A COMBINATORIAL PROBLEM RELATED TO HALF-FACTORIAL SETS AND CONNECTIONS TO GRAPH THEORY

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Contents

1. Notation

1.1. **Half-factorial monoids.** Let H be a monoid, that is a commutative cancellative semigroup with identity element $1_H \in H$. The subgroup of units (invertible elements) of H is denoted by $H^\times$. An element $a \in H \backslash H^\times$ is called an *atom* (or irreducible) if a has no non-trivial divisor, i.e. $a = bc$, $b, c \in H$ implies $b \in H^\times$ or $c \in H^\times$. Let $\mathcal{A}(H)$ be the set of atoms of H. Let $a \in H$. If $a = u_1 \dots u_l$ with $u_1, \dots, u_l \in \mathcal{A}(H)$ we say that this is a *factorization of length l of a*. We define the *set of lengths* of $a \in H \backslash H^\times$:

$$\mathsf{L}_H(a) := \{l \in \mathbb{N} \mid a \text{ has a factorization of length } l\} \subset \mathbb{N}.$$

If $a \in H^\times$ we set $\mathsf{L}_H(a) = \{0\}$. We may also write $\mathsf{L}(a)$ when it is clear.

Definition 1.1. *A monoid H is called half-factorial if $|\mathsf{L}_H(a)| = 1$ for every $a \in H$. Equivalently, H is half-factorial if each element $a \in H \backslash H^\times$ has a factorization into atoms and all factorizations of a have the same length.*

1.2. **Finite abelian groups.** Let G be a finite abelian group. We will only use additive notation. For a subset $G_0 \subset G$, resp. elements $e_1, \dots, e_r \in G$, we denote by $\langle G_0 \rangle$, resp. $\langle e_1, \dots, e_r \rangle$, the subgroup generated by G_0, resp. $e_1, \dots, e_r$. For $g \in G$ we denote by $\mathrm{ord}(g)$ the order of g. For $n \in \mathbb{N}$, let C_n denote the cyclic group with n elements. For a prime p and an integer $r \in \mathbb{N}$, we call an *elementary p-group of rank r* any group isomorphic to C_p^r.

1.3. **Sequences and block monoid.** Let G be a finite abelian group, and $G_0 \subset G$. We denote by $\mathcal{F}(G_0)$ the free abelian monoid generated by G_0, that is the set of commutative formal products

$$\mathcal{F}(G_0) = \{ \prod_{g \in G_0} g^{v_g} \mid v_g \in \mathbb{N} \}.$$

An element $S \in \mathcal{F}(G_0)$ is called a *sequence* in G_0. A divisor of S in $\mathcal{F}(G_0)$ is called a *subsequence* of S. For a sequence $S \in \mathcal{F}(G_0)$, there are unique integers $v_g(S)$, $g \in G_0$ such that $S = \prod_{g \in G_0} g^{v_g(S)}$. We denote by

- $|S| = \sum_{g \in G_0} v_g(S)$ the length of S,
- $\sigma(S) = \sum_{g \in G_0} v_g(S) g$ its sum,
- $\mathsf{k}(S) = \sum_{g \in G_0} \frac{v_g(S)}{\operatorname{ord}(g)}$ its *cross number*.

Then, $|\cdot| : \mathcal{F}(G_0) \to \mathbb{N}$, $\sigma : \mathcal{F}(G_0) \to G$ and $\mathsf{k} : \mathcal{F}(G_0) \to \mathbb{Q}_{\geq 0}$ are monoid homomorphisms. The kernel of σ is called the *block monoid* over G_0. We denote it by $\mathcal{B}(G_0)$, and we denote the set of its atoms by $\mathcal{A}(G_0)$ for conciseness. The elements of $\mathcal{A}(G_0)$ are simply the zero-sum sequences that do not have a proper zero-sum subsequence. It is clear that every element in $\mathcal{B}(G_0)$ has a factorization into atoms.

Definition 1.2. *Let G be a finite abelian group. A subset $G_0 \subset G$ is called a half-factorial set if its block monoid $\mathcal{B}(G_0)$ is a half-factorial monoid. We denote by $\mu(G)$ the maximum cardinality of a half-factorial set in G:*

$$\mu(G) := \max\{|G_0| \mid G_0 \subset G \textit{ is half-factorial}\}.$$

2. Results on half-factorial sets

2.1. **General results.** Let us first prove a very useful characterization of half-factorial sets, due to L. Skula [6] and A. Zaks.

Theorem 2.1. *A subset $G_0 \subset G$ is a half-factorial set if and only if $\mathsf{k}(A) = 1$ for every $A \in \mathcal{A}(G_0)$.*

Proof. Suppose that $\mathsf{k}(A) = 1$ for every $A \in \mathcal{A}(G_0)$. Let $S \in \mathcal{B}(G_0)$ and $S = \prod_{i=1}^n U_i = \prod_{i=1}^m U_i'$ be two factorizations of S into atoms, $U_i, U_i' \in \mathcal{A}(G_0)$. Then

$$\mathsf{k}(S) = \sum_{i=1}^{n} \underbrace{\mathsf{k}(U_i)}_{=1} = \sum_{i=1}^{m} \underbrace{\mathsf{k}(U_i')}_{=1}.$$

Thus $n = m$ and G_0 is half-factorial.
Conversely, assume G_0 is half-factorial. Let $A \in \mathcal{A}(G_0)$. We set $G_0 = \{g_1, \ldots, g_r\}$, $m = \prod_{i=1}^r \operatorname{ord}(g_i)$ and $m_j = \frac{m}{\operatorname{ord}(g_j)} v_{g_j}(A)$ for $1 \leq j \leq r$. Note that the sequences $g_i^{\operatorname{ord}(g_i)}$, for $1 \leq i \leq r$, belong to $\mathcal{A}(G_0)$. Then

$$A^m = \left(\prod_{j=1}^{r} g_j^{v_{g_j}(A)} \right)^m = \prod_{j=1}^{r} \left(g_j^{\operatorname{ord}(g_j)} \right)^{m_j}.$$

But $A \in \mathcal{A}(G_0)$ and $g_j^{\operatorname{ord}(g_j)} \in \mathcal{A}(G_0)$, so we have two factorizations of the same block into atoms. Since G_0 is half-factorial, we have $m = \sum_{j=1}^r m_j$, and dividing by m we get

$$\sum_{j=1}^{r} \frac{v_{g_j}(A)}{\operatorname{ord}(g_j)} = \mathsf{k}(A) = 1.$$

□

2.2. **Elementary p-groups.** A. Geroldinger and J. Kacorowski determined the exact value of $\mu(G)$ when G is a elementary p-group with even rank [9]. Then, A. Plagne and W. Schmid determined the exact value of $\mu(G)$ for any elementary p-group, as well as the structure of half-factorial sets of maximal cardinality [7].

Theorem 2.2 ([7])**.** *Let G be an elementary p-group of rank r. Then,*

$$\mu(G) = \begin{cases} 2 + \frac{r-1}{2}p & \text{if } r \text{ is odd,} \\ 1 + \frac{r}{2}p & \text{if } r \text{ is even.} \end{cases}$$

We only state a weaker version of [7, Theorem 1.2].

Theorem 2.3 ([7])**.** *Let G be an elementary p-group of rank r, and $G_0 \subset G$ a half-factorial set with $|G_0| = \mu(G)$. Then there exists a basis $\{f_1, \ldots, f_r\} \subset G$ such that*

$$G_0 = \bigcup_{i=1}^{\lfloor \frac{r}{2} \rfloor} \{jf_{2i-1} + (p+1-j)f_{2i} \mid j \in [1,p]\} \cup \{f_r, 0\}.$$

3. The constant $\psi_k(G)$ for elementary p-groups with even rank

3.1. **Definition and useful results.** Throughout this section, G denotes a finite abelian group with $|G| \geq 3$. For a subset $G_0 \subset G$ and a sequence $S \in \mathcal{F}(G \backslash G_0)$ we set

$$\Omega(G_0, S) = S\mathcal{F}(G_0) \cap \mathcal{B}(G) = \{B \in \mathcal{B}(G) \mid v_g(B) = v_g(S) \text{ for all } g \in G \backslash G_0\}.$$

Let also $\mathcal{G}_k = \{B \in \mathcal{B}(G_0) \mid |\mathsf{L}(B)| \leq k\}$ be the set of blocks that have at most k different lengths of factorization. Now we are ready for the central definition of this section.

Definition 3.1. *Let $k \in \mathbb{N}$. Then*

$$\psi_k(G) = \max\{|S| \mid G_0 \subset G \text{ half-factorial with } |G_0| = \mu(G) \text{ and } S \in \mathcal{F}(G\backslash G_0) \text{ with } \emptyset \neq \Omega(G_0, S) \subset \mathcal{G}_k\}.$$

Now let us prove a Lemma from [1] which will be very useful. First, we recall a classic result of additive combinatorics.

Lemma 3.2. *Let $A, B \subset \mathbb{R}$ be finite sets of real numbers. Let $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$. Then $|A + B| \geq |A| + |B| - 1$.*

Proof. Suppose $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ with $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_m$. Then we have the following inequalities:

$$a_1 + b_1 < a_1 + b_2 < \ldots < a_1 + b_m < a_2 + b_m < \ldots < a_n + b_m$$

and all those numbers belong to $A+B$. There are $|A|+|B|-1$ of them, so $|A+B| \geq |A|+|B|-1$. □

Lemma 3.3 ([1], Lemma 4.1)**.** *Let $\emptyset \neq G_0 \subset G$ and $S, S' \in \mathcal{F}(G \backslash G_0)$. Let $k, l \in \mathbb{N}$ be such that $\Omega(G_0, S) \not\subset \mathcal{G}_k$ and $\Omega(G_0, S') \not\subset \mathcal{G}_l$. Then*

$$\Omega(G_0, SS') \not\subset \mathcal{G}_{k+l}.$$

Proof. Let $B \in \Omega(G_0, S)$ with $|\mathsf{L}(B)| \geq k + 1$ and $B' \in \Omega(G_0, S')$ with $|\mathsf{L}(B')| \geq l + 1$. Then $BB' \in \Omega(G_0, SS')$, and $\mathsf{L}(B) + \mathsf{L}(B') \subset \mathsf{L}(BB')$, so by Lemma 3.2,

$$|\mathsf{L}(BB')| \geq |\mathsf{L}(B) + \mathsf{L}(B')| \geq |\mathsf{L}(B)| + |\mathsf{L}(B')| - 1 \geq k + l + 1.$$

Thus $\Omega(G_0, SS') \not\subset \mathcal{G}_{k+l}$. □

From now until the end of Section 3, we assume that G is an elementary p-group with even rank $2r$. By Theorem 2.3, all half-factorial subsets of G are equal up to automorphisms of the group. Thus it suffices to investigate $\Omega(G_0, \cdot)$ for one fixed half-factorial subset $G_0 \subset G$ of maximal cardinality. The case $p = 2$ has been completely solved in [2], so we assume $p \geq 3$. We fix a basis $\{e_1, e'_1, \ldots, e_r, e'_r\}$ of G and a half-factorial set

$$G_0 = \{0\} \cup \bigcup_{i=1}^{r} \{je_i + e'_i \mid j \in [0, p-1]\}$$

such that $|G_0| = \mu(G)$. Note that we made the following change of basis in comparison to Theorem 2.3: $e'_i = f_{2i}$ and $e_i = f_{2i-1} - f_{2i}$ for $1 \leq i \leq r$. For $1 \leq i \leq r$, let π_i denote the projection on $\langle e_i \rangle$, π'_i the projection on $\langle e'_i \rangle$ and $G_0^i = (\pi_i + \pi'_i)(G_0) = \{0\} \cup \{je_i + e'_i \mid j \in [0, p-1]\}$. Let us conclude this subsection with a crucial result [2, Proposition 4.4] for finding upper bounds on $\psi_k(G)$.

Proposition 3.4. *Let $p \geq 3$. Then $\Omega(G_0, S) \not\subset \mathcal{G}_1$ for each of the following choices of $S \in \mathcal{F}(G \backslash G_0)$.*

(1) Let $S = g$ with $g \in G \backslash (\langle e_1, \ldots, e_r \rangle + \sum_{i=1}^{r} G_0^i)$.

(2) Let $S \in \mathcal{F}((\langle e_1, \ldots, e_r \rangle + \sum_{i=1}^{r} G_0^i) \backslash \sum_{i=1}^{r} G_0^i)$ such that $(\pi_m + \pi'_m)(S) \in \mathcal{A}(\langle e_m \rangle \backslash \{0\})$ for some $m \in [1, r]$.

(3) Let $S = gh$ with $g, h \in \langle e_1, \ldots, e_r \rangle + \sum_{i=1}^{r} G_0^i$ such that $\pi'_j(g) = \pi'_j(h) = e'_j$ and $\pi'_m(g) = \pi'_m(h) = e'_m$ for distinct $j, m \in [1, r]$.

(4) Let $S = \prod_{j=1}^{s} g_j \in \mathcal{F}(\langle e_1, \ldots, e_r \rangle + \sum_{i=1}^{r} G_0^i)$ with $s \geq 3$ and $I_j = \{i \in [1, r] \mid \pi'_i(g_j) = e'_i\}$ such that for every $J \subset [1, r]$ with $|J| \geq 2$

$$|\bigcap_{j \in J} I_j| = \begin{cases} 1 & \text{if } J = \{j, j'\} \text{ and } j - j' = \pm 1 \mod s \\ 0 & \text{otherwise.} \end{cases}$$

3.2. **Alternative proof of Theorem 4.5 using hypergraphs.**

Proposition 3.5 ([3], 2.1 and [4], §2, Proposition 5)**.** *A hypergraph $\mathcal{H} = (E_i)_{i \in I}$ is cycle-free if and only if for every non-empty subset $J \subset I$, the following inequality holds:*

$$|J| > \sum_{i \in J} |E_i| - |\bigcup_{i \in J} E_i|.$$

Proof. Suppose $\mathcal{H}$ has a cycle $v_1 E_1 v_2 E_2 \ldots v_s E_s$. Then $\{v_1, \ldots, v_s\} \subset \bigcup_{i \in [1,s]} E_i$ and each v_i is counted at least twice in the sum $\sum_{i \in [1,s]} |E_i|$. Thus

$$\sum_{i \in [1,s]} |E_i| - |\bigcup_{i=1}^{s} E_i| \geq s.$$

Suppose $\mathcal{H}$ is cycle-free. Then any sub-hypergraph $(\bigcup_{i \in J} E_i, (E_i)_{i \in J})$ with $J \subset I$ is cycle-free too. Consider $G(J)$ the bipartite graph with vertex set $\bigcup_{i \in J} E_i \cup \{E_i \mid i \in J\}$ and such that $\{v, E_i\}$ is an edge if and only if $v \in E_i$, with $v \in \bigcup_{i \in J} E_i$, $i \in J$. Then $(\bigcup_{i \in J} E_i, (E_i)_{i \in J})$ is cycle-free if and only if $G(J)$ is a forest. But $G(J)$ has $|\bigcup_{i \in J} E_i| + |J|$ vertices, $\sum_{i \in J} |E_i|$ edges, and since it is a forest, we have

$$\sum_{i \in J} |E_i| = |J| + |\bigcup_{i \in J} E_i| - p$$

with $p \geq 1$ its number of connected components. In the end, we have

$$|J| > \sum_{i \in J} |E_i| - |\bigcup_{i \in J} E_i|.$$

□

Corollary 3.6. *Let $\mathcal{H}$ be a hypergraph on r vertices. If $\mathcal{H}$ has at least r edges, then $\mathcal{H}$ has a cycle.*

Proof. Suppose $\mathcal{H}$ is cycle-free and consider $J \subset I$ such that $|J| = r$. By Proposition 3.5 we have

$$r > \sum_{i \in J} \underbrace{|E_i|}_{\geq 2} - \underbrace{|\bigcup_{i \in J} E_i|}_{\leq r} \geq 2r - r = r$$

which is a contradiction. ☐

Theorem 3.7 ([2], Theorem 4.5)**.** *Let $p \geq 3$, $k \in \mathbb{N}$ and G be an elementary p-group with even rank $2r$. Then*

$$\psi_k(G) \leq rp - 1 + (k-1)\max\{p, r\}.$$

Proof. We proceed as in [2] for the first 3 steps, and we give an argument using hypergraphs for Step 4. Let us recall some notation. At this point we have $S = \prod_{i=1}^{l} g_i \in \mathcal{F}(\langle e_1, \ldots, e_r \rangle + \sum_{i=1}^{r} G_0^i)$ with $l \geq rp$, $S'|S$ the subsequence consisting of the elements $g|S$ such that there exists some $j_g \in [1, r]$ such that $(\pi_{j_g} + \pi'_{j_g})(g) \in \langle e_{j_g} \rangle \backslash \{0\}$. We set $T = S'^{-1}S$ and assume $|T| \geq r$. Furthermore, for $g|T$ let $I_g = \{i \in [1, r] \mid \pi'_i(g) = e'_i\}$ and assume that $gh|T \Rightarrow |I_g \cap I_h| \leq 1$. Now consider the hypergraph $([1, r], (I_g)_{g|T})$. By our assumptions, this hypergraph has at least r edges so by Corollary 3.6, it contains a cycle. Let g_j, $j = 1, \ldots, s$, be the indexes of the edges of this cycle. Since $gh|T \Rightarrow |I_g \cap I_h| \leq 1$, we have also $|I_{g_i} \cap I_{g_j}| = 1$ if $i - j = \pm 1 \mod s$. However we may have pairs i, j with $i - j \neq \pm 1 \mod s$ such that $|I_{g_i} \cap I_{g_j}| = 1$. This means that we can find a shorter cycle in the hypergraph, by omitting the edges $I_{g_{i+1}}, \ldots, I_{g_{j-1}}$. We can repeat this process until we get a cycle with no proper shorter cycle, i.e. elements g'_j, $1 \leq j \leq s'$ such that $r \geq s \geq s' \geq 3$ and for every $J \subset [1, s']$ with $|J| \geq 2$,

$$|\bigcap_{j \in J} I_{g'_j}| = \begin{cases} 1 & \text{if } J = \{j, j'\} \text{ and } j - j' = \pm 1 \mod s' \\ 0 & \text{otherwise} \end{cases}$$

Note that we have indeed $s' \geq 3$ because the condition $gh|T \Rightarrow |I_g \cap I_h| \leq 1$ implies that a cycle in the hypergraph cannot have length 2. Finally, we got exactly the condition we need to apply Proposition 3.4 and conclude. ☐

3.3. **Upper bound on $\psi_k(G)$ depending on $\mathsf{p}(k)$.** Let us introduce some constants from extremal graph theory, that we will connect to our problem to get an upper bound on $\psi_k(G)$.

Definition 3.8. *Let $k, n \in \mathbb{N}$.*

(1) $\mathsf{p}(k)$ denotes the smallest integer l with the following property: every graph with v vertices, for some $v \in \mathbb{N}$, and $v + l$ edges contains at least k edge disjoint cycles.
(2) $\mathsf{p}(k, n)$ denotes the smallest integer l with the following property: every graph with n vertices and $n + l$ edges contains at least k edge disjoint cycles.

By definition, $\mathsf{p}(k, n) \leq \mathsf{p}(k)$ for any $k, n \in \mathbb{N}$, and $\mathsf{p}(k, n) \leq \mathsf{p}(k + 1, n)$.

Definition 3.9. *Let $S = \prod_{j=1}^{l} g_j \in \mathcal{F}(G \backslash G_0)$ and for $1 \leq j \leq l$, $I_j = \{i \in [1, r] \mid \pi'_i(g_j) = e'_i\}$. The associated hypergraph of S is the hypergraph with vertex set $[1, r]$ and edges $(I_j)_{1 \leq j \leq l}$. An associated graph of S is a graph $([1, r], (E_j)_{1 \leq j \leq l})$ such that for all $1 \leq j \leq l$, $|E_j| = 2$ and $E_j \subseteq I_j$.*

Remark 3.10. *To make this definition rigorous, we need the sequence S to be such that $|I_j| \geq 2$ for all j. We will verify this condition each time we need to consider an associated graph.*

Definition 3.11. *Let $\Gamma = (V, E)$ be a hypergraph. We call a hypercycle of length s a set of edges $(E_j)_{j\in[1,s]} \subseteq E$ such that for all $J \subset [1, s]$ with $|J| \geq 2$,*

$$|\bigcap_{j\in J} E_j| = \begin{cases} 1 & \text{if } J = \{j, j'\} \text{ and } j - j' = \pm 1 \mod s \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.12. *This definition is not the same as the usual definition of a hypercycle we can find in hypergraphs literature.*

Lemma 3.13. *Let $T \in \mathcal{F}(\langle e_1, \ldots, e_r\rangle + \sum_{i=1}^r G_0^i)$ such that*

(1) $|I_g| \geq 2$ for all $g|T$;
(2) $|T| \geq r + \mathsf{p}(k)$;
(3) if $gh|T$ then $|I_g \cap I_h| \leq 1$.

Then the associated hypergraph of T has k edge disjoint hypercycles of length at least 3.

Proof. Note that (3) implies that any hypercycle in the associated hypergraph has a length of at least 3. Now, (1) allows us to consider an associated graph $([1, r], (E_g)_{g|T})$ of T, and (2) implies that this graph has k edge disjoint cycles (in the usual sense). Those cycles induce k cycles in the associated hypergraph (not necessarily hypercycles), and since there is a bijection between the edges of the associated graph and the ones of the associated hypergraph, they are edge disjoint in the hypergraph. Let $E_{g_1}, \ldots, E_{g_s}$ be the edges of such a cycle, so that $|E_{g_i} \cap E_{g_j}| = 1$ if $i - j = \pm 1 \mod s$, and $|E_{g_i} \cap E_{g_j}| = 0$ otherwise. Since for all $g|T$, $E_g \subseteq I_g$, by (3) we have also $|I_{g_i} \cap I_{g_j}| = 1$ if $i - j = \pm 1 \mod s$. However we may have pairs i, j with $i - j \neq \pm 1 \mod s$ such that $|I_{g_i} \cap I_{g_j}| = 1$. This means that we can find a shorter cycle in the hypergraph, by omitting the edges $I_{g_{i+1}}, \ldots, I_{g_{j-1}}$. We can repeat this process until we get a cycle with no proper shorter cycle. By doing this for each of the k edge disjoint cycles, we finally get k edge disjoint hypercycles.

□

Lemma 3.14. *Let $S = \prod_{i=1}^s g_i \in \mathcal{F}(\langle e_1, \ldots, e_r\rangle + \sum_{i=1}^r G_0^i)$ such that the associated hypergraph has k edge disjoint hypercycles of length at least 3. Then $\Omega(G_0, S) \not\subset \mathcal{G}_k$.*

Proof. We proceed by induction on k. The case $k = 1$ is exactly (4) in Proposition 3.4, we do not recall it here. Let $k \geq 2$ and let us assume the statement holds for all $1 \leq k' < k$. Let $S = \prod_{i=1}^l g_i \in \mathcal{F}(\langle e_1, \ldots, e_r\rangle + \sum_{i=1}^r G_0^i)$ such that the associated hypergraph has k edge disjoint hypercycles. Let $(I_c)_{c\in C}$ be the edges of such a hypercycle. Then $S = S'S''$ with $S' = \prod_{c\in C} g_c$ and $S'' = S'^{-1}S$. We have that the associated hypergraph of S'' has $k - 1$ edge disjoint hypercycles of length at least 3. By the induction hypothesis, $\Omega(G_0, S') \not\subset \mathcal{G}_1$ and $\Omega(G_0, S'') \not\subset \mathcal{G}_{k-1}$, so by Lemma 3.3, $\Omega(G_0, S) \not\subset \mathcal{G}_k$. □

Theorem 3.15. $\psi_k(G) \leq (k - 1 + r)p - 1 + \mathsf{p}(k)$.

Proof. We proceed by induction on k, as in [2, Theorem 4.5]. Note that the case $k = 1$ has already been done in the proof of this theorem, we do not recall it here. Let $k \geq 2$ and assume that $\psi_{k-1}(G) \leq (k - 2 + r)p - 1 + \mathsf{p}(k - 1)$. Let $S \in \mathcal{F}(G\backslash G_0)$ such that $|S| = (k - 1 + r)p + \mathsf{p}(k)$. We want to prove that $\Omega(G_0, S) \not\subset \mathcal{G}_k$.

Step 1: Suppose there exists some $g|S$ such that $g \in G\backslash(\langle e_1, \ldots, e_r\rangle + \sum_{i=1}^r G_0^i)$. We know by Proposition 3.4 that $\Omega(G_0, g) \not\subset \mathcal{G}_1$. Consider $R = g^{-1}S$. Then $|R| = (k - 1 + r)p + \mathsf{p}(k) - 1 \geq (k - 2 + r)p + \mathsf{p}(k - 1)$ (because $p \geq 2$ and $\mathsf{p}(k)$ is increasing). By induction hypothesis, $\Omega(G_0, R) \not\subset \mathcal{G}_{k-1}$, so by Lemma 3.3, we have $\Omega(G_0, S) \not\subset \mathcal{G}_k$. Now we assume that for all $g|S$, $g \in \langle e_1, \ldots, e_r\rangle + \sum_{i=1}^r G_0^i$.

Step 2: Consider $S'|S$ the subsequence consisting of the elements $g|S$ such that there exists some $j_g \in [1, r]$ with $(\pi_{j_g} + \pi'_{j_g})(g) \in \langle e_{j_g}\rangle\backslash\{0\}$. Suppose $|S'| \geq r(p - 1) + 1$. Then by the pigeonhole

principle, there is some $m \in [1,r]$ and a subsequence $S''|S'$ such that $|S''| \geq p$ and $(\pi_m + \pi'_m)(S'') \in \mathcal{F}(\langle e_m \rangle \backslash \{0\})$. Since $\langle e_m \rangle \cong C_p$ and $\mathsf{D}(C_p) = p$, there exists some $A|S''$ such that $(\pi_m + \pi'_m)(A) \in \mathcal{A}(\langle e_m \rangle \backslash \{0\})$ (with $|A| \leq p$). By Proposition 3.4 this implies that $\Omega(G_0, A) \not\subset \mathcal{G}_1$. We set $R = A^{-1}S$, so $|R| \geq |S| - p = (k-2+r)p - 1 + \mathsf{p}(k) \geq (k-2+r)p - 1 + \mathsf{p}(k-1)$. By induction hypothesis we get $\Omega(G_0, R) \not\subset \mathcal{G}_{k-1}$ and by Lemma 3.3, $\Omega(G_0, S) \not\subset \mathcal{G}_k$. Now we assume that $|S'| \leq r(p-1)$.
Step 3: Let $T = S'^{-1}S$ and for each $g|T$, $I_g := \{i \in [1,r] \mid \pi'_i(g) = e'_i\}$. By definition of T, every $g|T$ can be written $g = \sum_{i=1}^r a_i e_i + b_i e'_i$ with $a_i \in [0, p-1]$, $b_i \in \{0,1\}$ and $b_i = 0 \Rightarrow a_i = 0$. It follows that $|I_g| \geq 2$ for all $g|T$. Suppose there exist $g, h|T$ such that $|I_g \cap I_h| \geq 2$. By Proposition 3.4, $\Omega(G_0, gh) \not\subset \mathcal{G}_1$. We set $R = (gh)^{-1}S$, so that $|R| = (k-1+r)p + \mathsf{p}(k) - 2 \geq (k-2+r)p + \mathsf{p}(k-1)$. We can apply the induction hypothesis on R, and again by Lemma 3.3 we conclude that $\Omega(G_0, S) \not\subset \mathcal{G}_k$. Thus we assume if $gh|T$ then $|I_g \cap I_h| \leq 1$.
Step 4: At this point, T satisfies the hypotheses of Lemma 3.13. Indeed, (1) and (3) are clear and (2) holds because $|T| = |S| - |S'| \geq (k-1+r)p + \mathsf{p}(k) - r(p-1) \geq r + \mathsf{p}(k)$. Thus Lemma 3.13 and Lemma 3.14 provide the result. □

Remark 3.16. *This proof does not give the best bound possible, which is investigated in the next section. However, with the lower bound given by* [2, Theorem 4.5]*, we get*

$$(k-1+r)p - 1 \leq \psi_k(G) \leq (k-1+r)p - 1 + \mathsf{p}(k).$$

Hence the gap between those bounds depends only on k, and not on G.

3.4. **Improving this bound.** The bound in Theorem 3.15 is not optimal. Indeed it suffices to see that the last inequality in Step 4 of the proof is not tight. In this section we investigate the best bound we can get with our argument, and deduce the exact value of $\psi_k(G)$ in some cases. First, note we can replace $\mathsf{p}(k)$ by $\mathsf{p}(k,r)$, which is smaller in some cases, since all our graphs have r vertices.

Theorem 3.17.

$$\psi_k(G) \leq rp - 1 + \max_{j=0,\ldots,k-1} \{pj + \mathsf{p}(k-j, r)\}.$$

Proof. Let $\varphi_k(G) = \max_{j=0,\ldots,k-1}\{pj + \mathsf{p}(k-j,r)\}$. Note that if $k \geq 2$, we have

$$\varphi_k(G) = \max\{\mathsf{p}(k,r), p + \varphi_{k-1}(G)\}. \tag{3.1}$$

In particular $\varphi_k(G)$ is increasing. We proceed as in the proof of Theorem 3.15 by induction on k. Once again, the case $k=1$ has been done in [2, Theorem 4.5] ($\varphi_1(G) = 0$), and we do not recall it here. Let $k \geq 2$ and assume that $\psi_{k-1}(G) \leq rp - 1 + \varphi_{k-1}(G)$. Let $S \in \mathcal{F}(G \backslash G_0)$ such that $|S| = rp + \varphi_k(G)$. We want to prove that $\Omega(G_0, S) \not\subset \mathcal{G}_k$.
Step 1: Suppose there exists some $g|S$ such that $g \in G \backslash (\langle e_1, \ldots, e_r \rangle + \sum_{i=1}^r G_0^i)$. We know by Proposition 3.4 that $\Omega(G_0, g) \not\subset \mathcal{G}_1$. Consider $R = g^{-1}S$. Then $|R| = rp - 1 + \varphi_k(G) > rp - 1 + \varphi_{k-1}(G)$, and by induction hypothesis, $\Omega(G_0, R) \not\subset \mathcal{G}_{k-1}$. By Lemma 3.3, $\Omega(G_0, S) \not\subset \mathcal{G}_k$.
Step 2: Consider $S'|S$ as in the proof of Theorem 3.15 and suppose $|S'| \geq r(p-1) + 1$. As in the previous proof and using Proposition 3.4, we can find an atom $A|S$ such that $|A| \leq p$ and $\Omega(G_0, A) \not\subset \mathcal{G}_1$. We set $R = A^{-1}S$, and we have $|R| \geq |S| - p = rp + \varphi_k(G) - p > rp - 1 + \varphi_{k-1}(G)$ by (3.1). By induction hypothesis, and Lemma 3.3 we have $\Omega(G_0, S) \not\subset \mathcal{G}_k$. Now we assume $|S'| \leq r(p-1)$, and that S' has no zerosum subsequence.
Step 3: We define T and the sets I_g as in the previous proof. Likewise, $|I_g| \geq 2$ for all $g|T$. Similarly, if there exist $g, h|T$ such that $|I_g \cap I_h| \geq 2$ we set $R = (gh)^{-1}S$. Then $|R| = |S| - 2 > rp - 1 + \varphi_{k-1}(G)$ by (3.1), since $p > 3$, and the induction hypothesis, together with Proposition 3.4 and Lemma 3.3 imply $\Omega(G_0, S) \not\subset \mathcal{G}_k$. Now we assume that if $gh|T$ then $|I_g \cap I_h| \leq 1$.
Step 4: At this point T clearly satisfies the hypotheses (1) and (3) of Lemma 3.13. And (2) holds

because $|T| = |S| - |S'| \geq rp + \varphi_k(G) - r(p-1) \geq r + \mathsf{p}(k,r)$ by (3.1). Then Lemma 3.13 and Lemma 3.14 provide the result. $\square$

Now we will apply this refined bound to compute the exact value of $\psi_k(G)$ in some cases. Indeed, we know $\mathsf{p}(k)$ for small values of k.

Theorem 3.18 ([5], Chap. III.3, Theorem 3.5)**.** *(i)* $\mathsf{p}(1) = 0$.
(ii) $\mathsf{p}(2) = 4$.
(iii) $\mathsf{p}(3) = 10$.
(iv) $\mathsf{p}(4) = 18$.

Corollary 3.19. *Let $p \geq 3$, and G be an elementary p-group with even rank $2r$. Then*
(1) $\psi_1(G) = rp - 1$.
(2) $\psi_2(G) = (r+1)p - 1$ *if* $p \geq 5$, *and* $\psi_2(G) \in \{3r+2, 3r+3\}$ *if* $p = 3$.
(3) $\psi_3(G) = (r+2)p - 1$ *if* $p \geq 5$, *and* $\psi_3(G) \in [3r+5, 3r+9]$ *if* $p = 3$.
(4) $\psi_4(G) = (r+3)p-1$ *if* $p \geq 7$, $\psi_4(G) \in [3r+8, 3r+17]$ *if* $p = 3$, *and* $\psi_4(G) \in [5r+14, 5r+17]$ *if* $p = 5$.

Proof. We use the lower bound in [2, Theorem 4.5], and the upper bound of Theorem 3.17 that we can compute thanks to the values of Corollary 3.18 (and obviously the fact that $\mathsf{p}(k,r) \leq \mathsf{p}(k)$). $\square$

3.5. $\psi_2(G)$ **for 3-elementary groups with even rank.** As we can see in Corollary 3.19, our bounds are not good enough for small values of p. In this section we investigate the exact value of $\psi_2(G)$ when $p = 3$. We already know that $\psi_2(G) \in \{3r+2, 3r+3\}$. Throughout this section, G is a 3-elementary group with rank $2r$.

Lemma 3.20. *If $r \leq 5$, then $\psi_2(G) = 3r + 2$.*

Proof. For $r \leq 3$, this is just the particular case of [2, Theorem 4.5]. For $r \in \{4, 5\}$, it suffices to note that $\mathsf{p}(2,4) = \mathsf{p}(2,5) = 3$, and then Theorem 3.17 gives the desired upper bound $\psi_2(G) \leq 3r+2$. $\square$

Lemma 3.21. $\psi_2(C_3^{12}) = 20$.

Proof. We already know that $\psi_2(C_3^{12}) \geq 20$. For the upper bound, we proceed as in the proof of Theorem 3.15. Let $S \in \mathcal{F}(G \backslash G_0)$ such that $|S| = 21$. We want to prove that $\Omega(G_0, S) \not\subset \mathcal{G}_2$. Everything is similar until Step 4.
Step 4: At this point, T satisfies hypotheses (1) and (3) of Lemma 3.13. Since $|S| = 21$ and $|S'| \leq 12$, we know that $|T| \geq 9$. If $|T| \geq 10 = 6 + \mathsf{p}(2,6)$, then we can conclude as in Theorem 3.15. Now we assume $|T| = 9$, and thus $|S'| = 12$.
Step 5: It is known [5, Chap III.3, Theorem 3.5] that the complete bipartite graph $K^{3,3}$ is the only graph with 6 vertices and 9 edges not containing two edge disjoint cycles. Thus any hypergraph with 6 vertices and 9 edges not containing two edge disjoint hypercycles has $K^{3,3}$ as an associated graph. However, note that adding one vertex to any edge of $K^{3,3}$ creates necessarily two edge disjoint hypercycles. So $K^{3,3}$ is also the only hypergraph without two edge disjoint hypercycles. So we can assume the associated hypergraph of T is $K^{3,3}$. Without loss of generality we assume that the disjoint vertices sets of $K^{3,3}$ are $\{1,2,3\}$ and $\{4,5,6\}$. So there exist $x_1, \ldots, x_6 \in \{0,2\}$ and elements $b_{i,j} \in \langle e_1, \ldots, e_6 \rangle$, for $(i,j) \in \{1,2,3\} \times \{4,5,6\}$ such that

$$S' = \prod_{i=1}^{6} e_i^{x_i} (2e_i)^{2-x_i}$$

and

$$T = \prod_{(i,j) \in \{1,2,3\} \times \{4,5,6\}} (b_{i,j} + e'_i + e'_j).$$

We will construct a sequence $B \in \Omega(G_0, S)$ such that $|\mathsf{L}(B)| \geq 3$. We do it only in the particular case where $x_i = 2$ for all $1 \leq i \leq 6$ and $b_{i,j} = 0$ for all $(i,j) \in \{1,2,3\} \times \{4,5,6\}$, but we can make similar constructions for the general case. We set

$$B = S \prod_{i=1}^{6} (e_i + e_i') e_i'^2.$$

Here are three factorizations of B in atoms, of length respectively 4, 5 and 7:

$$\begin{aligned}
B =& [(e_1' + e_4')(e_4' + e_2')(e_2' + e_5')(e_5' + e_1')e_1^2(e_1 + e_1')e_2^2(e_2 + e_2')e_4^2(e_4 + e_4')e_5^2(e_5 + e_5')] \\
& [(e_1' + e_6')(e_2' + e_6')(e_3' + e_6')e_1'^2 e_2'^2 e_3'^2] \\
& [(e_3' + e_4')(e_3' + e_5')e_3^2(e_3 + e_3')e_4'^2 e_5'^2] \\
& [e_6^2(e_6 + e_6')e_6'^2] \\
=& [(e_1' + e_4')(e_4' + e_2')(e_2' + e_5')(e_5' + e_1')(e_3' + e_5')(e_3' + e_6')(e_2' + e_6') \\
& \qquad e_1^2(e_1 + e_1')e_4^2(e_4 + e_4')e_3^2(e_3 + e_3')e_6^2(e_6 + e_6')] \\
& [e_2^2(e_2 + e_2')e_2'^2] \\
& [e_5^2(e_5 + e_5')e_5'^2] \\
& [(e_1' + e_6')e_1'^2 e_6'^2] \\
& [(e_3' + e_4')e_3'^2 e_4'^2] \\
=& T \prod_{i=1}^{6} [e_i^2(e_i + e_i')e_i'^2].
\end{aligned}$$

This proves that $|\mathsf{L}(B)| \geq 3$ and finishes the proof. □

Theorem 3.22. *For all $r \geq 1$, $\psi_2(C_3^{2r}) = 3r + 2$.*

Proof. We proceed as in Lemma 3.21, we already know that $\psi_2(C_3^{2r}) \geq 3r + 2$. Let $S \in \mathcal{F}(G \backslash G_0)$ such that $|S| = 3r + 3$. We want to prove that $\Omega(G_0, S) \not\subset \mathcal{G}_2$.
Step 4: At this point, T satisfies hypotheses (1) and (3) of Lemma 3.13. Since $|S| = 3r + 3$ and $|S'| \geq 2r$, we know that $|T| \geq r + 3$. If $|T| \geq r + 4 = r + \mathsf{p}(2, r)$, then we can conclude as in Theorem 3.15. Now we assume $|T| = r + 3$, and thus $|S'| = 2r$.
Step 5: By definition of S', there exist $x_1, \ldots, x_r \in \{0, 2\}$ such that

$$S' = \prod_{i=1}^{r} e_i^{x_i}(2e_i)^{2-x_i}.$$

Again by [5, Chap III.3, Theorem 3.5], and a similar argument as in Lemma 3.21, subdivisions of $K^{3,3}$ are the only hypergraphs with r vertices and $r + 3$ edges not containing two edge disjoint hypercycles. Thus we can assume that T is of the form

$$T = \prod_{(i,j) \in \{1,2,3\} \times \{4,5,6\}} \prod_{m=1}^{m_{i,j}} (b_{i,j}^m + e'_{l_{i,j}^m} + e'_{l_{i,j}^{m+1}})$$

where for all $(i,j) \in \{1,2,3\} \times \{4,5,6\}$, $m_{i,j} \geq 1$, for all $2 \leq m \leq m_{i,j} - 1$, $l_{i,j}^{m+1} = l_{i,j}^m + 1$ with $l_{i,j}^1 = i$ and $l_{i,j}^{m_{i,j}+1} = j$, $b_{i,j}^m \in \langle e_1, \ldots, e_r \rangle$ and $\sum_{(i,j) \in \{1,2,3\} \times \{4,5,6\}} m_{i,j} = r + 3$. Once again we will do the construction only for $x_i = 2$ for all i and $b^m_{i,j} = 0$ for all (i,j) and m. We set

$$B = S \prod_{i=1}^{6} e_i'^2 \prod_{i=1}^{r} (e_i + e_i').$$

Here are three factorizations of B into atoms of length respectively 4,5 and 7:

$$\begin{aligned}
B = & \left[\prod_{(i,j)\in\{1,2\}\times\{4,5\}} \left(\prod_{m=1}^{m_{i,j}} (e'_{l^m_{i,j}} + e'_{l^{m+1}_{i,j}}) \prod_{m=2}^{m_{i,j}} e^2_{l^m_{i,j}} (e_{l^m_{i,j}} + e'_{l^m_{i,j}}) \right) \prod_{i\in\{1,2,4,5\}} e_i^2 (e_i + e_i') \right] \\
& \left[e_1'^2 e_2'^2 e_3'^2 \prod_{(i,j)\in\{1,2,3\}\times\{6\}} \prod_{m=1}^{m_{i,j}} (e'_{l^m_{i,j}} + e'_{l^{m+1}_{i,j}}) \prod_{m=2}^{m_{i,j}} e^2_{l^m_{i,j}} (e_{l^m_{i,j}} + e'_{l^m_{i,j}}) \right] \\
& \left[e_4'^2 e_5'^2 e_3^2 (e_3 + e_3') \prod_{(i,j)\in\{3\}\times\{4,5\}} \prod_{m=1}^{m_{i,j}} (e'_{l^m_{i,j}} + e'_{l^{m+1}_{i,j}}) \prod_{m=1}^{m_{i,j}} e^2_{l^m_{i,j}} (e_{l^m_{i,j}} + e'_{l^m_{i,j}}) \right] \\
& [e_6^2 (e_6 + e_6') e_6'^2] \\
= & \left[\prod_{(i,j)\notin\{(1,6),(3,4)\}} \left(\prod_{m=1}^{m_{i,j}} (e'_{l^m_{i,j}} + e'_{l^{m+1}_{i,j}}) \prod_{m=2}^{m_{i,j}} e^2_{l^m_{i,j}} (e_{l^m_{i,j}} + e'_{l^m_{i,j}}) \right) \prod_{i\in\{1,3,4,6\}} e_i^2 (e_i + e_i') \right] \\
& [e_2^2 (e_2 + e_2') e_2'^2] \\
& [e_5^2 (e_5 + e_5') e_5'^2] \\
& \left[e_1' e_6' \prod_{m=1}^{m_{1,6}} (e'_{l^m_{1,6}} + e'_{l^{m+1}_{1,6}}) \prod_{m=2}^{m_{1,6}} e^2_{l^m_{1,6}} (e_{l^m_{1,6}} + e'_{l^m_{1,6}}) \right] \\
& \left[e_3' e_4' \prod_{m=1}^{m_{3,4}} (e'_{l^m_{3,4}} + e'_{l^{m+1}_{3,4}}) \prod_{m=2}^{m_{1,6}} e^2_{l^m_{3,4}} (e_{l^m_{3,4}} + e'_{l^m_{3,4}}) \right] \\
= & \left[\prod_{(i,j)\in\{1,2,3\}\times\{4,5,6\}} \prod_{m=1}^{m_{i,j}} (e'_{l^m_{i,j}} + e'_{l^{m+1}_{i,j}}) \prod_{m=2}^{m_{i,j}} e^2_{l^m_{i,j}} (e_{l^m_{i,j}} + e'_{l^m_{i,j}}) \right] \\
& \prod_{i=1}^{6} [e_i^2 (e_i + e_i') e_i'^2].
\end{aligned}$$

This proves that $|\mathsf{L}(B)| \geq 3$ and finishes the proof. □

4. The constant $\psi_k(G)$ for elementary p-groups with odd rank

Let $p \geq 3$ and G be an elementary p-group with odd rank $2r+1$. By Theorem 2.3, all half-factorial sets of G are equal up to automorphisms of the group. Thus it suffices to investigate $\Omega(G_0, \cdot)$ for one fixed half-factorial subset $G_0 \subset G$ of maximal cardinality. We fix a basis $\{e_1, e_1', \ldots, e_r, e_r', f\}$ of G and a half-factorial set

$$G_0 = \{0, f\} \cup \bigcup_{i=1}^{r} \{je_i + e_i' \mid j \in [0, p-1]\}$$

such that $|G_0| = \mu(G)$. Note that we made the following change of basis in comparison with Theorem 2.3: $e_i' = f_{2i}$ and $e_i = f_{2i-1} - f_{2i}$ for $1 \leq i \leq r$ and $f = f_{2r+1}$. For $1 \leq i \leq r$, let π_i denote the projection on $\langle e_i \rangle$, π_i' the projection on $\langle e_i' \rangle$ and $G_0^i = (\pi_i + \pi_i')(G_0) = \{0\} \cup \{je_i + e_i' \mid j \in [0, p-1]\}$. Let also π_f denote the projection on $\langle f \rangle$.

Lemma 4.1. *Let $g = cf + \sum_{i=1}^r a_i e_i + b_i e_i' \in G \backslash G_0$, with $a_i, b_i, c \in [0, p-1]$ and $A \in \Omega(G_0, g) \cap \mathcal{A}(G)$. Then*

$$\mathsf{k}(A) = \frac{1}{p} + |I_1| \frac{p-1}{p} + |I_2| + |I_3| - \frac{c_g}{p} + m_A + \frac{p-c}{p} \delta_{c \neq 0}$$

where $I_1 = \{i \in [1, r] \mid b_i = 1\}$, $I_2 = \{i \in [1, r] \mid b_i = 0$ and $a_i \neq 0\}$, $I_3 = \{i \in [1, r] \mid b_i \notin [0, 1]\}$, $c_g = \sum_{i \in I_3} b_i$ and $m_A \in [0, |I_3|]$. Moreover,

$$|\mathsf{k}(\Omega(G_0, g) \cap \mathcal{A}(G))| = 1 + |\{i \in [1, r] \mid b_i \notin [0, 1]\}|$$

Proof. Let $B \in \Omega(G_0, g)$. Then $B = g 0^v f^w \prod_{i=1}^r S_i$ with $v, w \in \mathbb{N}$ and $S_i \in \mathcal{F}(G_0^i \backslash \{0\})$. Then $B \in \mathcal{A}(G)$ if and only if $v = 0$, $(cf) f^w$ is an atom and $(a_i e_i + b_i e_i') S_i$ is an atom for each $1 \leq i \leq r$. For each $1 \leq i \leq r$, let $g_i = (\pi_i + \pi_i')(g) = a_i e_i + b_i e_i'$ and $F_i \in \mathcal{F}(G_0^i)$ such that $A = g f^w \prod_{i=1}^r F_i$. Note that $(cf) f^w$ is an atom if and only if $c \neq 0$ and $c + w = p$ or $c = w = 0$. This last observation, together with the same investigation as in [2, Lemma 4.2], leads to

$$\mathsf{k}(A) = \frac{1}{\operatorname{ord}(g)} + \sum_{i=1}^r \mathsf{k}(F_i) + \frac{w}{\operatorname{ord}(f)} = \frac{1}{p} + |I_1| \frac{p-1}{p} + |I_2| + |I_3| - \frac{c_g}{p} + |I_4| + \frac{p-c}{p} \delta_{c \neq 0}$$

with $I_4 = \{i \in I_3 \mid \mathsf{k}(F_i) = 2p - b_i\}$. Every quantity in this equality is fixed with g except $|I_4| \in [0, |I_3|]$ which can take at most $1 + |I_3|$ different values. Thus $|\mathsf{k}(\Omega(G_0, g) \cap \mathcal{A}(G))| \leq 1 + |I_3|$. To prove the remaining part, it suffices to consider the same sequences as in [2, Lemma 4.2], and the sequence f^{p-c} if $c \neq 0$, or the empty sequence if $c = 0$ for the last component. □

Proposition 4.2. *$\Omega(G_0, S) \not\subset \mathcal{G}_1$ for each of the following choices of $S \in \mathcal{F}(G \backslash G_0)$:*

(1) Let $S = g$ with $g \in G \backslash (\langle e_1, \ldots, e_r, f \rangle + \sum_{i=1}^r G_0^i)$.

(2) Let $S \in \mathcal{F}((\langle e_1, \ldots, e_r, f \rangle + \sum_{i=1}^r G_0^i) \backslash (\langle f \rangle + \sum_{i=1}^r G_0^i))$ such that $(\pi_m + \pi_m')(S) \in \mathcal{A}(\langle e_m \rangle \backslash \{0\})$ for some $m \in [1, r]$.

(3) Let $S = gh$ with $g, h \in \langle e_1, \ldots, e_r, f \rangle + \sum_{i=1}^r G_0^i$ such that $\pi_j'(g) = \pi_j'(h) = e_j'$ and $\pi_m'(g) = \pi_m'(h) = e_m'$ for distinct $j, m \in [1, r]$.

(4) Let $S = \prod_{j=1}^s g_j \in \mathcal{F}(\langle e_1, \ldots, e_r, f \rangle + \sum_{i=1}^r G_0^i)$ with $s \geq 3$ and $I_j = \{i \in [1, r] \mid \pi_i'(g_j) = e_i'\}$ such that for every $J \subset [1, r]$ with $|J| \geq 2$

$$|\bigcap_{j \in J} I_j| = \begin{cases} 1 & \text{if } J = \{j, j'\} \text{ and } j - j' = \pm 1 \mod s \\ 0 & \text{otherwise.} \end{cases}$$

Proof. 1. By Lemma 4.1, we have

$$|\mathsf{k}(\Omega(G_0, g) \cap \mathcal{A}(G))| > 1$$

since $\{g = cf + \sum_{i=1}^r a_i e_i + b_i e_i' \mid b_i \in [0, 1]$ for all $i\} = \langle e_1, \ldots, e_r, f \rangle + \sum_{i=1}^r G_0^i$.

2. Let $S = \prod_{j=1}^l g_j$. We have that for each $j \in [1, l]$ there exists some atom $A_j \in \Omega(G_0, g_j)$. By Lemma 4.1 (we can investigate component by component), we get $A_j = g_j F_j F_j'$ with $F_j \in \mathcal{F}(G_0^m)$, $F_j' \in \mathcal{F}(G_0 \backslash G_0^m)$ and $|F_j| = p$, since $\pi_m'(g_j) = 0$ and $\pi_m(g_j) \neq 0$. We consider the block $B = \prod_{i=1}^l A_j \in \Omega(G_0, S)$. Clearly $l \in \mathsf{L}(B)$. Since $\sigma((\pi_m + \pi_m')(S)) = 0$ we have $\sigma(\prod_{j=1}^l F_j) = 0$. Since $F_j \in \mathcal{F}(G_0^m)$ and G_0^m is half-factorial, $\{l\} = \mathsf{L}(\prod_{j=1}^l F_j)$. Consequently, $B = (\prod_{j=1}^l F_j)(\prod_{j=1}^l g_j F_j')$ and $l + \mathsf{L}(\prod_{j=1}^l g_j F_j') \subset \mathsf{L}(B)$. Clearly, $\prod_{j=1}^l g_j F_j' \neq 1$ so $\mathsf{L}(\prod_{j=1}^l g_j F_j') \neq \{0\}$, which implies $|\mathsf{L}(B)| > 1$.

3. As in [2, Proposition 4.4 (3)], it suffices to show that there exist atoms $A, A' \in \Omega(G_0, S)$ with $\mathsf{k}(A) \neq \mathsf{k}(A')$. The same constructions as in the proof work, the only difference is that $g_1, h_1 \in \langle e_3, e_3', \ldots, e_r, e_r', f \rangle$.

4. Similarly, the same constructions as in [2, Proposition 4.4 (4)] work, by setting $F \in \mathcal{F}(G_0)$ zero-sumfree such that $\sigma(F) = -\sum_{m=s+1}^r (\pi_m + \pi_m')(\sigma(S)) - \pi_f(\sigma(S))$. □

To generalize Theorem 3.17, we need some more results, because it is unclear that $|I_g| \geq 2$ for all $g|T$ at the end of Step 3. Now we investigate elements g such that $|I_g| \in \{0,1\}$, namely $g \in \langle f\rangle\backslash\{0\} + G_0\backslash\{f\}$. The following lemma is about elements $g \in \langle f\rangle\backslash\{0\}$, namely those such that $|I_g| = 0$.

Lemma 4.3. *Let $S \in \mathcal{F}(\langle f\rangle\backslash\{0, f\})$ be such that $|S| \geq (p-1)(p-2)+1$. Then $\Omega(G_0, S) \not\subset \mathcal{G}_1$.*

Proof. By the pigeonhole principle, there exists some $c \in [2, p-1]$ such that $(cf)^p|S$. Note that $\{(cf)^p, f^p, f^{p-c}(cf)\} \subset \mathcal{A}(G)$. Then, $(f^{p-c}(cf))^p = (f^p)^{p-c}(cf)^p \in \Omega(G_0, (cf)^p)$ are two factorizations of the same sequence, of length respectively p and $p-c+1$, which are different by hypothesis on c. So $\Omega(G_0, (cf)^p) \not\subset \mathcal{G}_1$ and $\Omega(G_0, S) \not\subset \mathcal{G}_1$. □

Corollary 4.4. $\psi_1(C_p) \leq (p-1)(p-2)$.

Corollary 4.5. $\psi_k(C_p) \leq k(p-1)(p-2)+k-1$

Proof. By induction on k, using Corollary 4.4 and [2, Lemma 3.2]. □

Remark 4.6. *In* [2, Corollary 4.9]*, the lower bound $\psi_k(C_p) \geq pk - 1 + \frac{p-1}{2}$ if $p \geq 5$ and $\psi_k(C_3) = 3k-1$ is given. There is a big gap between this lower bound and the upper bound of Corollary 4.5.*

Now the sequences S with elements g such that $|I_g| = 1$ (and possibly $|I_g| = 0$ too) remain.

Lemma 4.7. *Let $S \in \mathcal{F}((\langle f\rangle + G_0)\backslash G_0)$ be such that there exist $c \in [2, p-1]$ and $g_j \in G_0$, $1 \leq j \leq p$ with $|\{i \in [1,r] \mid \exists j,\ \pi'_i(g_j) = e'_i\}| = p$ such that $\prod_{j=1}^{p}(cf + g_j)|S$. Then $\Omega(G_0, S) \not\subset \mathcal{G}_1$.*

Proof. The same proof as in Lemma 4.3 works:

$$\prod_{j=1}^{p}[(cf+g_j)g_j^{p-1}f^{p-c}] = (f^p)^{p-c}\prod_{j=1}^{p}(cf+g_j)g_j^{p-1} \in \Omega\left(G_0, \prod_{j=1}^{p}(cf+g_j)\right)$$

are two factorizations of the same sequence, of length respectively p and $p-c+1$, which are different by hypothesis on c. Indeed, $\prod_{j=1}^{p}(cf+g_j)g_j^{p-1}$ is an atom because all the g_js live on a different component. □

Remark 4.8. *The hypothesis is close to the condition $|S| \geq (p-1)(p-2)+1$, but we need the g_js to live on different components, otherwise the length of the second factorization can increase. In particular we need $r \geq p$. Also, this result does not consider elements of the form $f+g$, with $g \in G_0$, which are new in comparison with Lemma 4.3.*

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