Enumeration in the lattice of q-decreasing words

ALEA 2026

Nathanaël HASSLER
joint work with Jean-Luc BARIL and Sergey KIRGIZOV

March 12, 2026

UNIVERSITÉ BOURGOGNE EUROPE
LIB Laboratoire Informatique de Bourgogne

q-decreasing words

Let $q > 0$ be a real number and $n \geq 0$ an integer.

Definition

A *q-decreasing word* of length n is a binary word of length n such that:

every maximal factor of the form $0^a 1^b$ satisfies either $a = 0$ or $q \cdot a > b$.

Let $\mathcal{W}_n^q$ be the set of q-decreasing words of length n.

Example: $\mathcal{W}_4^1 = \{0000, 0001, 0010, 1000, 1001, 1100, 1110, 1111\}$.

q-decreasing words

Let $q > 0$ be a real number and $n \geq 0$ an integer.

Definition

A *q-decreasing word* of length n is a binary word of length n such that:

every maximal factor of the form $0^a 1^b$ satisfies either $a = 0$ or $q \cdot a > b$.

Let $\mathcal{W}_n^q$ be the set of q-decreasing words of length n.

Example: $\mathcal{W}_4^1 = \{\underset{4>0}{0000}, 0001, 0010, 1000, 1001, 1100, 1110, 1111\}$.

q-decreasing words

Let $q > 0$ be a real number and $n \geq 0$ an integer.

Definition

A *q-decreasing word* of length n is a binary word of length n such that:

every maximal factor of the form $0^a 1^b$ *satisfies either* $a = 0$ *or* $q \cdot a > b$.

Let $\mathcal{W}_n^q$ be the set of q-decreasing words of length n.

Example: $\mathcal{W}_4^1 = \{0000, \underset{3>1}{0001}, 0010, 1000, 1001, 1100, 1110, 1111\}$.

q-decreasing words

Let $q > 0$ be a real number and $n \geq 0$ an integer.

Definition

A *q-decreasing word* of length n is a binary word of length n such that:

every maximal factor of the form $0^a 1^b$ *satisfies either* $a = 0$ *or* $q \cdot a > b$.

Let $\mathcal{W}_n^q$ be the set of q-decreasing words of length n.

Example: $\mathcal{W}_4^1 = \{0000, 0001, \underset{2>1}{0010}, 1000, 1001, 1100, 1110, 1111\}$.

q-decreasing words

Let $q > 0$ be a real number and $n \geq 0$ an integer.

Definition

A *q-decreasing word* of length n is a binary word of length n such that:

every maximal factor of the form $0^a 1^b$ satisfies either $a = 0$ or $q \cdot a > b$.

Let $\mathcal{W}_n^q$ be the set of q-decreasing words of length n.

Example: $\mathcal{W}_4^1 = \{0000, 0001, 00\underset{1>0}{10}, 1000, 1001, 1100, 1110, 1111\}$.

q-decreasing words

Let $q > 0$ be a real number and $n \geq 0$ an integer.

Definition

A *q-decreasing word* of length n is a binary word of length n such that:

every maximal factor of the form 0^a1^b satisfies either $a = 0$ or $q \cdot a > b$.

Let $\mathcal{W}_n^q$ be the set of q-decreasing words of length n.

Example: $\mathcal{W}_4^1 = \{0000, 0001, 0010, \underbrace{1}_{a=0}000, 1001, 1100, 1110, 1111\}$.

q-decreasing words

Let $q > 0$ be a real number and $n \geq 0$ an integer.

Definition

A *q-decreasing word* of length n is a binary word of length n such that:

every maximal factor of the form 0^a1^b satisfies either $a = 0$ or $q \cdot a > b$.

Let $\mathcal{W}_n^q$ be the set of q-decreasing words of length n.

Example: $\mathcal{W}_4^1 = \{0000, 0001, 0010, 1000, 1001, 1100, 1110, 1111\}$.

Proposition

When q is a positive integer, $\mathcal{W}_n^q$ is enumerated by the $(q+1)$-generalized Fibonacci numbers F_{n+1}^{q+1}, where F_n^q is defined by

$$F_n^q = F_{n-1}^q + F_{n-2}^q + \ldots + F_{n-q}^q,$$

with initial conditions $F_n^q = 0$ for $n < 0$ and $F_0^q = 1$.

Context

- J.-L Baril, S. Kirgizov and V. Vajnovszki (2022).
  - Introduced q-decreasing words
  - Bijection between $\mathcal{W}_n^q$ and binary words with length n avoiding 1^{q+1} when q is an **integer**
  - 1-Gray code for $\mathcal{W}_n^1$ (answering a conjecture of Eğecioğlu and Iršič)

Context

- J.-L Baril, S. Kirgizov and V. Vajnovszki (2022).
  - Introduced q-decreasing words
  - Bijection between $\mathcal{W}_n^q$ and binary words with length n avoiding 1^{q+1} when q is an **integer**
  - 1-Gray code for $\mathcal{W}_n^1$ (answering a conjecture of Eğecioğlu and Iršič)
- E. Barcucci, A. Bernini, S. Bilotta and R. Pinzani (2025).
  - Bijection between $\mathcal{W}_n^q$ and binary words with length n avoiding some patterns when q is a **rational number**

Context

- J.-L Baril, S. Kirgizov and V. Vajnovszki (2022).
  - Introduced q-decreasing words
  - Bijection between $\mathcal{W}_n^q$ and binary words with length n avoiding 1^{q+1} when q is an **integer**
  - 1-Gray code for $\mathcal{W}_n^1$ (answering a conjecture of Eğecioğlu and Iršič)
- E. Barcucci, A. Bernini, S. Bilotta and R. Pinzani (2025).
  - Bijection between $\mathcal{W}_n^q$ and binary words with length n avoiding some patterns when q is a **rational number**
- S. Dovgal and S. Kirgizov (2025).
  - Growth rate of $|\mathcal{W}_n^q|$ for any positive real number q

q-decreasing words

Theorem - Dovgal, Kirgizov (2025)

For any real positive q, the number of q-decreasing words of length n satisfies as $n \to \infty$

$$|\mathcal{W}_n^q| \sim C_q \cdot \Phi(q)^n$$

for some constant C_q and where Φ is a function such that

- Φ is strictly increasing,
- Φ is discontinuous at every positive rational point,
- for any integer k, $\Phi(k)$ is the $(k+1)$-bonacci constant.

Example: $\Phi(1) = \frac{1+\sqrt{5}}{2}$, $\Phi(2)$ is the tribonacci constant.

The boolean lattice

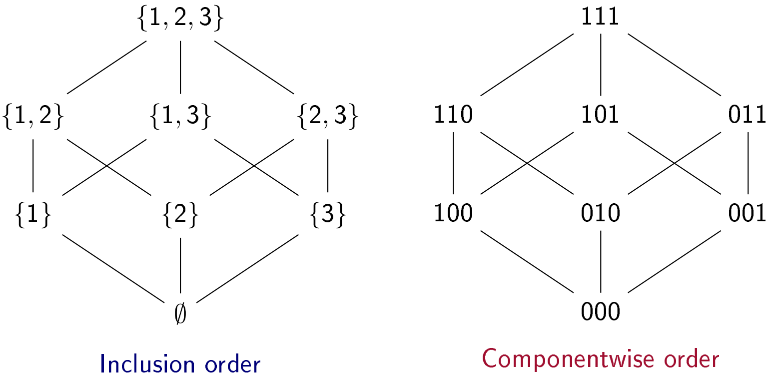

Figure: The Hasse diagram of the boolean lattice of size 3.

The poset of q-decreasing words

For $v = v_1 v_2 \ldots v_n$ and $w = w_1 w_2 \ldots w_n$ two q-decreasing words, we have

$$v \leq w \iff v_i \leq w_i \text{ for all } 1 \leq i \leq n.$$

Theorem

The poset $\mathbb{W}_n^q := (\mathcal{W}_n^q, \leq)$ is a lattice.

Remark: This is not a sublattice of the boolean lattice. Indeed, in $\mathbb{W}_3^1$,

$$100 \vee 001 = 111 \neq 101.$$

An example

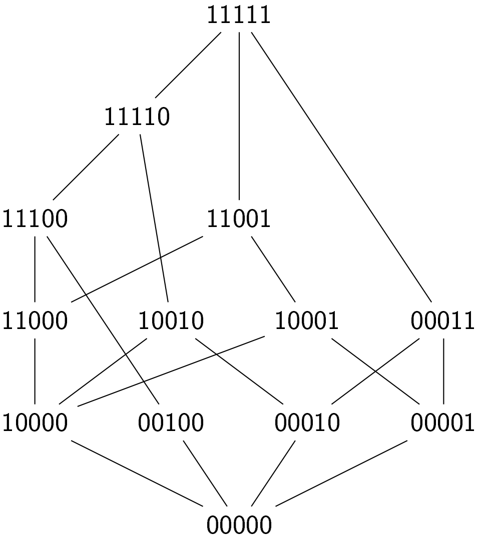

Figure: The lattice $\mathbb{W}_5^1$.

An example

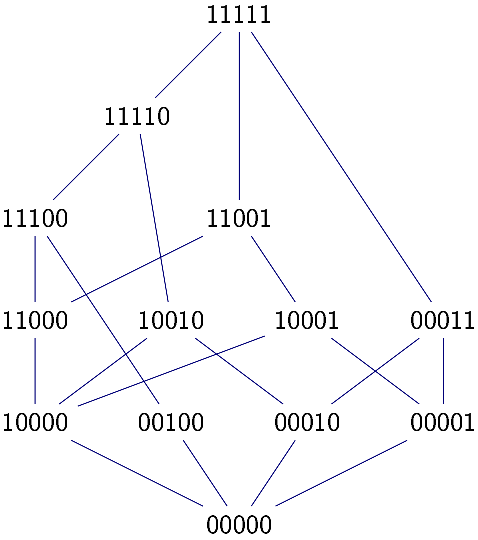

Figure: Coverings.

An example

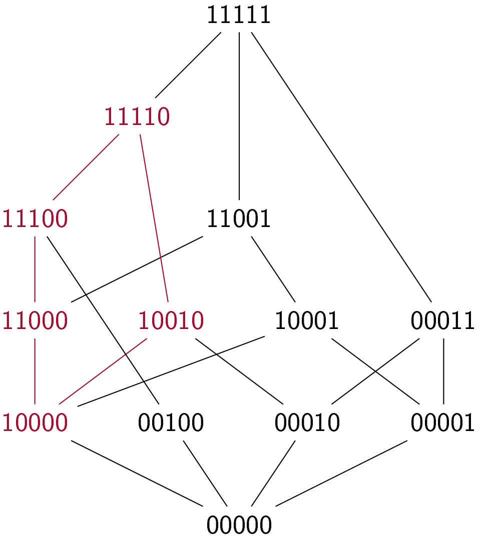

Figure: Intervals.

An example

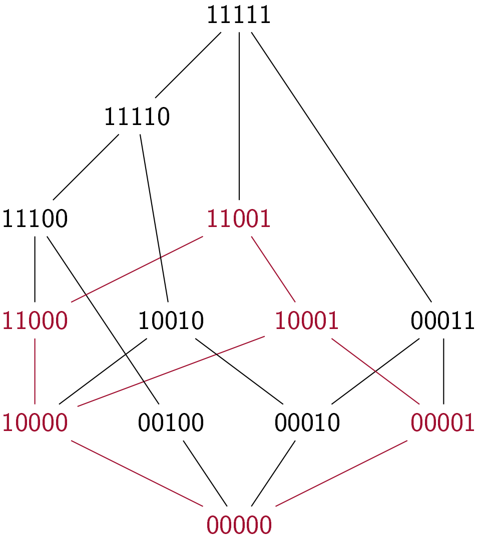

Figure: Intervals.

An example

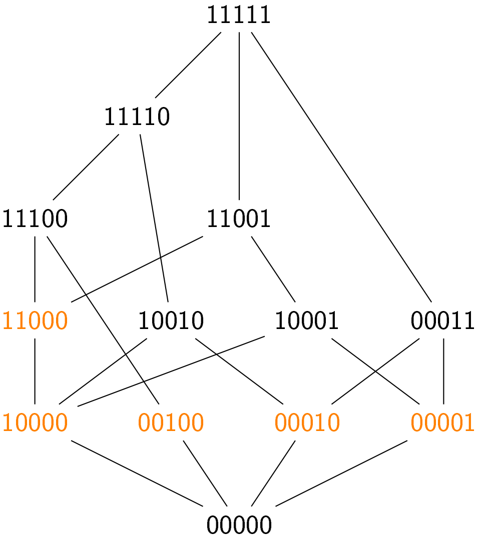

Figure: Join-irreducible elements.

An example

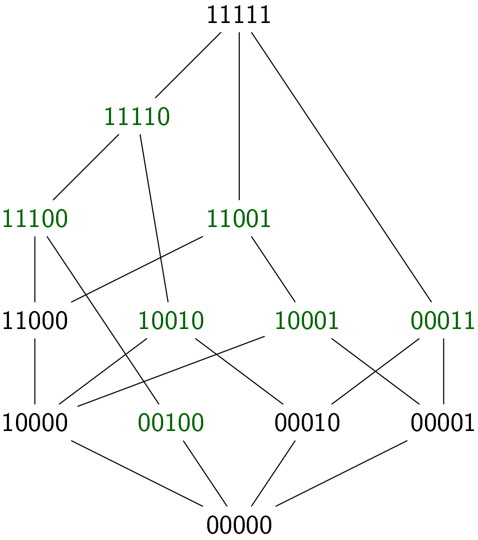

Figure: Meet-irreducible elements.

Structure of q-decreasing words

Any word $w \in \mathcal{W}_n^q$ can be written

$$w = 1^m 0^{a_1} 1^{b_1} \dots 0^{a_k} 1^{b_k} 0^\ell,$$

with

- $m, \ell \geq 0$,
- $q \cdot a_i > b_i \geq 1$ for $1 \leq i \leq k$,

Structure of q-decreasing words

Any word $w \in \mathcal{W}_n^q$ can be written

$$w = 1^m \underbrace{0^{a_1} 1^{b_1} \ldots 0^{a_k} 1^{b_k}}_{\text{prime factors}} 0^\ell,$$

with

- $m, \ell \geq 0$,
- $q \cdot a_i > b_i \geq 1$ for $1 \leq i \leq k$,

Structure of q-decreasing words

Any word $w \in \mathcal{W}_n^q$ can be written

$$w = 1^m \underbrace{0^{a_1} 1^{b_1} \ldots 0^{a_k} 1^{b_k}}_{\text{prime factors}} 0^{\ell},$$

with

- $m, \ell \geq 0$,
- $q \cdot a_i > b_i \geq 1$ for $1 \leq i \leq k$,

Theorem - Kirgizov (2022)

The generating function $W_q(x)$ for the number of q-decreasing words depending on their length is given by

$$W_q(x) = \frac{1}{(1-x)\left(1 - \sum_{i=0}^{\infty} x^{1+i+\left\lfloor \frac{i}{q} \right\rfloor}\right)}.$$

Furthermore, when $q = c/d$ is a rational number, we have

$$W_{c/d}(x) = \frac{1 - x^{c+d}}{(1-x)\left(1 - x^{c+d} - \sum_{i=0}^{c-1} x^{1+i+\left\lfloor \frac{di}{c} \right\rfloor}\right)}.$$

Intervals

Definition

An interval $I = [v, w]$ is called *prime* whenever w is prime, i.e. $w = 0^a 1^b$ with $q \cdot a > b \geq 1$.
Let $P_q(x)$ be the generating function for the number of prime intervals.

Intervals

Definition

An interval $I = [v, w]$ is called *prime* whenever w is prime, i.e. $w = 0^a 1^b$ with $q \cdot a > b \geq 1$.
Let $P_q(x)$ be the generating function for the number of prime intervals.

$$
\begin{array}{cccccccc}
w & = & 1\ldots1 & 0^{a_1}1^{b_1} & \cdots & 0^{a_k}1^{b_k} & 0\ldots0 \\
v & = & \nu & v_1 & \cdots & v_k & 0\ldots0
\end{array}
$$

Intervals

Definition

An interval $I = [v, w]$ is called *prime* whenever w is prime, i.e. $w = 0^a 1^b$ with $q \cdot a > b \geq 1$.
Let $P_q(x)$ be the generating function for the number of prime intervals.

$$
\begin{array}{cccccccc}
w & = & \boxed{1 \dots 1} & 0^{a_1} 1^{b_1} & \cdots & 0^{a_k} 1^{b_k} & 0 \dots 0 \\
v & = & \nu & v_1 & \cdots & v_k & 0 \dots 0
\end{array}
$$

$$\nu \leq 1 \dots 1$$

Intervals

Definition

An interval $I = [v, w]$ is called *prime* whenever w is prime, i.e. $w = 0^a 1^b$ with $q \cdot a > b \geq 1$.
Let $P_q(x)$ be the generating function for the number of prime intervals.

w	$=$	$1\dots1$	$0^{a_1}1^{b_1}$	$\cdots$	$0^{a_k}1^{b_k}$	$0\dots0$
v	$=$	ν	v_1	$\cdots$	v_k	$0\dots0$
		$\nu \leq 1\dots1$	prime intervals			

Intervals

Definition

An interval $I = [v, w]$ is called *prime* whenever w is prime, i.e. $w = 0^a 1^b$ with $q \cdot a > b \geq 1$.
Let $P_q(x)$ be the generating function for the number of prime intervals.

w	$=$	$1 \ldots 1$	$0^{a_1} 1^{b_1}$	$\cdots$	$0^{a_k} 1^{b_k}$	$0 \ldots 0$
v	$=$	ν	v_1	$\cdots$	v_k	$0 \ldots 0$
		$\nu \leq 1 \ldots 1$	prime intervals			

Intervals

Definition

An interval $I = [v, w]$ is called *prime* whenever w is prime, i.e. $w = 0^a 1^b$ with $q \cdot a > b \geq 1$.
Let $P_q(x)$ be the generating function for the number of prime intervals.

$w =$	$1 \ldots 1$	$0^{a_1} 1^{b_1}$	$\cdots$	$0^{a_k} 1^{b_k}$	$0 \ldots 0$
$v =$	ν	v_1	$\cdots$	v_k	$0 \ldots 0$
	$\nu \leq 1 \ldots 1$	prime intervals			

Proposition

For any real $q > 0$, the generating function $I_q(x)$ for the number of intervals in $\mathbb{W}_n^q$ is given by

$$I_q(x) = \frac{W_q(x)}{(1 - x)(1 - P_q(x))}.$$

Structure of prime intervals

Let $w = 0^a 1^b$ and $v \leq w$. Then we have either

Structure of prime intervals

Let $w = 0^a 1^b$ and $v \leq w$. Then we have either

1. $v = 0^{a+b}$, or

Structure of prime intervals

Let $w = 0^a 1^b$ and $v \leq w$. Then we have either

1. $v = 0^{a+b}$, or
2. $v = 0^{a+k} 1 \nu$, with $k \in [0, b-1]$ and $\nu \in \mathcal{W}^q_{b-k-1}$.

Let $w = 0^a 1^b$ and $v \leq w$. Then we have either

1. $v = 0^{a+b}$, or
2. $v = 0^{a+k} 1 \nu$, with $k \in [0, b-1]$ and $\nu \in \mathcal{W}^q_{b-k-1}$.

$$P_q(x) = \sum_{b=1}^{+\infty} \sum_{q \cdot a > b} \left(1 + \sum_{k=0}^{b-1} |\mathcal{W}^q_{b-k-1}| \right) x^{a+b}.$$

Structure of prime intervals

Let $w = 0^a 1^b$ and $v \leq w$. Then we have either

1. $v = 0^{a+b}$, or
2. $v = 0^{a+k}1\nu$, with $k \in [0, b-1]$ and $\nu \in \mathcal{W}^q_{b-k-1}$.

$$P_q(x) = \sum_{b=1}^{+\infty} \sum_{q \cdot a > b} \left(1 + \sum_{k=0}^{b-1} |\mathcal{W}^q_{b-k-1}| \right) x^{a+b}.$$

$$P_q(x) = \sum_{b=1}^{+\infty} \left(1 + \sum_{k=0}^{b-1} |\mathcal{W}^q_k| \right) \frac{x^{1+b+\left\lfloor \frac{b}{q} \right\rfloor}}{1-x}.$$

Structure of prime intervals

Let $w = 0^a 1^b$ and $v \leq w$. Then we have either

1. $v = 0^{a+b}$, or
2. $v = 0^{a+k} 1 \nu$, with $k \in [0, b-1]$ and $\nu \in \mathcal{W}^q_{b-k-1}$.

$$P_q(x) = \sum_{b=1}^{+\infty} \sum_{q \cdot a > b} \left(1 + \sum_{k=0}^{b-1} |\mathcal{W}^q_{b-k-1}| \right) x^{a+b}.$$

$$P_q(x) = \sum_{b=1}^{+\infty} \left(1 + \sum_{k=0}^{b-1} |\mathcal{W}^q_k| \right) \frac{x^{1+b+\lfloor \frac{b}{q} \rfloor}}{1-x}.$$

Intervals

Theorem

Let $q = c/d$ be a positive rational number, and define the polynomial $\Pi_{c/d}(x) \;=\; 1 - x^{c+d} - \sum_{k=0}^{c-1} x^{1+k+\left\lfloor \frac{dk}{c} \right\rfloor}$. Let $\rho = e^{\frac{2i\pi}{c}}$, then

$$P_q(x) = \frac{\sum_{k=0}^{c(c+d+2)} a_k x^{1+k+\left\lfloor \frac{dk}{c} \right\rfloor}}{(1-x)(1-x^{c+d})^2 \prod_{k=0}^{c-1} \Pi_q(\rho^k x^{1+\frac{d}{c}})},$$

where

$$\sum_{k=0}^{c(c+d+2)} a_k x^k = x\left(1 - x^{c+d} + (1-x)\Pi_q(x)\right)(1+\ldots+x^{c-1})^2 \prod_{k=1}^{c-1} \Pi_q(\rho^k x).$$

Corollary

Let $r = e^{\frac{2i\pi}{c+d}}$. There exists a nonzero polynomial $T \in \mathbb{C}[X]$ with degree at most $c + d - 1$ such that

$$[x^n]P_q(x) \underset{n\to\infty}{\sim} T(r^n) \cdot \Phi(q)^{\frac{cn}{c+d}}.$$

Example:

$$P_1(x) = \frac{x^3(x^4 - 2)}{(x-1)^2(x+1)(x^4 + x^2 - 1)}$$

$$[x^n]P_1(x) \sim c \cdot (1 + (-1)^n) \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{n/2}$$

Corollary

Let $r = e^{\frac{2i\pi}{c+d}}$. There exists a nonzero polynomial $T \in \mathbb{C}[X]$ with degree at most $c+d-1$ such that

$$[x^n]P_q(x) \underset{n\to\infty}{\sim} T(r^n) \cdot \Phi(q)^{\frac{cn}{c+d}}.$$

Example:

$$P_1(x) = \frac{x^3(x^4-2)}{(x-1)^2(x+1)(x^4+x^2-1)}$$

$$[x^n]P_1(x) \sim c \cdot (1+(-1)^n) \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{n/2}$$

with $c = \frac{1}{40}\left(30 + 14\sqrt{5} + (25+11\sqrt{5})\sqrt{-2+2\sqrt{5}}\right)$.

Meet-irreducible elements when $0 < q \leq 1$

Fact

A word $w \in \mathbb{W}_n^q$ is meet-irreducible if and only if it has exactly one upper cover.

Meet-irreducible elements when $0 < q \leq 1$

Fact

A word $w \in \mathbb{W}_n^q$ is meet-irreducible if and only if it has exactly one upper cover.

$$\mathtt{B} = \{0^a 1^b \mid 0^{a-1} 1^b \text{ is } q\text{-decreasing}\}$$

Meet-irreducible elements when $0 < q \leq 1$

Fact

A word $w \in \mathbb{W}_n^q$ is meet-irreducible if and only if it has exactly one upper cover.

$$\mathtt{B} = \{0^a 1^b \mid 0^{a-1} 1^b \text{ is } q\text{-decreasing}\}$$

$$\mathtt{C} = \{0^a 1^b \mid 0^a 1^{b+1} \text{ is } q\text{-decreasing}\}$$

Meet-irreducible elements when $0 < q \leq 1$

Fact

A word $w \in \mathbb{W}_n^q$ is meet-irreducible if and only if it has exactly one upper cover.

$$\mathtt{B} = \{0^a 1^b \mid 0^{a-1} 1^b \text{ is } q\text{-decreasing}\}$$

$$\mathtt{C} = \{0^a 1^b \mid 0^a 1^{b+1} \text{ is } q\text{-decreasing}\}$$

Lemma

A word w is meet-irreducible if and only if its prime factor decomposition avoids the pattern $\mathtt{CB}$.

Meet-irreducible elements when $0 < q \leq 1$

Fact

A word $w \in \mathbb{W}_n^q$ is meet-irreducible if and only if it has exactly one upper cover.

$$\mathtt{B} = \{0^a 1^b \mid 0^{a-1} 1^b \text{ is } q\text{-decreasing}\}$$

$$\mathtt{C} = \{0^a 1^b \mid 0^a 1^{b+1} \text{ is } q\text{-decreasing}\}$$

Lemma

A word w is meet-irreducible if and only if its prime factor decomposition avoids the pattern $\mathtt{CB}$.

Example.

$$\underbrace{000011}_{\in \mathtt{C}}\underbrace{0001}_{\in \mathtt{B}}$$

Meet-irreducible elements when $0 < q \leq 1$

Fact

A word $w \in \mathbb{W}_n^q$ is meet-irreducible if and only if it has exactly one upper cover.

$$\mathtt{B} = \{0^a 1^b \mid 0^{a-1} 1^b \text{ is } q\text{-decreasing}\}$$

$$\mathtt{C} = \{0^a 1^b \mid 0^a 1^{b+1} \text{ is } q\text{-decreasing}\}$$

Lemma

A word w is meet-irreducible if and only if its prime factor decomposition avoids the pattern $\mathtt{CB}$.

Example.

$$\underbrace{000011}_{\in \mathtt{C}}\underbrace{0001}_{\in \mathtt{B}} \leq \quad 1111110001$$

Meet-irreducible elements when $0 < q \leq 1$

Fact

A word $w \in \mathbb{W}_n^q$ is meet-irreducible if and only if it has exactly one upper cover.

$$\mathtt{B} = \{0^a 1^b \mid 0^{a-1}1^b \text{ is } q\text{-decreasing}\}$$

$$\mathtt{C} = \{0^a 1^b \mid 0^a 1^{b+1} \text{ is } q\text{-decreasing}\}$$

Lemma

A word w is meet-irreducible if and only if its prime factor decomposition avoids the pattern $\mathtt{CB}$.

Example.

$$\underbrace{000011}_{\in \mathtt{C}}\underbrace{0001}_{\in \mathtt{B}} \leq \begin{matrix} 1111110001 \\ 0000111001 \end{matrix}$$

Meet-irreducible elements when $0 < q \leq 1$

Fact

A word $w \in \mathbb{W}_n^q$ is meet-irreducible if and only if it has exactly one upper cover.

$$\mathtt{B} = \{0^a 1^b \mid 0^{a-1}1^b \text{ is } q\text{-decreasing}\}$$

$$\mathtt{C} = \{0^a 1^b \mid 0^a 1^{b+1} \text{ is } q\text{-decreasing}\}$$

Lemma

A word w is meet-irreducible if and only if its prime factor decomposition avoids the pattern $\mathtt{CB}$.

Example.

$$\underbrace{000011}_{\in\mathtt{C}}\,\underbrace{0001}_{\in\mathtt{B}} \leq \begin{array}{c} 1111110001 \\ \underbrace{0000111}_{\in\mathcal{W}^q}001 \end{array}$$

Meet-irreducible elements when $0 < q \leq 1$

Fact

A word $w \in \mathbb{W}_n^q$ is meet-irreducible if and only if it has exactly one upper cover.

$$\mathtt{B} = \{0^a 1^b \mid 0^{a-1} 1^b \text{ is } q\text{-decreasing}\}$$

$$\mathtt{C} = \{0^a 1^b \mid 0^a 1^{b+1} \text{ is } q\text{-decreasing}\}$$

Lemma

A word w is meet-irreducible if and only if its prime factor decomposition avoids the pattern $\mathtt{CB}$.

Example.

$$\underbrace{000011}_{\in \mathtt{C}}\underbrace{0001}_{\in \mathtt{B}} \leq \begin{matrix} 1111110001 \\ 0000111\underbrace{001}_{\in \mathcal{W}^q} \end{matrix}$$

Meet-irreducible elements when $0 < q \leq 1$

Theorem

Let $q = c/d$ be a rational number with $0 < q \leq 1$. The generating function $M(x)$ for the number of meet-irreducible elements in $\mathbb{W}_n^q$ is

$$M(x) = \frac{x^{2+\lfloor d/c \rfloor}(x-1)A - (x-1)A^2 + x^2}{(x-1)\left((1 - Ax^{2+\lfloor d/c \rfloor} + Ax^{3+\lfloor d/c \rfloor} + (x-1)(x+A-A^2)\right)},$$

with

$$A = \frac{\sum_{i=1}^{c} x^{1+i+\left\lfloor \frac{id}{c} \right\rfloor}}{1 - x^{c+d}}.$$

Meet-irreducible elements when $q > 1$

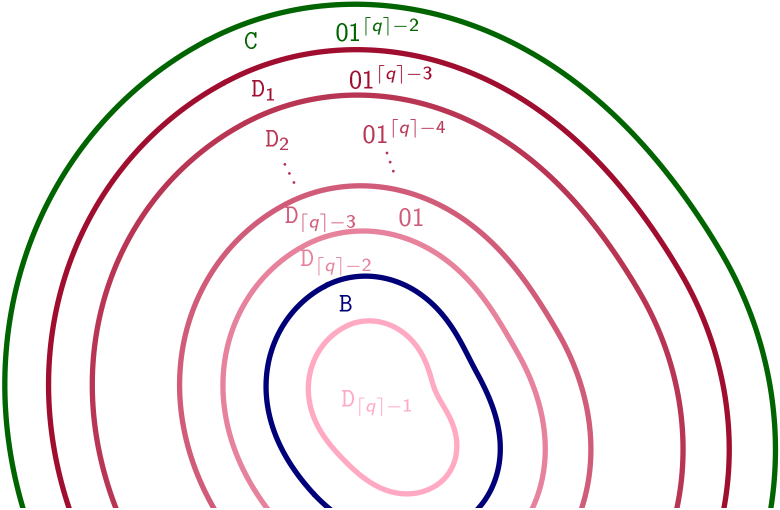

Figure: More sets and more forbidden patterns → words over an alphabet of $2\lceil q \rceil + 1$ letters avoiding $\lceil q \rceil^2 + 2\lceil q \rceil - 1$ patterns

Biblio

1. J.-L Baril, S. Kirgizov and V. Vajnovszki, *Gray codes for Fibonacci q-decreasing words*, Theoretical Computer Science, 2022.
2. S. Kirgizov, *$\mathbb{Q}$-bonacci words and numbers*, Fibonacci Quarterly, 2022.
3. S. Dovgal and S. Kirgizov, *Structure and growth of $\mathbb{R}$-bonacci words*, The Electronic Journal of Combinatorics, 2025.
4. E. Barcucci, A. Bernini, S. Bilotta and R. Pinzani, *Pattern avoiding and q-decreasing binary words*, RAIRO-Theor. Inf. Appl.,2025.
5. J.-L. Baril, N. Hassler, S. Kirgizov, *Enumeration in the lattice of q-decreasing words*, https://arxiv.org/abs/2511.09480.

Biblio

1. J.-L Baril, S. Kirgizov and V. Vajnovszki, *Gray codes for Fibonacci q-decreasing words*, Theoretical Computer Science, 2022.
2. S. Kirgizov, *$\mathbb{Q}$-bonacci words and numbers*, Fibonacci Quarterly, 2022.
3. S. Dovgal and S. Kirgizov, *Structure and growth of $\mathbb{R}$-bonacci words*, The Electronic Journal of Combinatorics, 2025.
4. E. Barcucci, A. Bernini, S. Bilotta and R. Pinzani, *Pattern avoiding and q-decreasing binary words*, RAIRO-Theor. Inf. Appl.,2025.
5. J.-L. Baril, N. Hassler, S. Kirgizov, *Enumeration in the lattice of q-decreasing words*, https://arxiv.org/abs/2511.09480.

Thank you!