-b}).$$

Using Remark 2.2, we can check that it takes indeed its values in $\textsc{ji}_n^p$. To define its reciprocal, for $x \in \mathbb{F}_n^p$, let k_x (resp. m_x) be maximal such that x starts (resp. ends) with k_x (resp. m_x) consecutive 1s. The reciprocal of $\tilde{\Psi}_n^p$ is then defined by

$$\tilde{\Phi}_n^p(x) = \{k_x + 1, n - m_x\}.$$

For $x \in \textsc{ji}_n^p$, $n - k_x - m_x - 1 \not\equiv 0 \bmod p$, hence $\tilde{\Phi}_n^p$ takes its values in $E(\mathcal{T}_n^p)$.

3. Some statistics on MI_n^p

The bijection Ψ_n^p from Section 2 allows us to express statistics on $\Psi_n^p(a,b)$ easily in terms of a and b. It therefore gives a method to compute the average value of statistics. In this section we give an illustration of this method on three statistics: the number of parts, the first part and the number of weak records. We start with a lemma computing some sums over $E(\mathcal{T}_n^p)$.

Lemma 3.1. *For a given $p \geq 2$ we have as $n \to \infty$*

$$\sum_{\substack{1\le a<b\le n\\ a\not\equiv b \bmod p}} a = \left(1-\frac{1}{p}\right)\frac{n^3}{6} + O(n^2),$$

$$\sum_{\substack{1\le a<b\le n\\ a\not\equiv b \bmod p}} b = \left(1-\frac{1}{p}\right)\frac{n^3}{3} + O(n^2).$$

Proof. First, let us compute the sum without the congruence constraint.

$$\begin{aligned} s(n) := \sum_{1\le a<b\le n} a &= \sum_{a=1}^{n-1}\sum_{b=a+1}^{n} a = \sum_{a=1}^{n-1}(n-a)a \\ &= \frac{n^2(n-1)}{2} - \frac{n(n-1)(2n-1)}{6} = \frac{n(n-1)(n+1)}{6}. \end{aligned}$$

Similarly,

$$\begin{aligned} S(n) := \sum_{1\le a<b\le n} b &= \sum_{b=2}^{n}\sum_{a=1}^{b-1} b = \sum_{b=2}^{n} b(b-1) \\ &= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n(n-1)(n+1)}{3}. \end{aligned}$$

Now if $0 \le i \le p-1$, let $\delta_i = 1$ if $i > n \bmod p$, and $\delta_i = 0$ otherwise. Then

$$\sum_{\substack{1\le a<b\le n\\ a\equiv b\equiv i \bmod p}} a = \sum_{a=0}^{\lfloor n/p\rfloor-\delta_i-1}\ \sum_{b=a+1}^{\lfloor n/p\rfloor-\delta_i}(pa+i) = p\cdot s(\lfloor n/p\rfloor) + O(n^2) = \frac{n^3}{6p^2} + O(n^2).$$

We deduce

$$\sum_{\substack{1\le a<b\le n\\ a\not\equiv b \bmod p}} a = s(n) - \sum_{i=0}^{p-1}\sum_{\substack{1\le a<b\le n\\ a\equiv b\equiv i \bmod p}} a = \frac{n^3}{6} + O(n^2) - p\left(\frac{n^3}{6p^2} + O(n^2)\right) = \left(1-\frac{1}{p}\right)\frac{n^3}{6} + O(n^2).$$

Similarly,

$$\sum_{\substack{1\le a<b\le n\\ a\equiv b\equiv i \bmod p}} b = \sum_{b=1}^{\lfloor n/p\rfloor-\delta_i}\ \sum_{a=0}^{b-1}(pb+i) = p\cdot S(\lfloor n/p\rfloor) + O(n^2) = \frac{n^3}{3p^2} + O(n^2),$$

and so

$$\sum_{\substack{1\le a<b\le n\\ a\not\equiv b \bmod p}} b = S(n) - \sum_{i=0}^{p-1} \sum_{\substack{1\le a<b\le n\\ a\equiv b\equiv i \bmod p}} b = \frac{n^3}{3} + O(n^2) - p\left(\frac{n^3}{3p^2} + O(n^2)\right) = \left(1-\frac{1}{p}\right)\frac{n^3}{3} + O(n^2).$$

□

3.1. **The number of parts.** For a composition $x = (x_1, \ldots, x_m) \in \mathbb{F}_n^p$ we denote by $\texttt{parts}(x)$ its number of parts m.

Proposition 3.2. *The average number of parts of a composition in* MI_n^p *satisfies as* $n \to \infty$

$$\frac{1}{|\mathrm{MI}_n^p|}\sum_{x\in\mathrm{MI}_n^p} \texttt{parts}(x) \sim \left(1+\frac{2}{p}\right)\frac{n}{3}.$$

Proof. From Eq. (2.3), for every $a \not\equiv b \bmod p$ we have

$$\begin{aligned}\texttt{parts}(\Psi_n^p(a,b)) &= \left\lfloor\frac{b-a}{p}\right\rfloor + 1 + a - 1 + \left\lfloor\frac{n-b+1}{p}\right\rfloor + \mathbb{1}_{b\equiv n+1 \bmod p}\\ &= \frac{n}{p} + \left(1-\frac{1}{p}\right)a + O(1).\end{aligned}$$

The total number of parts in all compositions of MI_n^p is then

$$\begin{aligned}\sum_{x\in\mathrm{MI}_n^p} \texttt{parts}(x) = \sum_{\substack{1\le a<b\le n\\ a\not\equiv b \bmod p}} \texttt{parts}(\Psi_n^p(a,b)) &= \sum_{\substack{1\le a<b\le n\\ a\not\equiv b \bmod p}} \left[\frac{n}{p} + \left(1-\frac{1}{p}\right)a\right] + O(n^2)\\ &= a_p(n)\cdot\frac{n}{p} + \left(1-\frac{1}{p}\right)^2\frac{n^3}{6} + O(n^2),\end{aligned}$$

by Lemma 3.1. From Eq. (1.1), $a_p(n) = \left(1-\frac{1}{p}\right)\frac{n^2}{2} + O(1)$, so we deduce

$$\begin{aligned}\frac{1}{|\mathrm{MI}_n^p|}\sum_{x\in\mathrm{MI}_n^p} \texttt{parts}(x) &= \frac{1}{a_p(n)}\left(a_p(n)\cdot\frac{n}{p} + \left(1-\frac{1}{p}\right)^2\frac{n^3}{6} + O(n^2)\right)\\ &= \frac{n}{p} + \left(1-\frac{1}{p}\right)\frac{n}{3} + O(1)\\ &= \left(1+\frac{2}{p}\right)\frac{n}{3} + O(1).\end{aligned}$$

□

3.2. **The first part.** For a composition $x = (x_1, \ldots, x_m) \in \mathbb{F}_n^p$ we denote by $\texttt{first}(x)$ the value of its first part x_1.

Proposition 3.3. *The average value of the statistic* $\texttt{first}$ *on the compositions of* MI_n^p *satisfies as* $n \to \infty$

$$\frac{1}{|\mathrm{MI}_n^p|}\sum_{x\in\mathrm{MI}_n^p} \texttt{first}(x) = p\left(1 - \frac{p}{n} + \frac{p(p+1)}{3n^2} + O\left(\frac{1}{n^3}\right)\right).$$

Proof. From Eq. (2.3), for every $a \not\equiv b \bmod p$ we have

$$\texttt{first}(\Psi_n^p(a,b)) = \begin{cases} p & \text{if } b-a \geq p \\ (b-a) \bmod p & \text{otherwise.} \end{cases}$$

The sum of $\texttt{first}(x)$ over all compositions $x \in \mathrm{MI}_n^p$ is then

$$\sum_{x \in \mathrm{MI}_n^p} \texttt{first}(x) = \sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p}} \texttt{first}(\Psi_n^p(a,b)) = \sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p \\ b-a \geq p}} p + \sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p \\ b-a < p}} (b-a) \bmod p.$$

Let us first compute the following sum when $n \geq p-1$.

$$\sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p \\ b-a < p}} 1 = \sum_{a=1}^{n-p+1} \sum_{b=a+1}^{a+p-1} 1 + \sum_{a=n-p+2}^{n-1} \sum_{b=a+1}^{n} 1 = \sum_{a=1}^{n-p+1} (p-1) + \sum_{a=n-p+2}^{n-1} (n-a) \tag{3.1}$$

$$= (n-p+1)(p-1) + \sum_{a=1}^{p-2} a = (n-p+1)(p-1) + \frac{(p-2)(p-1)}{2}. \tag{3.2}$$

We deduce

$$\sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p \\ b-a \geq p}} p = \sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p}} p - \sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p \\ b-a < p}} p = p\left(a_p(n) - (n-p+1)(p-1) - \frac{(p-2)(p-1)}{2}\right).$$

Now observe that if $1 \leq a < b \leq n$ with $b-a < p$, then $b-a = (b-a) \bmod p$. Then for $n \geq p-1$ we have

$$\sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p \\ b-a < p}} (b-a) \bmod p = \sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p \\ b-a < p}} (b-a) = \sum_{a=1}^{n-p+1} \sum_{b=a+1}^{a+p-1} (b-a) + \sum_{a=n-p+2}^{n-1} \sum_{b=a+1}^{n} (b-a)$$

$$= \sum_{a=1}^{n-p+1} \left[\frac{(a+p)(a+p-1) - a(a+1)}{2} - (p-1)a\right] + \sum_{a=n-p+2}^{n-1} \left[\frac{n(n+1) - a(a+1)}{2} - (n-a)a\right]$$

$$= \sum_{a=1}^{n-p+1} \frac{p(p-1)}{2} + \sum_{a=n-p+2}^{n-1} \left[\frac{(n-a)^2 + n - a}{2}\right]$$

$$= \frac{p(p-1)}{2}\left(n-p+1+\frac{p-2}{3}\right).$$

Finally,

$$\sum_{x \in \mathrm{MI}_n^p} \texttt{first}(x) = p\left(a_p(n) - (n-p+1)(p-1) - \frac{(p-2)(p-1)}{2}\right) + \frac{p(p-1)}{2}\left(n-p+1+\frac{p-2}{3}\right)$$

$$= p\left(a_p(n) - n \cdot \frac{p-1}{2} + \frac{(p-1)(p+1)}{6}\right).$$

Using that $a_p(n) = \frac{(p-1)n^2}{2p} + O(1)$, we deduce

$$\frac{1}{|\mathrm{MI}_n^p|} \sum_{x \in \mathrm{MI}_n^p} \texttt{first}(x) = p\left(1 - \frac{p}{n} + \frac{p(p+1)}{3n^2} + O\left(\frac{1}{n^3}\right)\right).$$

□

3.3. **The number of weak records.** For $x = (x_1, \ldots, x_m) \in \mathbb{F}_n^p$, a *weak record* is a position $1 \leq j \leq m$ such that $x_i \leq x_j$ for each $1 \leq i \leq j$. We denote by $\texttt{wrec}(x)$ the number of weak records of x.

Proposition 3.4. *The average number of weak records of a composition in* MI_n^p *satisfies as* $n \to \infty$

$$\frac{1}{|\mathrm{MI}_n^p|} \sum_{x \in \mathrm{MI}_n^p} \texttt{wrec}(x) \sim \frac{2n}{3p}.$$

Proof. From Eq. (2.3), we have that

$$\texttt{wrec}(\Psi_n^p(a,b)) = \left\lfloor \frac{b-a}{p} \right\rfloor + \left\lfloor \frac{n-b+1}{p} \right\rfloor$$

when $b - a \geq p$. We do not compute it in details for $b - a < p$ as we shall see the contribution of those pairs $\{a, b\}$ to the total number of weak records in MI_n^p is negligible. Indeed, we saw in Eq. (3.2) that the number of those pairs is $O(n)$, and since $\texttt{wrec}(\Psi_n^p(a,b)) \leq n$ for every pair $\{a, b\}$ we have that

$$\sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p \\ b-a<p}} \texttt{wrec}(\Psi_n^p(a,b)) = O(n^2).$$

Then,

$$\sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p}} \texttt{wrec}(\Psi_n^p(a,b)) = \sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p \\ b-a \geq p}} \left(\left\lfloor \frac{b-a}{p} \right\rfloor + \left\lfloor \frac{n-b+1}{p} \right\rfloor\right) + O(n^2)$$

$$= \frac{1}{p} \sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p \\ b-a \geq p}} (n-a) + O(n^2).$$

Using Eq. (3.2) and Lemma 3.1, we have

$$\sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p \\ b-a \geq p}} a = \sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p}} a - O(n^2) = \frac{n a_p(n)}{3} - O(n^2).$$

Finally,

$$\sum_{\substack{1 \leq a < b \leq n \\ a \not\equiv b \bmod p}} \texttt{wrec}(\Psi_n^p(a,b)) = \frac{1}{p}\left(n a_p(n) - \frac{n a_p(n)}{3}\right) + O(n^2) = \frac{2n a_p(n)}{3p} + O(n^2),$$

which concludes. □

References

[1] M. Aigner, Turán's graph theorem. *Am. Math. Monthly*, 102 (1995), 808–816.
[2] P. Turán, On an extremal problem in graph theory (in Hungarian). *Math. Fiz. Lapok*, 48 (1941), 436—452.
[3] T. Brylawski, The lattice of integer partitions, *Discrete Mathematics*, 6-(3), 201-219 (1973).
[4] B. Bollobás, *Extremal graph theory*, Academic Press, 1978.
[5] S. Heubach and T. Mansour, Compositions of n with parts in a set. *Congressus Numerantium*, 168, 127-143 (2004).
[6] J.-L. Baril and N. Hassler, Intervals in a family of Fibonacci lattices, https://arxiv.org/abs/2411.17628, (2025).
[7] R.P. Stanley, *Enumerative Combinatorics, Volume 1*, Cambridge University Press, 1997.

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