$\mathbb{A}^1$-contractible curves

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Abstract

The aim of this internship is to characterize the affine line as the only smooth $\mathbb{A}^1$-contractible curve, the starting point being the article [As19]. In the two first parts, we build a precise definition of curve and give some fundamental results. In the third part, we introduce the Picard group, a useful tool to classify varieties. Then, the fourth part aims to explain the meaning of the adjective "$\mathbb{A}^1$-contractible", and this requires to build a homotopy category. Finally, the fifth part details the proof that the affine line is the only smooth $\mathbb{A}^1$-contractible curve.

Contents

1 Schemes

In this section, we introduce the fundamental objects of algebraic geometry : schemes. They are often described as a generalization of algebraic varieties, because the Nullstellensatz theorem makes a link between classical algebraic varieties and some specific schemes. A more exhaustive construction of schemes can be found in [Har77] or [Liu02].

1.1 Schemes

We begin by defining the simplest type of scheme, called "affine scheme" : they are the elementary bricks from which we will build the object "scheme".

Definition 1.1 Let R be a ring. The spectrum of R is a set, defined by :

$$\mathrm{Spec}(R) := \{\mathfrak{p} \text{ prime ideal of } R\}$$

It can be endowed with the Zariski topology, where the closed subsets are the following ones : for I an ideal of R,

$$V(I) := \{\mathfrak{p} \in \mathrm{Spec}(R),\ I \subset \mathfrak{p}\}$$

Remark 1.1 The following formulas ensure that this defines a topology, called the Zariski topology :

(i) $V(I) \cup V(J) = V(I \cap J)$

(ii) $\displaystyle\bigcap_{\lambda\in\Lambda} V(I_\lambda) = V\left(\sum_{\lambda\in\Lambda} I_\lambda\right)$

(iii) $V(\mathrm{Spec}(R)) = \emptyset$ and $V(\{0\}) = \mathrm{Spec}(R)$

Remark 1.2 This topology is not separated in general : indeed, if the ring R is an integral domain then the ideal $\mathfrak{p}_0 = \{0\}$ is prime, so it is a point of the space $\mathrm{Spec}(R)$. Yet, the singleton $\{\mathfrak{p}_0\}$ is not closed in $\mathrm{Spec}(R)$... Actually, its closure is the whole space : $\overline{\{\mathfrak{p}_0\}} = \mathrm{Spec}(R)$. We call such a point a generic point.

Proposition 1.1 Let R be a ring and I, J two ideals of R. Then the following formulas hold :

- $V(\sqrt{I}) = V(I)$
- $V(I) \subset V(J)$ iff $\sqrt{J} \subset \sqrt{I}$
- $V(I \cap J) = V(IJ)$

Proof
• If $\mathfrak{p}$ is a prime ideal that contains $\sqrt{I}$, then it contains I (because $I \subset \sqrt{I}$), so $V(\sqrt{I}) \subset V(I)$. Conversely, let us suppose that $\mathfrak{p}$ is a prime ideal that contains I. Let $a \in \sqrt{I}$, and $n \in \mathbb{N}^*$ such that $a^n \in I$. Then $a^n \in \mathfrak{p}$. But $\mathfrak{p}$ is prime, so $a \in \mathfrak{p}$, and thus $\sqrt{I} \subset \mathfrak{p}$.

• If $\sqrt{J} \subset \sqrt{I}$, then $V(\sqrt{I}) \subset V(\sqrt{J})$ and by the previous point, $V(I) \subset V(J)$. Conversely, we can use the formula :

$$\bigcap_{\substack{\mathfrak{p}\text{ prime}\\ I\subset\mathfrak{p}}} \mathfrak{p} = \sqrt{I}$$

(see lemma 6.5 in appendices). Indeed, if $V(I) \subset V(J)$, then every prime ideal of R that contains I also contains J, and by the formula, $\sqrt{J} \subset \sqrt{I}$.

• As $IJ \subset I \cap J$, we have $V(I \cap J) \subset V(IJ)$. Conversely, if a prime ideal $\mathfrak{p}$ contains IJ, then it contains I or J because it is prime, so $\mathfrak{p} \in V(I) \cup V(J) = V(I \cap J)$. ■

In the Zariski topology, the open subsets are very big, in the sense of the following proposition.

Proposition 1.2 Let $X = \mathrm{Spec}(R)$ be a topological space as above. Let us suppose that X is irreducible (it can't be written as a union of two proper closed subsets). Then every open subset of X is dense in X.

Proof
Let U be an open subset of X. It exists an ideal I of R such that $U = V(I)^c$. The closure $\overline{U}$ is a closed subset of X, so it exists an ideal J of R such that $\overline{U} = V(J)$.
Then $V(J)^c \cap V(I)^c = \emptyset$, which can be rewrite as $V(J) \cup V(I) = X$. But X is irreducible, so either $V(I) = X$ either $V(J) = X$. In any case, U is dense in X. ■

Proposition 1.3 Let R be a ring.
Then, for $f \in R$, the set $D(f) := \{\mathfrak{p} \in \operatorname{Spec}(R),\ f \notin \mathfrak{p}\}$ is an open set for the Zariski topology on $\operatorname{Spec}(R)$.
Moreover, these open sets form a basis of open subsets of $\operatorname{Spec}(R)$.

Proof
It suffices to see that $D(f)^c = V((f))$. Indeed, if $\mathfrak{p}$ is a prime ideal of R such that $f \in \mathfrak{p}$, then the ideal generated by f is in $\mathfrak{p}$, so $\mathfrak{p} \in V((f))$. Conversely, if $\mathfrak{p}$ contains (f), then $f \in \mathfrak{p}$.

Now, let U be an open set in $\operatorname{Spec}(R)$, i.e $U^c = V(I)$ for a certain ideal I of R. Then we have $U = \bigcup_{f \in I} D(f)$. ■

Definition 1.2 Locally ringed space structure
Let R be a ring and $X := \operatorname{Spec}(R)$. For all $f \in R$, we define :

$$O_X(D(f)) := R_f$$

where R_f means the localization of R by the powers of f. Moreover, for any inclusion $D(f) \subset D(g)$, we define a restriction map by

$$\begin{array}{rccc} \operatorname{Res}_{f,g} : & R_f & \longrightarrow & R_g \\ & \dfrac{a}{f^m} & \longmapsto & q^m \dfrac{a}{g^{mm_0}} \end{array}$$

where $m_0 \in \mathbb{N}$ and $q \in R$ such that $fq = g^{m_0}$ (it exists because $D(f) \subset D(g)$).
Finally, let U be an open subset of X, such that $U = \bigcup_{\lambda \in \Lambda} D(f_\lambda)$. We have $D(f_\lambda) \cap D(f_\mu) = D(f_\lambda f_\mu)$ for all $\lambda, \mu \in \Lambda$, so it makes sense to define $O_X(U)$ by

$$O_X(U) = \left\{ (s_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} O_X(D(f_\lambda)) \mid \forall \lambda, \mu, \operatorname{Res}_{f_\lambda, f_\lambda f_\mu}(s_\lambda) = \operatorname{Res}_{f_\mu, f_\lambda f_\mu}(s_\mu) \right\}$$

Proposition 1.4 This endows X with a structure of locally ringed space.
The stalk of $\mathfrak{p} \in X$ is the localization $R_\mathfrak{p}$.

Proof see [Liu02], section 2.3.1. ■

The object (X, O_X) is an affine scheme. We will see later why the adjective "affine" is accurate.

Definition 1.3 Scheme
Let X be a topological space, and O_X a ring sheaf on X, such that every stalk of X is a local ring. If X can be covered by a family of open sets $(U_i)_{i \in I}$ such that each U_i is isomorphic (as a locally ringed space) to an affine scheme (i.e. the spectrum of a ring with a locally ringed space structure as above), then we say that X is a scheme.

Example 1.1 Projective schemes
Let R be a graded ring. Similarly to the construction of the spectrum of a ring, we can build the homogeneous spectrum of a graded ring, as follows :

$$\operatorname{Proj}(R) := \{\mathfrak{p} \text{ homogeneous prime ideal of } R, \text{ such that } \bigoplus_{d>0} R_d \not\subseteq \mathfrak{p}\}$$

Then $\operatorname{Proj}(R)$ can be endowed with a structure of scheme :

- The topology is given by the closed subsets $V_+(I) := \{\mathfrak{p} \in \operatorname{Proj}(R) \mid I \subset \mathfrak{p}\}$, for any homogeneous ideal I of R.
- A basis of open sets is given by the $D_+(f) := V_+((f))^c$, for any homogeneous element $f \in R$.

- For all $f \in R$ homogeneous, we define $O_{\text{Proj}(R)}(D_+(f)) := R_{(f)}$, the ring of elements of degree 0 in the localization R_f. This defines a sheaf of rings (see [Liu02], page 51)

- For any homogeneous element $f \in R$, we have $D_+(f) = \text{Spec}(R_{(f)})$, so $\text{Proj}(R)$ can be covered by open subsets that are isomorphic to the spectrum of a ring.

A scheme that is isomorphic to the homogeneous spectrum of a graded ring is called a projective scheme.

Remark 1.3 Unlike in the affine case, the closed points of a projective scheme $\text{Proj}(R)$ are not the maximal homogeneous ideals of the ring R. Indeed, there is a unique maximal homogeneous ideal in R, which is $\bigoplus_{d>0} R_d$. But in the definition, we took away this ideal, so the closed points are the prime homogeneous ideals "just below" this big ideal.
For instance, in $\text{Proj}(\mathbb{C}[T_0, T_1])$, the closed points are the ideals $(aT_0 - bT_1)$, with $a, b \in \mathbb{C}$.

Definition 1.4 Affine and projective spaces
Let $n \in \mathbb{N}$. Let k be a field.

- The affine space of dimension n over k is defined by

$$\mathbb{A}^n(k) := \text{Spec}(k[T_1, ..., T_n])$$

- The projective space of dimension n over k is defined by

$$\mathbb{P}^n(k) := \text{Proj}(k[T_0, ..., T_n])$$

Remark 1.4 These definitions generalize the usual affine space and projective space. Indeed, when k is algebraically closed, the closed points of the abstract spaces $\mathbb{A}^n(k)$ and $\mathbb{P}^n(k)$ are in one-to-one correspondence with the concrete affine and projective spaces, as explain below.

In the affine space, the closed points are the maximal ideals of $k[T_1, ..., T_n]$. When k is algebraically closed, the maximal ideals of $k[T_1, ..., T_n]$ are of the form $(T_1 - a_1, ..., T_n - a_n)$ (it is a weak version of the Nullstellensatz theorem), so they are in bijection with the points $(a_1, ..., a_n)$ of the affine space k^n.

In the projective space, it is the same idea : the closed points are the prime homogeneous ideals of the form $(T_0 - a_0T_i, \cdots, T_{i-1} - a_{i-1}T_i, T_{i+1} - a_{i+1}T_i, \cdots, T_n - a_nT_i)$, with $0 \leq i \leq n$, and we can associate uniquely to any of these ideal a point $[a_0 : \cdots : a_{i-1} : 1 : a_{i+1} : \cdots : a_n]$.

Definition 1.5 Dimension
Let X be a scheme. The dimension of X is its dimension as a topological space, i.e. its Krull dimension, which is the supremum of the lengths of chains of irreducible closed subsets of X.

Example 1.2 The dimension of $\mathbb{A}^n(k)$ and $\mathbb{P}^n(k)$ is n.

Definition 1.6 Let (X, O_X) be a scheme. We say that X is :

- connected if its underlying topological space is connected ;

- integral when its underlying topological space is irreducible and every stalk is a reduced ring ;

- noetherian when it can be covered by a finite union of affine schemes that are spectrum of noetherian rings.

Proposition 1.5 Let $X = \text{Spec}(R)$ (resp. $X = \text{Proj}(R)$) be an affine scheme (resp. a projective scheme). Then R is an integral domain iff X is integral.

Proof
• Let us suppose that R is an integral domain. If we can write X as union of two closed subsets, then $X = V(I_1) \cup V(I_2)$, with I_1, I_2 two ideals of R, then $X = V(I_1 \cap I_2)$, so every prime ideal of R contains $I_1 \cap I_2$. But R is an integral domain, so $\{0\}$ is a prime ideal, and thus $I_1 \cap I_2 = \{0\}$. If I_1 is not trivial, it exists $a \in I_1$ such that $a \neq 0$. For all $b \in I_2$, we have $ab \in I_1 \cap I_2$ so $ab = 0$, and as R is an integral domain, $b = 0$. So $I_2 = \{0\}$, and finally X is irreducible.
Moreover, for all $\mathfrak{p} \in \text{Spec}(R)$, the stalk $O_X(\mathfrak{p})$ is the localization of R by $\mathfrak{p}$. As R is an integral domain

and $R \setminus \mathfrak{p}$ does not contain 0, the localization $R_{\mathfrak{p}}$ is an integral domain, so it is a reduced ring. Therefore X is integral.

• Conversely, let us suppose that R is not an integral domain : it exits $a, b \in R$ such that $ab = 0$ and $a \neq 0$, $b \neq 0$. Then $(a) \cap (b) = \sqrt{(0)}$, so $X = V((a)) \cup V((b))$, with $V((a)) \neq X$ and $V((b)) \neq X$, because $a \neq 0$ and $b \neq 0$. So X is not irreducible.

The proof in the projective case is the same. ∎

Proposition 1.6 Let X be an integral scheme. Then there is a unique generic point in X, and its stalk is a field.

Proof
Let us cover X by open affine subschemes : $X = \bigcup_{\lambda \in \Lambda} U_\lambda$, with $U_\lambda = \mathrm{Spec}(R_\lambda)$. Since X is integral, each ring R_λ is an integral domain. Thus the ideal $\{0_\lambda\} \subset R_\lambda$ is a generic point of U_λ. Yet, U_λ is dense in X because X is irreducible, so $\{0_\lambda\}$ is a generic point of X.
Now, let x_1, x_2 be two generic points of X, and let $U_1 = \mathrm{Spec}(R_1)$, $U_2 = \mathrm{Spec}(R_2)$ be two affine open subsets such that $x_1 \in U_1$ and $x_2 \in U_2$. From the above, we must have $x_1 = \{0_1\}$ and $x_2 = \{0_2\}$.
However, the set $U_1 \cap U_2$ is an open subset of X, so it is dense in X : a fortiori, it is non empty. Hence, $U_1 \cap U_2$ is a non empty open subset of U_1, so it contains the ideal $\{0_1\}$. Thus $\{0_1\}$ is also a point of U_2, and it is a generic point of U_2, so it coincides with $\{0_2\}$. Finally $x_1 = x_2$.

Furthermore, we can compute the stalk of x_1 : it is the localization of R_1 by the ideal $\{0_1\}$, so it is the fraction field $\mathrm{Frac}(R_1)$. ∎

Definition 1.7 Let X be an integral scheme, and ξ be its generic point. The field $O_X(\xi)$ is called the function field of X, and denoted $\kappa(X)$.

1.2 Morphisms of schemes

Definition 1.8 Morphism of schemes
Let X and Y be two schemes. A morphism between X and Y is a couple $(f, \hat{f})$ such that

- $f : X \longrightarrow Y$ is a continuous map ;
- for all open subset V of Y, $\hat{f}(V) : O_Y(V) \longrightarrow O_X(f^{-1}(V))$ is a morphism of rings compatible with the restrictions, i.e. such that for all open $W \subset V$, the following diagram commutes :

$$\begin{array}{ccc} O_Y(V) & \xrightarrow{\hat{f}(V)} & O_X(f^{-1}(V)) \\ \Big\downarrow {\scriptstyle \mathrm{Res}_{V,W}} & & \Big\downarrow {\scriptstyle \mathrm{Res}_{f^{-1}(V),f^{-1}(W)}} \\ O_Y(W) & \xrightarrow{\hat{f}(W)} & O_X(f^{-1}(W)) \end{array}$$

Proposition 1.7 Let $\phi : A \longrightarrow B$ be a morphism of rings. Then ϕ induces a morphism of schemes $f : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ such that $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$. If ϕ is surjective, then the morphism f is injective. Moreover, the closure of $\mathrm{Im}(f)$ in $\mathrm{Spec}(A)$ is $V(\mathrm{Ker}(\phi))$.

Proof
The map f is well defined as the inverse image of a prime ideal by a morphism of rings is a prime ideal. Then for all $a \in A$, we have $f^{-1}(D(a)) = D(\phi(a))$ so f is continuous. Finally, for all $a \in A$, as ϕ induces a morphism of rings $A_a \longrightarrow B_{\phi(a)}$, there is a morphism of rings $O_{\mathrm{Spec}(A)}(D(a)) \longrightarrow O_{\mathrm{Spec}(B)}(f^{-1}(D(a)))$, and if $D(a) \subset D(a')$, than $D(\phi(a)) \subset D(\phi(a'))$ and the diagram commutes :

$$\begin{array}{ccc} A_a & \xrightarrow{\phi} & B_{\phi(a)} \\ \uparrow \scriptstyle{\mathrm{Res}_{a,a'}} & & \uparrow \scriptstyle{\mathrm{Res}_{\phi(a),\phi(a')}} \\ A_{a'} & \xrightarrow{\phi} & B_{\phi(a')} \end{array}$$

Therefore f is a morphism of schemes.

If ϕ is surjective, then for two ideals $\mathfrak{p}$, $\mathfrak{q}$ in $\mathrm{Spec}(B)$ such that $\phi^{-1}(\mathfrak{p}) = \phi^{-1}(\mathfrak{q})$, we have $\phi(\phi^{-1}(\mathfrak{p})) = \mathfrak{p} = \mathfrak{q} = \phi(\phi^{-1}(\mathfrak{q}))$, so f is injective.

Now, let us show that $\overline{\mathrm{Im}(f)} = V(\mathrm{Ker}(\phi))$ (where $\overline{\mathrm{Im}(f)}$ is the closure of $\mathrm{Im}(f)$ for the Zariski topology on $\mathrm{Spec}(A)$).
Let $\mathfrak{q} \in \mathrm{Im}(f)$: it exists a prime ideal $\mathfrak{p} \in \mathrm{Spec}(B)$ such that $\phi^{-1}(\mathfrak{p}) = \mathfrak{q}$. As $0 \in \mathfrak{p}$, we have $\phi^{-1}(0) \subset \phi^{-1}(\mathfrak{p})$ so $\mathrm{Ker}(\phi) \subset \mathfrak{q}$. Thus $\mathrm{Im}(f) \subset V(\mathrm{Ker}(\phi))$, and by definition of closure, $\overline{\mathrm{Im}(f)} \subset V(\mathrm{Ker}(\phi))$.
Conversely, let $\mathfrak{q} \in V(\mathrm{Ker}(\phi))$. Let us notice that

$$\overline{\mathrm{Im}(f)} = \bigcap_{\substack{I \text{ ideal of } A \\ \mathrm{Im}(f) \subset V(I)}} V(I)$$

Let I be such an ideal of A. Then $I \subset \bigcap_{\mathfrak{p} \in \mathrm{Im}(f)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(B)} \phi^{-1}(\mathfrak{p}) = \phi^{-1}\left(\bigcap_{\mathfrak{p} \in \mathrm{Spec}(B)} \mathfrak{p}\right)$.
Yet, we have $\bigcap_{\mathfrak{p} \in \mathrm{Spec}(B)} \mathfrak{p} = \sqrt{0_B}$, so

$$I \subset \phi^{-1}(\sqrt{0_B}) = \sqrt{\mathrm{Ker}(\phi)} = \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ \mathrm{Ker}(\phi) \subset \mathfrak{p}}} \mathfrak{p}$$

As $\mathfrak{q}$ is a prime ideal that contains $\mathrm{Ker}(\phi)$, we have $I \subset \mathfrak{q}$.
Finally $\mathfrak{q} \in \overline{\mathrm{Im}(f)}$. ■

Definition 1.9 Morphisms of finite type
Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is locally of finite type when for every point $x \in X$, it exists an open affine neighbourhood $\mathrm{Spec}(A)$ of x in X and an open affine $\mathrm{Spec}(B)$ in Y such that $f(\mathrm{Spec}(A)) \subset \mathrm{Spec}(B)$ and A is a finite type B-algebra for the induced map $B \longrightarrow A$, given by the morphism $\hat{f} : O_B(\mathrm{Spec}(B)) \longrightarrow O_A(\mathrm{Spec}(A))$.
Moreover, we say that f is of finite type when f is locally of finite type and for every open V of Y that is quasi-compact, the inverse image $f^{-1}(V)$ is quasi-compact.

Proposition 1.8 Glueing schemes
Let $(X_i)_{i \in I}$ be a family of schemes over k, such that for all $i \in I$, for all $j \in I$, it is given a subscheme X_{ij} of X_i, and an isomorphism of schemes over k, $f_{ij} : X_{ij} \longrightarrow X_{ji}$ such that :

(i) $f_{ii} = \mathrm{id}_{X_{ii}}$

(ii) $f_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$

(iii) $f_{jk} \circ f_{ij} = f_{ik}$ on $X_{ij} \cap X_{ik}$.

Then there exists a unique (up to isomorphisms) scheme over k, X, with open immersions $g_i : X_i \longrightarrow X$ such that

$$X = \bigcup_{i \in I} g_i(X_i) \text{ and } g_i = g_j \circ f_{ij} \text{ on } X_{ij}$$

Proof see [Liu02], page 50. ■

Example 1.3 The projective space $\mathbb{P}^n(k)$, for any field k, is the glueing of a family $(X_i)_{0 \leq i \leq n}$, where

$$X_i := D_+(T_i) \cong \mathbb{A}^n(k)$$

Definition 1.10 Affine and finite morphisms
Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is affine when for all affine open subset V of Y, the subset $f^{-1}(V)$ is affine.
Moreover, we say that f is finite when f is affine and for all open affine subset V of Y, the ring $O_X(f^{-1}(V))$ is finite as a $O_Y(V)$-module.

Remark 1.5 A finite morphism $f : X \longrightarrow Y$ is a locally finite type morphism.
Indeed, let $x \in X$. Then we can find an affine open $\text{Spec}(R) \subset Y$ such that $f(x) \in \text{Spec}(R)$. As f is finite, the inverse image $f^{-1}(\text{Spec}(R))$ is affine : $f^{-1}(\text{Spec}(R)) = \text{Spec}(A)$, and the ring A is a finite R-module.

Proposition 1.9 A closed immersion is a finite morphism.

Proof
Let $f : X \longrightarrow Y$ be a closed immersion, which means that we can factor f in $i \circ \phi$ where $i : Z \longrightarrow Y$ is the inclusion of a closed subscheme of Y, and $\phi : X \longrightarrow Z$ is an isomorphism. As an isomorphism, ϕ is affine. Moreover, if $V := \text{Spec}(R)$ is an affine open of Y, we have $i^{-1}(V) = Z \cap V$, so it is a closed subset of V in Y. Therefore it exists an ideal I of R such that $V(I) = Z \cap V$, and we have $V(I) \cong \text{Spec}(R/I)$ so $Z \cap V$ is affine. Thus i is an affine morphism, and by composition, f is an affine morphism. Finally, we have $O_X(f^{-1}(V)) = O_X(\phi^{-1}(Z \cap V)) \cong O_Z(Z \cap V) = O_Z(\text{Spec}(R/I)) = R/I$, and it is a finite R-module. Hence f is a finite morphism. ■

Definition 1.11 Scheme over a field
Let (X, O_X) be a scheme. We say that X is a scheme over a field k when we consider a morphism of scheme $f : X \longrightarrow \text{Spec}(k)$.
In that case, the function field $\kappa(X)$ can be denoted $k(X)$.

Definition 1.12 Scheme of finite type
We say that a scheme X over k is of finite type when there is a finite covering $X = \bigcup_{i=1}^{n} \text{Spec}(R_i)$ where each R_i is a finite-type k-algebra.

Definition 1.13 Unramified, flat and étale morphisms
Let $f : X \longrightarrow Y$ be a morphism of schemes over k, such that X and Y are noetherian schemes of finite type. Let $x \in X$, and $y := f(x)$.

- We say that f is unramified at x if the morphism of rings $\hat{f} : O_{Y,y} \longrightarrow O_{X,x}$ induced by f satisfies $\hat{f}(\mathfrak{m}_y)O_{X,x} = \mathfrak{m}_x$, and if the extension $k(y) \longrightarrow k(x)$ is separable.
- We say that f is flat at x if the morphism of rings $\hat{f} : O_{Y,y} \longrightarrow O_{X,x}$ is flat, i.e. it makes flat $O_{X,x}$ as a $O_{Y,y}$-module (see appendices for the definition of flat module).
- We say that f is étale when for all $x \in X$, it is unramified and flat at x.

Proposition 1.10 The composition of two étale morphisms is étale.

Proof
Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two étale morphisms. Let $x \in X$. Then the morphism $\hat{f} : O_Y(f(x)) \longrightarrow O_X(x)$ is such that $\hat{f}(\mathfrak{m}_{f(x)})O_X(x) = \mathfrak{m}_x$, and the morphism $\hat{g} : O_Z(g(f(x)) \longrightarrow O_Y(f(x))$ is such that $\hat{g}(\mathfrak{m}_{g(f(x))}O_Y(f(x)) = \mathfrak{m}_{f(x)}$. Then $\hat{f}(\hat{g}(\mathfrak{m}_{g(f(x))})O_Y(f(x)))O_X(x) = \mathfrak{m}_x$, so $\widehat{g \circ f}(\mathfrak{m}_{g \circ f(x)})O_X(x) = \mathfrak{m}_x$.
Moreover, the extension $k(g(f(x))) \longrightarrow k(x)$ is the composition of two separable extensions so it is separable. Therefore $g \circ f$ is unramified at x.
Finally, $O_X(x)$ is flat as a $O_Y(f(x))$-module via $\hat{f}$, and $O_Y(f(x))$ is flat as a $O_Z(g(f(x)))$-module via $\hat{g}$, so $O_X(x)$ is flat as a $O_Z(g(f(x)))$-module, via $\hat{f} \circ \hat{g}$ (see proposition 6.4 in appendices). Then $g \circ f$ is flat at x.
Hence $g \circ f$ is étale at x, for all $x \in X$. ■

Proposition 1.11 Any immersion is étale.

Proof
Let $i : X \longrightarrow Y$ be an inclusion of schemes over k, with X and Y locally Noetherian and of finite type. Let $x \in X$. Then $O_X(x) = O_Y(x)$ so the morphism $\hat{i} : O_Y(x) \longrightarrow O_X(x)$ is the identity. Therefore i is unramified and flat at x.
As an immersion is the composition of an inclusion and an isomorphism, and since any isomorphism is étale, we get that any immersion is étale. ■

Proposition 1.12 Let $f : X \longrightarrow Y$ be a morphism of schemes. There is a unique morphism $\Delta : X \longrightarrow X \times_Y X$ such that $\pi_1 \circ \Delta = \pi_2 \circ \Delta = \mathrm{id}_X$ (where π_i are the projections $X \times_Y X \longrightarrow X$).

Proof
The following diagram is commutative :

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

By universal property of fibred product, it exists a unique morphism $\Delta : X \longrightarrow X \times_Y X$ such that $\pi_1 \circ \Delta = \pi_2 \circ \Delta = \mathrm{id}_X$. ■

Definition 1.14 We say that f as above is separated when Δ is a closed immersion.
If X is a scheme over k a field, we say that X is separated when the morphism $X \longrightarrow \mathrm{Spec}(k)$ is a separated morphism.

Definition 1.15 Categories of schemes
We will denote by Sch_k the category of noetherian schemes over k, of finite dimension, and Sm_k the category of smooth, separated schemes of finite type over k. (See [Lei16] to have the basic notions of category theory).

1.3 Base change and smoothness

Now we describe a way to change the field over which a scheme is defined, and this will allow us to talk about smoothness of schemes.

Definition 1.16 Base change
Let k be a field and $k \subset k'$ an extension. Let X be a scheme over k.
The base change of X consists in building a scheme over k' from X, by setting

$$X_{k'} := X \times_k \mathrm{Spec}(k')$$

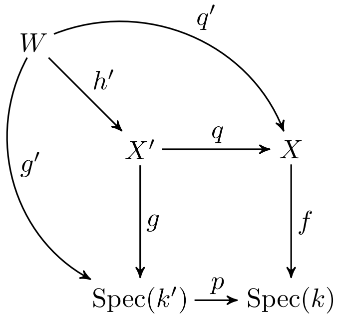
Hence, $X_{k'}$ is endowed with a structure of scheme over k' by the projection $X \times_k \mathrm{Spec}(k') \longrightarrow \mathrm{Spec}(k')$.

Proposition 1.13 Let $k \subset k'$ be a field extension. Let $X = \mathrm{Spec}(k[T_1, ..., T_n]/I)$ (resp. $\mathrm{Proj}(k[T_0, ..., T_n]/\bar{I})$) be an affine (resp. projective) variety over k. Then the base change $X_{k'}$ of X over k' is isomorphic to $\mathrm{Spec}(k'[T_1, ..., T_n]/I)$ (resp. $\mathrm{Proj}(k'[T_0, ..., T_n]/\bar{I})$).

Proof
By definition, $X_{k'} = X \times_k \mathrm{Spec}(k')$. Let us check that $X' = \mathrm{Spec}(k'[T_1, ..., T_n]/I)$ satisfies the universal property of fibred product. The inclusion $i : k \longrightarrow k'$ induces a morphism of rings $q : X' \longrightarrow X$. Moreover, X' has naturally a structure of scheme over k', so there is a morphism $g : X' \longrightarrow \mathrm{Spec}(k')$. Let us denote by $p : \mathrm{Spec}(k') \longrightarrow \mathrm{Spec}(k)$ the morphism of rings induced by $k \subset k'$, and by $f : X \longrightarrow \mathrm{Spec}(k)$ the morphism induced by the structure of scheme over k of X. Then the diagram commutes :

$$\begin{array}{ccc} X' & \xrightarrow{q} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(k') & \xrightarrow{p} & \mathrm{Spec}(k) \end{array}$$

Moreover, let W be a scheme and morphisms $q' : W \longrightarrow X$ and $g' : W \longrightarrow \mathrm{Spec}(k')$ such that $f \circ q' = p \circ g'$. Then g' induces a morphism of rings $\phi : k' \longrightarrow O_W(W)$, and the map q' induces a morphism of rings $\psi : k[T_1, ..., T_n]/I \longrightarrow O_W(W)$. Together, they define a morphism of rings $k'[T_1, ..., T_n]/I \longrightarrow O_W(W)$. Moreover, as $f \circ q' = p \circ g'$, we have then $\phi \circ i = \psi \circ j$, and therefore ϕ and ψ coincide on k. Then we have a morphism of rings $\eta : k'[T_1, ..., T_n]/I \longrightarrow O_W(W)$ such that $\eta_{|k'} = \phi$ and $\eta_{|k[T_1,...,T_n]/I} = \psi$. This induces a morphism of schemes $h' : W \longrightarrow X'$ Then the following diagram commutes :



Finally X' satisfy the universal property of fibred product, so $X' \cong X \times_k \mathrm{Spec}(k')$.

The proof is the same in the projective case. ■

Definition 1.17 Regular and smooth schemes
Let X be a noetherian scheme over k.
• We say that X is regular when every closed stalk of X is a regular local ring, i.e. for every closed point $x \in X$, we have $\dim_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2) = \dim(X)$.
• We say that X is smooth when the base change $X_{\bar{k}} = X \times_k \mathrm{Spec}(\bar{k})$ is locally of finite type and regular.

Proposition 1.14 A smooth scheme over k is regular over k. The converse is false.

Proposition 1.15 Let X be a smooth scheme of finite type over k. Let $k \subset k'$ be a field extension of k. Then the base change $X_{k'} := X \times_k \mathrm{Spec}(k')$ is still a smooth scheme of finite type (over k').

Proof
First, by definition of smoothness, the base change of a smooth scheme is still smooth.
Let us show that $X_{k'}$ is of finite type over k'. Let $z \in X_{k'}$. Let us denote by π the projection $X_{k'} \longrightarrow X$. Since X is of finite type, it exists an open affine neighbourhood $\mathrm{Spec}(A)$ of $\pi(z)$ in X, such that A is a finite type k-algebra, so we can write $A = k[T_1, ..., T_n]/I$. Then $\pi^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(k'[T_1, ..., T_n]/I')$, where I' is the ideal of $k'[T_1, ..., T_n]$ generated by I. Thus we have an open affine neighbourhood of z, such that the ring associated is a finite type k'-algebra. Therefore $X_{k'}$ is of finite type. ■

Remark 1.6 Unfortunately, if k' is an extension of k, the base change of an integral scheme X over k toward k' is not always an integral scheme over k'...
For instance, let $k = \mathbb{Q}$, $k' = \mathbb{Q}(i)$ and $X = \mathrm{Spec}(\mathbb{Q}(i))$. Then the fibred product $X \times_k \mathrm{Spec}(k')$ is equal to $\mathrm{Spec}(\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{Q}(i))$. The ring $\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{Q}(i)$ is not an integral domain, as shown by the following little calculus :

$$(1 \otimes i + i \otimes 1) * (1 \otimes i - i \otimes 1) = -1 \otimes 1 - i \otimes i + i \otimes i + 1 \otimes 1 = 0$$

Therefore $X_{\mathbb{Q}(i)}$ is not integral.
Due to this loss of integrity, the notion of abstract variety below is not stable by base change, which will cause some difficulties later. Thus in what follows, we will try to specify when the results work even if the scheme is not integral.

1.4 Varieties

Here we will make the link between schemes and classical algebraic varieties (defined as the zero-locus of a set of polynomials).

From this point, let k denote a given field.

Definition 1.18 Abstract variety
Let X be a scheme over k. We say that X is an abstract variety over k when the two conditions hold :

(i) X is integral

(ii) The morphism $X \longrightarrow \text{Spec}(k)$ is separated and of finite type.

In the case of an algebraically closed field, an abstract variety can be viewed as a classical variety, thanks to the following proposition.

Proposition 1.16 Suppose that k is an algebraically closed field. Let $F_1, ..., F_s$ be irreducible polynomials in $k[T_1, ..., T_n]$, and V be the variety $\{(a_1, ..., a_n) \in k^n \mid F_i(a_1, ..., a_n) = 0 \ \ \forall i\}$.
If we denote by $A := k[T_1, ..., T_n]/(F_1, ..., F_r)$ the coordinate ring of V, then $\text{Spec}(A)$ is an affine variety and there is a one-to-one correspondence between V and the closed points of $\text{Spec}(A)$.

Proof see corollary 1.1.15, page 31 of [Liu02]. ■

Proposition 1.17 Let $n \in \mathbb{N}$. The affine space of dimension n over k is a variety over k, and its function field is $k(T_1, ..., T_n)$, the field of fractions of $k[T_1, ..., T_n]$.

Proof
As $k[T_1, ..., T_n]$ is an integral domain and by proposition 1.3, $\mathbb{A}^n(k)$ is integral.
As the inclusion $k \longrightarrow k[T_1, ..., T_n]$ is a finite type ring morphism, the morphism $f : \mathbb{A}^n(k) \longrightarrow \text{Spec}(k)$ is locally of finite type. Moreover, the only open subset of $\text{Spec}(k)$ is $\text{Spec}(k)$, and it is quasi-compact. Let us show that $f^{-1}(\text{Spec}(k)) = \mathbb{A}^n(k)$ is quasi-compact : let us consider an open covering $\mathbb{A}^n(k) = \bigcup_{\lambda \in \Lambda} U_\lambda$. By definition, for all $\lambda \in \Lambda$, it exists an ideal $I_\lambda \subset k[T_1, ..., T_n]$ such that $U_\lambda = V(I_\lambda)^c$. Then $\bigcup_{\lambda \in \Lambda} U_\lambda = V\left(\sum I_\lambda\right)^c$. Yet $k[T_1, ..., T_n]$ is noetherian, so we can extract a finite number of ideals I_{λ_i}, $i = 1, ..., r$, such that $\sum I_\lambda = \sum_{i=1}^{r} I_{\lambda_i}$, and then we get $\mathbb{A}^n(k) = \bigcup_{i=1}^{r} U_{\lambda_i}$. Therefore $\mathbb{A}^n(k)$ is of finite type over k.
Finally, $\mathbb{A}^n(k) \times_k \mathbb{A}^n(k) = \mathbb{A}^{2n}$ (see appendices on fibred product), so the map $\mathbb{A}^n(k) \longrightarrow \mathbb{A}^{2n}(k)$ is a closed immersion, induced by the morphism of rings $k[T_1, ..., T_{2n}] \longrightarrow k[T_1, ..., T_n]$ that maps T_i over T_i and T_{i+n} over 1 for $i \in [\![1, n]\!]$. This shows that $\mathbb{A}^n(k)$ is a variety over k.

As $\mathbb{A}^n(k)$ is affine, we can choose $\mathbb{A}^n(k)$ for the affine open subscheme of the definition of function field. We have $O_{\mathbb{A}^n(k)}(\mathbb{A}^n(k)) = k[T_1, ..., T_n]$ and the fraction field of this ring is $k(T_1, ..., T_n)$. ■

Proposition 1.18 Let $n \in \mathbb{N}$. The projective space of dimension n over k is a variety over k, and its function field is $k^h(T_0, ..., T_n)$, the homogeneous fractions of $k[T_0, ..., T_n]$.

Proof
It suffices to adapt the proof of the affine case. ■

Definition 1.19 Affine and projective varieties
• An affine variety over k is a variety over k that is isomorphic to a closed subscheme of $\mathbb{A}^n(k)$, for one $n \in \mathbb{N}$.
• A projective variety over k is a variety over k that is isomorphic to a closed subscheme of $\mathbb{P}^n(k)$, for one $n \in \mathbb{N}$.

Remark 1.7 Fortunately, the spaces $\mathbb{A}^n(k)$ and $\mathbb{P}^n(k)$ are respectively an affine variety and a projective variety over k.

Proposition 1.19 Let X be a variety over k.
Then X is an affine scheme iff X is an affine variety over k.

Remark 1.8 This explains why we call "affine scheme" a scheme that is isomorphic to the spectrum of a ring.

Proof
• First, if X is an affine variety, then X is (isomorphic to) a closed subscheme of $\mathbb{A}^n(k)$, so it exists an ideal I of $k[T_1, ..., T_n]$ such that $X \cong V(I)$.
But $V(I)$ is the set of all prime ideals of $k[T_1, ..., T_n]$ that contains I, and it is isomorphic to the set of prime ideals of $k[T_1, ..., T_n]/I$. Then $X \cong \text{Spec}(k[T_1, ..., T_n]/I)$, so X is an affine scheme.

• Conversely, let us suppose that X is an affine scheme : $X = \text{Spec}(R)$. As X is a variety over k, the morphism $f : X \longrightarrow \text{Spec}(k)$ is of finite type, so for all $x \in X$, it exists an affine open neighbourhood

$U_x = \mathrm{Spec}(R_x)$ of x such that R_x is a finite type k-algebra. Thus we have an open covering $X = \bigcup_{x\in X} U_x$. Moreover, since f is of finite type, the inverse image $f^{-1}(\mathrm{Spec}(k)) = X$ is quasi-compact, so we can find a finite number of points $x_1, ..., x_r$ in X such that

$$X = \bigcup_{i=1}^{r} U_{x_i}$$

Let us write $U_{x_i} = \mathrm{Spec}(R_i)$, with R_i a finite type k-algebra. Then X is a finite union of spectrum of finite type k-algebra, so R is a finite type k-algebra itself : $R = k[T_1, ..., T_n]/I$. The projection $k[T_1, ..., T_n] \longrightarrow R$ induces an immersion $i : X \longrightarrow \mathbb{A}^n(k)$, and we have $i(X) = V(I)$ so X is isomorphic to a closed subscheme of $A^n(k)$. ■

Example 1.4 The multiplicative group $\mathbb{G}_m$.
We denote by $k[T, T^{-1}] := k[T_1, T_2]/(T_1T_2 - 1)$. The multiplicative group scheme is

$$\mathbb{G}_m = \mathrm{Spec}(k[T, T^{-1}])$$

endowed with the following group structure :
Consider the morphism of rings $\phi : k[T, T^{-1}] \longrightarrow k[T, T^{-1}] \otimes_k k[T, T^{-1}]$ such that $\phi(T) = T \otimes T$. This induces a morphism of schemes $m : \mathrm{Spec}(k[T, T^{-1}] \otimes_k k[T, T^{-1}]) \longrightarrow \mathrm{Spec}(k[T, T^{-1}])$, i.e. a morphism of schemes

$$m : \mathbb{G}_m \times_k \mathbb{G}_m \longrightarrow \mathbb{G}_m$$

that gives a "multiplicative law" on $\mathbb{G}_m$.

In the same way, we define an inverse morphism $i : \mathbb{G}_m \longrightarrow \mathbb{G}_m$ induced by $T \longrightarrow T^{-1}$ and a neutral morphism $e : \mathrm{Spec}(k) \longrightarrow \mathbb{G}_m$ induced by $T \longrightarrow 1$.

Now, let us show that $\mathbb{G}_m$ is an affine variety.
First, as $T_1T_2 - 1$ is irreducible, the ring $k[T_1, T_2]/(T_1T_2 - 1)$ is an integral domain, and thus $\mathbb{G}_m$ is integral. Furthermore, we have

$$\mathbb{G}_m \times_k \mathbb{G}_m \cong \mathrm{Spec}(k[T_1, T_2, T_3, T_4]/(T_1T_2 - 1, T_3T_4 - 1))$$

Therefore the morphism $\mathbb{G}_m \longrightarrow \mathbb{G}_m \times_k \mathbb{G}_m$ is a closed immersion, induced by the morphism of rings $k[T_1, T_2, T_3, T_4]/(T_1T_2 - 1, T_3T_4 - 1) \longrightarrow k[T_1, T_2]/(T_1T_2 - 1)$ which maps T_1 on T_1, T_2 on T_2 and T_3, T_4 on 1. Then $\mathbb{G}_m$ is separated. Finally, as $k[T_1, T_2]/(T_1T_2 - 1)$ is a finite type k-algebra and a noetherian ring, $\mathbb{G}_m$ is of finite type over k.
Moreover, the projection $k[T_1, T_2] \longrightarrow k[T_1, T_2]/(T_1T_2 - 1)$ gives a closed immersion of $\mathbb{G}_m$ in $\mathbb{A}^2(k)$, so $\mathbb{G}_m$ is an affine variety.

Moreover, when the field k is algebraically closed, one can see $\mathbb{G}_m$ as the pointed line $\mathbb{A}^n(k) \setminus \{0\}$: this needs some computations, detailed below.
As k is algebraically closed, the nonzero prime ideals of $k[T_1, T_2]$ are the maximal ideals $(T_1 - a, T_2 - b)$ for $a, b \in k$ and the prime ideals (P) where P is irreducible. Then the prime ideals that contains $(T_1T_2 - 1)$ are the maximal ones and $(T_1T_2 - 1)$. Let $\mathfrak{p} = (T_1 - a, T_2 - b)$ such that $T_1T_2 - 1 \in \mathfrak{p}$. It exists two polynomials P, Q such that

$$T_1T_2 - 1 = (T_1 - a)P + (T_2 - b)Q$$

Case 1 : if $\deg(P) > \deg(Q)$, then the degree of $(T_1 - a)P + (T_2 - b)Q$ is $\deg(P) + 1$, so $\deg(P) = 1$. We can write $P = p_1T_1 + p_2T_2 + p_0$ and $Q = q_0$ with $p_1, p_2, p_0, q_0 \in k$. Thus

$$T_1T_2 = p_1T_1^2 + p_2T_1T_2$$

Hence $p_1 = 0$ and $p_2 = 1$, and after some computation, we have $p_0 = 0$, $q_0 = a$ and $bq_0 = 1$. Necessarily, $ab = 1$. Then $\mathfrak{p} = (T_1 - a, T_2 - \frac{1}{a})$.

Case 2 : if $\deg(P) = \deg(Q) = d$, then we can write $P = \sum_{i=0}^{d} P_i$, with P_i a homogeneous polynomial of degree i, and the same for Q. We get the following system :

$$\begin{cases} 1 = aP_0 + bQ_0 \\ 0 = T_1P_0 + T_2Q_0 - aP_1 - bQ_1 \\ T_1T_2 = T_1P_1 + T_2Q_1 - aP_2 - bQ_2 \\ 0 = T_1P_i + T_2Q_i - aP_{i+1} - bQ_{i+1} \text{ for } i \leq 2 \end{cases}$$

Actually, when working on these equations, we end on $ab = 1$, which means that $\mathfrak{p} = (T_1 - a, T_2 - \frac{1}{a})$. Therefore the closed points of $\mathbb{G}_m$ are in one-to-one correspondence with the points along the line $\mathbb{A}^n(k)$ except 0.

Definition 1.20 Projectivization
Let X an affine variety over k, say $X := \mathrm{Spec}(k[T_1, ..., T_n]/I)$. The projectivization of X, denoted by $\overline{X}$, is the scheme $\mathrm{Proj}(k[T_0, ..., T_n]/\overline{I})$, where $\overline{I}$ is the ideal generated by

$$\left\{ T_0^d F\left(\frac{T_1}{T_0}, ..., \frac{T_n}{T_0}\right) \mid F \in I, \deg(F) = d \right\}$$

Remark 1.9 We could have defined like this the projectivization of any affine scheme of finite type over k (indeed we don't need the scheme to be integral neither separated in this definition).

Example 1.5 The projectivization of $\mathbb{A}^n(k)$ is $\mathbb{P}^n(k)$.

Example 1.6
For an affine curve, the projectivization of X consists in adding a point at infinity. For instance, if $X := \mathrm{Spec}(\mathbb{C}[T_1, T_2]/(T_2 - T_1^2))$, we have

$$\overline{X} = \mathrm{Proj}(\mathbb{C}[T_0, T_1, T_2]/(T_2T_0 - T_1^2))$$

Hence the closed points of $\overline{X}$ are of two kinds :

- The closed points $[1 : x_1 : x_2]$ where $x_1^2 = x_2$, which corresponds to the closed points of X ;
- The point "at infinity" $[0 : 0 : 1]$.

Proposition 1.20 Let X be an affine variety over k of dimension 1. Then $\overline{X}$ is a projective variety over k of dimension 1, and $\overline{X} \cong X \sqcup \{x_1, ..., x_m\}$ where the x_i's are closed points.
Moreover, if X is smooth, then $\overline{X}$ is smooth too.

Proof
Let us write $X := \mathrm{Spec}(k[T_1, ..., T_n]/I)$. By construction, $\overline{X}$ is a subset of $\mathbb{P}^n(k)$. Moreover, as X is a variety, the ideal I is prime in $k[T_1, ..., T_n]$, so $\overline{I}$ is prime in $k[T_0, ..., T_n]$. Then, by proposition 1.4, the scheme $\mathrm{Proj}(k[T_0, ..., T_n]/\overline{I})$ is integral.
Let us denote by $\pi_{\overline{I}}$ the canonical projection $k[T_0, ..., T_n] \longrightarrow k[T_0, ..., T_n]/\overline{I}$. Then we can cover $\overline{X}$ by finitely many open charts :

$$\overline{X} = \bigcup_{i=0}^{n} D_+(\pi_{\overline{I}}(T_i))$$

We can show that these charts are affine. Indeed, $D_+(\pi_{\overline{I}}(T_i)) \cong \mathrm{Spec}(k[Y_0, ..., Y_{i-1}, Y_{i+1}, ..., Y_n]/J_i)$ where $Y_l = T_l/T_i$ and J_i is the ideal generated by $\left\{\overline{F}(Y_0, ..., Y_{i-1}, 1, Y_{i+1}, ..., Y_n) \mid F \in I\right\}$.
We can notice that $k[Y_1, ..., Y_n]/J_0 \cong k[T_1, ..., T_n]/I$, so $D_+(\pi_{\overline{I}}(T_0)) \cong X$.
Therefore there is an open immersion $j : X \longrightarrow \overline{X}$. Furthermore, as X has dimension one, the Krull dimension of the ring $k[T_1, ..., T_n]/I$ is one, so the ring $k[T_0, ..., T_n]/\overline{I}$ also has dimension one. This allows to compute $\overline{X} \setminus X = \overline{X} \setminus D_+(\pi_{\overline{I}}(T_0))$. Indeed, this is a proper closed subset of $\overline{X}$, so this is simply a finite set of closed points. Hence $\overline{X} \setminus X = \{x_1, ...x_m\}$.
Then $\overline{X}$ is an integral projective scheme of dimension 1, of finite type over k. Furthermore, as X is a variety, the morphism $\Delta : X \longrightarrow X \times_k X$ is a closed immersion. Then we can build from it a closed immersion $\overline{\Delta} : \overline{X} \longrightarrow \overline{X} \times_k \overline{X}$, such that $\overline{\Delta}_{|X} = \Delta$ and for all $i \in [\![1, m]\!]$, $\overline{\Delta}(x_i) = (x_i, x_i) \in \overline{X} \times_k \overline{X}$. Hence $\overline{X}$ is a separated scheme over k, and finally it is a projective variety.

Moreover, if X is smooth, then $\overline{X}$ is the glueing union of $n + 1$ affine smooth charts (because they are isomorphic to X which is smooth), so $\overline{X}$ is smooth. ■

Remark 1.10 If we want to work without integrity, the proof above works : if X is just an affine scheme of finite type over k and of dimension 1, then $\overline{X}$ is a projective scheme of finite type over k, of dimension 1.

Example 1.7 The projectivization of $\mathbb{G}_m$
We suppose in this example that k is algebraically closed. Recall that the multiplicative group $\mathbb{G}_m$ is defined as the spectrum of $k[T_1, T_2]/(T_1T_2 - 1)$. By definition, its projectivization is then

$$\text{Proj}(k[T_0, T_1, T_2]/(T_1T_2 - T_0^2))$$

and we have $\mathbb{G}_m \cong D_+(\pi(T_0))$, where π is the canonical projection $k[T_0, T_1, T_2] \longrightarrow k[T_0, T_1, T_2]/(T_1T_2 - T_0^2)$.
Therefore $\overline{\mathbb{G}_m} \setminus \mathbb{G}_m \cong V_+(\pi(T_0))$. Let us compute this set. By definition, $V_+(\pi(T_0))$ is the set of all prime homogeneous ideals of $k[T_0, T_1, T_n]/(T_1T_2 - T_0^2)$ that contains $\pi(T_0)$, and that does not contain all the elements of positive degree.
First, the homogeneous ideal generated by $\pi(T_0)$ is not prime. Indeed, we have $\pi(T_1T_2) = \pi(T_0^2)$ so $\pi(T_1)\pi(T_2)$ is in $(\pi(T_0))$, but neither $\pi(T_1)$ nor $\pi(T_2)$ is in $(\pi(T_0))$. Therefore $(\pi(T_0))$ is not in $V_+(\pi(T_0))$.
Then any ideal $\mathfrak{p}$ in $V_+(\pi(T_0))$ is generated by $\pi(T_0)$ and at least another polynomial $\pi(P)$. We can suppose that P is a homogeneous, irreducible polynomial in T_1, T_2, because $\pi(T_0)$ is already in $\mathfrak{p}$. Furthermore, every occurrence of T_1T_2 in P can be removed, because $\pi(T_1T_2) = \pi(T_0)^2$ is already in $\mathfrak{p}$. Therefore we can suppose that $P = P_1 + P_2$ where P_1 is a homogeneous polynomial in T_1, of degree d, and P_2 a homogeneous polynomial in T_2 of degree d too. But then $\pi(T_1^d)\pi(P) = \pi(T_1^dP_1) + \pi(T_1^dP_2)$, and $\pi(T_1^dP_2)$ can be seen as a polynomial in T_0 because $\pi(T_1T_2) = \pi(T_0^2)$. Hence $\pi(T_1^dP_2)$ is in $\mathfrak{p}$, and so is $\pi(T_1^dP_1)$. Finally, we can actually suppose that P is a polynomial only in T_1. Yet, we said that it was a homogeneous prime polynomial, and as k is algebraically closed, we get $P = T_1$, up to multiplication by a nonzero constant. As T_1 and T_2 play symmetric roles, we can also have $P = T_2$.
We can check that the ideals $(\pi(T_0), \pi(T_1))$ and $(\pi(T_0), \pi(T_2))$ are prime ideals, and they don't contain all the polynomials of positive degree.
Therefore the projectivization of $\mathbb{G}_m$ only consists in "adding two points at infinity". We could have guessed this by thinking of $\mathbb{G}_m$ as the pointed line $\mathbb{A}^1(k) \setminus \{0\}$: to complete it we need a point at zero and a point at infinity.

Proposition 1.21 Let X be an affine variety over k. The projectivization of $X \times_k X$ is $\overline{X} \times_k \overline{X}$.

Proof
Say $X = \text{Spec}(k[T_1, ..., T_n]/I)$, with I a prime ideal of $k[T_1, ..., T_n]$. Then

$$X \times_k X = \text{Spec}(k[T_1, ..., T_n/I \otimes_k k[S_1, ..., S_n]/J) \cong \text{Spec}(k[T_1, ..., T_n, S_1, ..., S_n]/(I, J))$$

where J is the ideal I with polynomials in S_i instead of T_i.
Therefore the projectivization of $X \times_k X$ is $\text{Proj}(k[T_0, T_1, ..., T_n, S_1, ..., S_n]/(\overline{I}, \overline{J}))$.
On an other hand, we have

$$\overline{X} \times_k \overline{X} = \text{Proj}(k[T_0, ..., T_n]/\overline{I} \otimes k[S_0, ..., S_n]/\overline{J}) \cong \text{Proj}(k[T_0, ..., T_n, S_0, ..., S_n]/(\overline{I}, \overline{J}, S_0 - T_0) \text{(see appendices)}$$

Therefore $\overline{X} \times_k \overline{X} = \overline{X \times_k X}$. ■

1.5 Rational points

A little focus on some special points of a scheme, which will be important in the final section.

Definition 1.21 Rational point
Let X be a scheme over k. A point $x \in X$ is rational when the residue field $k(x) := O_X(x)/\mathfrak{m}_x$ is equal to k.

Example 1.8 Rational points of the line
The prime ideals of $k[T]$ are the principal ideals (P) with P an irreducible polynomial, and the ideal $\{0\}$. It is easy to show that $\{0\}$ is not a rational point : indeed, the stalk $O_{\mathbb{A}^1(k)}(\{0\})$ is the fraction field $k(T)$, so the residue field is $k(T)$ which is not isomorphic to k.

Moreover, if P is an irreducible polynomial, the stalk at the point (P) is the localization $k[T]_{(P)} := (k[T] \setminus (P))^{-1}k[T]$. The maximal ideal of this ring is the image of (P), and then the residue field at (P) is

$$k((P)) = k[T]_{(P)}/\mathfrak{m}_{(P)} \cong k[T]/(P)$$

Therefore the rational points of $\mathbb{A}^1(k)$ are exactly the ideals (P) with P a polynomial of degree 1.

In particular, when k is algebraically closed, the irreducible polynomials all have degree 1, so all the points of $\mathbb{A}^1(k)$ except $\{0\}$ are rational points.

Definition 1.22 Sections
Let X be a scheme over k, and $f : X \longrightarrow \mathrm{Spec}(k)$ the associated morphism. A section of X is a morphism of k-schemes $\sigma : \mathrm{Spec}(k) \longrightarrow X$, such that $f \circ \sigma = \mathrm{id}_{\mathrm{Spec}(k)}$.
The set of sections of X is denoted by $X(k)$.

Proposition 1.22 Let X be a scheme over k, and $f : X \longrightarrow \mathrm{Spec}(k)$ the associated morphism. Then $X(k)$ can be identified with the set of rational points of X.

Proof Let $\sigma \in X(k)$. As k is a field, we have $\mathrm{Spec}(k) = \{\xi\}$, with ξ the zero ideal of k. Then σ is just the choice of one point in $X : \sigma(\xi) = x$. But σ is a morphism of schemes, so it induces a morphism of rings $k(x) \longrightarrow k$. Since $k(x)$ is a field, this morphism of rings is injective, so k is an extension of $k(x)$. But $k(x)$ is also an extension of k, so $k = k(x)$, and x is rational.

Conversely, let $x \in X$ be a rational point. Then we have a canonical surjection $O_X(x) \longrightarrow k$, which induces an immersion $\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(O_X(x))$. It suffices to find a morphism $\mathrm{Spec}(O_X(x)) \longrightarrow X$ and we will have a section of X. Let U be an affine subset that contains x (it exists by definition of scheme). We have a restriction morphism $O_X(U) \longrightarrow O_X(x)$, that induces a morphism of schemes $\mathrm{Spec}(O_X(x)) \longrightarrow \mathrm{Spec}(O_X(U))$. But U is affine, so $\mathrm{Spec}(O_X(U)) = U$, and thus we have a morphism $\mathrm{Spec}(O_X(x)) \longrightarrow U$, that we can compose with the inclusion $U \longrightarrow X$ to have a morphism $\mathrm{Spec}(O_X(x)) \longrightarrow X$. ■

Example 1.9 Rational points of an elliptic curve
Let C be an elliptic curve over $\mathbb{Q}$, i.e. a smooth projective curve over $\mathbb{Q}$ of genus 1.
The $\mathbb{Q}$-rational points of C form a group, and the Mordell-Weil theorem states that it is abelian and finitely generated.

Proposition 1.23 When X is a variety and k is algebraically closed, the rational points are exactly the closed points.

Proof
Let us suppose that k is algebraically closed, and let X be a variety over k.

• Let $x \in X$ be a closed point. As X is a variety over k, it exists an affine open subset $\mathrm{Spec}(R)$ of x such that R is a finite type k-algebra : $R = k[T_1, ..., T_n]/I$. Then x can be seen as a maximal ideal $\mathfrak{p}$ of $k[T_1, ..., T_n]$ that contains I. Yet, k is algebraically closed, so by the weak Nullstellensatz, we have $R/\mathfrak{p} \cong k$. Thus the residue field of x is

$$O_X(x)/\mathfrak{m}_x \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong R/\mathfrak{p} \cong k$$

Then x is a rational point.

• Conversely, let us suppose that x is a rational point. Let us denote again by $\mathrm{Spec}(R)$ an affine neighbourhood of x, with $R \cong k[T_1, ..., T_n]/I$, and $\mathfrak{p}$ the prime ideal associated. We have, by hypothesis :

$$O_X(x)/\mathfrak{m}_x \cong \mathrm{Frac}(R/\mathfrak{p}) \cong k$$

We can embed $R/\mathfrak{p}$ into its field of fraction $\mathrm{Frac}(R/\mathfrak{p})$. Thus

$$R/\mathfrak{p} \subset \mathrm{Frac}(R/\mathfrak{p}) \cong k$$

On an other hand, as R is a k-algebra, we have an canonical inclusion $k \subset R/\mathfrak{p}$. Therefore $R/\mathfrak{p} \cong k$, so $\mathfrak{p}$ is a maximal ideal of R, and then x is a closed point in X. ■

2 Curves

Since the aim of this report is to prove something about curves, this section begins to study these objects and gives tools like the genus or the divisors of a curve.

Definition 2.1 A curve is a variety over k of dimension 1.

2.1 Classification of curves

In this subsection, we will focus on showing that a curve is either an affine variety either a projective variety. Then let us take a curve X over k. The main idea is to embed X in a projective curve and say that if X is not projective, then it must be an affine chart of this projective curve.

Lemma 2.1 There exists a projective curve $\overline{X}$ and an open immersion $j : X \longrightarrow \overline{X}$ such that :

$$\overline{X} = X \sqcup \{x_1, ..., x_n\}$$

with each x_i a closed point in $\overline{X}$.

Proof
Let us write $X = \bigcup_{i=1}^r U_i$, an affine open covering of X. As X is a curve, each open subscheme U_i is an affine curve over k. Then by proposition 1.20, the projectivization $\overline{U}_i$ of U_i is a projective curve over k, and $\overline{U_i} = U_i \sqcup \{x_1^{(i)}, ..., x_{m_i}^{(i)}\}$ for some closed points $x_j^{(i)}$.

We want to apply the glueing lemma (proposition 1.8). In this aim, we set $U_{ij} := \overline{U_i} \cap \overline{U_j}$ for all $i, j \in [\![1, r]\!]$ and $f_{ij} = \mathrm{id}_{U_{ij}}$. Then the conditions (i), (ii) and (iii) of the glueing lemma are satisfied, so it exists a unique scheme over k, $\overline{X}$, such that $\overline{X} = \bigcup_{i=1}^r \overline{U_i}$. Then :

$$\overline{X} = \bigcup_{i=1}^r \overline{U_i} = \bigcup_{i=1}^r U_i \sqcup \{x_1^{(i)}, ..., x_{m_i}^{(i)}\} = X \sqcup \{x_1^{(1)}, ..., x_{m_1}^{(1)}, ..., x_1^{(r)}, ..., x_{m_r}^{(r)}\}$$

with each $x_j^{(i)}$ a closed point of $\overline{X}$.

Let us show that $\overline{X}$ is a projective curve over k. First, $\overline{X}$ is reduced, because each $\overline{U_i}$ is reduced. Then, let us suppose that $\overline{X} = F_1 \cup F_2$ with F_1, F_2 two closed subsets of $\overline{X}$. We have $X = (X \cap F_1) \cup (X \cap F_2)$, and $X \cap F_i$ is a closed subset of X for $i = 1, 2$. But X is irreducible, so we have (for instance) $X = F_1 \cap X$, which means that $X \subset F_1$. Therefore $\overline{X} \setminus F_1 = \{y_1, ..., y_s\}$ is a finite set of closed points. As the complement of F_1, which is closed, the set $\{y_1, ..., y_s\}$ is an open subset of $\overline{X}$. Thus for all $i \in [\![1, r]\!]$, $\{y_1, ..., y_s\} \cap \overline{U_i}$ is an open and closed subset of $\overline{U_i}$. But $\overline{U_i}$ is an integral scheme, so its underlying topological space is connected. Thus we have $\{y_1, ..., y_s\} \cap \overline{U_i} = \emptyset$ or $\overline{U_i}$. But if $\{y_1, ..., y_s\} \cap \overline{U_i} = \overline{U_i}$, then the dimension of $\overline{U_i}$ is 0 : that is absurd. Therefore for all $i \in [\![1, r]\!]$, we have $\{y_1, ..., y_s\} \cap \overline{U_i} = \emptyset$, and thus $\{y_1, ..., y_s\} = \emptyset$, so $F_1 = \overline{X}$.
This shows that $\overline{X}$ is irreducible.

Now, we have to show that the morphism $f : \overline{X} \longrightarrow \mathrm{Spec}(k)$, induced be the morphisms $\overline{U_i} \longrightarrow \mathrm{Spec}(k)$, is separated and of finite type. As $\overline{X}$ is a finite union of finite type schemes over k, $\overline{X}$ is of finite type over k. Finally, as X is a variety, the morphism $\Delta : X \longrightarrow X \times_k X$ is a closed immersion. By proposition 1.21, we have $\overline{X} \times_k \overline{X} = \overline{X \times_k X}$, so $\overline{X} \times_k \overline{X} = X \times_k X \sqcup \{z_1, \cdots z_t\}$ for some closed points z_i, and then the closed immersion $\Delta : X \longrightarrow X \times_k X$ induces a closed immersion $\overline{X} \longrightarrow \overline{X} \times_k \overline{X}$.
We have shown that $\overline{X}$ is a variety over k. It is also a curve, because $\dim(\overline{X}) = \dim(X) = 1$, and it is a projective curve by construction. ■

Remark 2.1 If X was just a scheme of finite type over k of dimension 1, then the above proof can be adapted, so we still can show that there exists a projective scheme of finite type over k, of dimension 1 such that $\overline{X} = X \sqcup \{x_1, ..., x_n\}$.

Proposition 2.1 X is either affine or projective.

Proof
Let us suppose that X is not a projective curve. Then, by lemma 2.1 above, it exists an open immersion $j : X \longrightarrow \overline{X}$, with $\overline{X}$ a projective curve and $\overline{X} \setminus X = \{x_1, ..., x_n\}$.
As $\overline{X}$ is a projective curve, there is a closed immersion $f : \overline{X} \longrightarrow \mathbb{P}^n(k)$ for some $n \in \mathbb{N}^*$. Moreover, by construction of the projectivization, we have $f^{-1}(D_+(T_0)) = j(X)$. But f is a closed immersion, so it is a finite morphism. As $D_+(T_0)$ is an affine open subscheme of $\mathbb{P}^n(k)$, we conclude that $j(X)$ is affine, and so is X. ■

Remark 2.2 This is still true if X is just a finite type scheme over k of dimension 1.

2.2 Genus of a curve

We already know the genus of a topological space : it is an invariant that "counts the number of holes" of the space. Here, we define two different genus for a curve. Actually, when the curve is "sufficiently nice", all these genus match.

Definition 2.2 Cech cohomology
Let X be a variety over k. Let $\mathcal{U}$ be an affine open covering of X, such that $\mathcal{U} = (U_i)_{i\in I}$. For all $p \in \mathbb{N}$, we define the space of p-cochains of $\mathcal{U}$ by

$$C^p(\mathcal{U}) := \prod_{(i_0,\ldots,i_p)\in I^{p+1}} O_X(U_{i_0} \cap \ldots \cap U_{i_{p+1}})$$

The space of cochains is then $C(\mathcal{U}) = \bigoplus_{p\geq 0} C^p(\mathcal{U})$.

Moreover, for all $p \geq 0$, the differential is a map $d^p : C^p(\mathcal{U}) \longrightarrow C^{p+1}(\mathcal{U})$ such that, for $f \in C^p(\mathcal{U})$, and $(i_0, \cdots, i_{p+1}) \in I^{p+2}$,

$$(d^p f)_{i_0,\cdots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j f_{i_0,\cdots,\widehat{i_j},\cdots,i_{p+1}} \,{}_{|U_{i_0}\cap\cdots\cap U_{i_{p+1}}}$$

Finally, the Cech cohomology is defined by

$$H^p(\mathcal{U}) := \mathrm{Ker}(d^p)/\,\mathrm{Im}(d^{p-1})$$

It can be shown that $H^p(\mathcal{U})$ does not depend on the affine covering $\mathcal{U}$, so we can simply write

$$H^p(X) = \mathrm{Ker}(d^p)/\,\mathrm{Im}(d^{p-1})$$

Remark 2.3 If X can be covered by m affine open subsets, then for all $p \geq m$, $H^p(X) = 0$. (see [Liu02] p.186).

Definition 2.3 Arithmetic genus
Let X be a smooth projective curve over k. The arithmetic genus of X is defined by

$$p_a(X) := 1 - \chi_k(O_X) = 1 - \sum_{i\geq 0} (-1)^i \dim_k H^i(X)$$

Proposition 2.2 Sheaf of ideals
Let X be a variety over k. Let Y be a closed subset of X.
For all open subset U of X, we set

$$\mathcal{I}(U) := \{f \in O_X(U) \mid \forall P \in O_X(U \cap Y), f_{|P} \in \mathfrak{m}_P\}$$

Then $\mathcal{I}$ is an O_X-module, called the sheaf of ideals of X over Y.

Proof
First, for any open subset U of X, the set $\mathcal{I}(U)$ is an ideal of $O_X(U)$. Therefore it has a structure of $O_X(U)$-module. Then, if $V \subset U$, we have a restriction map $\mathrm{Res}^{\mathcal{I}}_{U,V} : \mathcal{I}(U) \longrightarrow O_X(V)$, given by the restriction map $O_X(U) \longrightarrow O_X(V)$. It is easy to check that the map $\mathrm{Res}^{\mathcal{I}}_{U,V}$ actually arrive in $\mathcal{I}(V)$.
Hence $\mathcal{I}$ is a subsheaf of O_X. ■

Definition 2.4 Let X be a variety over k, and $\Delta : X \longrightarrow X \times_k X$ the closed immersion of proposition 1.6. Let $\mathcal{I}$ be the sheaf of ideals of $X \times_k X$ over $\Delta(X)$. Then we define the sheaf of differentials by :

$$\Omega_{X/k} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$$

Definition 2.5 Geometric genus
Let X be a projective smooth variety over k. The geometric genus of X is defined by

$$p_g(X) = \dim_k \omega_X(X)$$

where $\omega_X := \bigwedge^n \Omega_{X/k}$, where $n = \dim(X)$.

Theorem 2.1 Let X be a smooth projective variety over k. Then the geometric genus and the arithmetic genus of X are the same, and coincide with the topological genus of X when $k = \mathbb{C}$.

Proof See [Har77] page 246, remark 7.12.2. ■

2.3 The Riemann-Roch theorem

Here, we will suppose that k is algebraically closed. The aim of the section is to show that the projective line is the only smooth projective curve of genus 0. To do this, we will use the famous Riemann-Roch theorem.

Definition 2.6 Divisors of a curve
Let X be a curve over k. A divisor of X is a formal sum $D := \sum_{i=1}^{r} n_i P_i$, for $n_i \in \mathbb{Z}$ and P_i a closed point of X.
Moreover, we say that D is effective when $n_i \geq 0$ for all i.
The degree of D is the integer $\sum_{i=1}^{r} n_i$.

Definition 2.7 Divisor of a function
Let X be a projective curve over k, and h a homogeneous polynomial. For a closed point P of X, with coordinates $[a_0 : ... : a_n]$ (see the remark 1.4 for the link between closed points of X and homogeneous coordinates), we define $\nu_P(h)$ to be the valuation of $h(T_0 - a_1, ..., T_n - a_n)$. We extend this definition to all $f \in \kappa(X)$, setting $\nu_P(f) = \nu_P(h_1) - \nu_P(h_2)$ if $f = h_1/h_2$, with h_1 and h_2 two homogeneous polynomials of same degree.
Then, for $f \in \kappa(X)^*$, we define

$$\text{Div}(f) := \sum \nu_P(f) P$$

The sum is over all closed points of X, but it is actually a finite sum because except for finitely many points, $\nu_P(f) = 0$.
When a point P is such that $\nu_P(f) > 0$, we say that P is zero of f, and when $\nu_P(f) < 0$, we say that P is a pole of f.

Definition 2.8 Let X be a projective curve over k. For all divisor D of X, we define

$$L(D) := \{f \in \kappa(X)^* \mid \text{Div}(f) + D \text{ is effective}\} \sqcup \{0\}$$

Lemma 2.2 Let X be a projective curve over k, and D a divisor of X. Then $L(D)$ is a k-vector space.

Proof
We have an inclusion $L(D) \subset \kappa(X)$, and $\kappa(X)$ is a k-vector space itself.
By definition $0 \in L(D)$. Let us write $D = \sum_Z n_Z Z$, with for almost all Z, $n_Z = 0$. If $f, g \in L(D)$, then $\text{Div}(f) + D$ and $\text{Div}(g) + D$ are effective, so for all Z, $n_Z + \nu_Z(f) \geq 0$ and $n_Z + \nu_Z(g) \geq 0$. Thus for all Z, $\min(\nu_Z(f), \nu_Z(g)) + n_Z \geq 0$, and so $\nu_Z(f+g) + n_Z \geq 0$. Therefore $f + g \in L(D)$.
If $a \in k$, then for all Z, $\nu_Z(af) = \nu_Z(f)$ because k is an integral domain, so $af \in L(D)$.
Therefore $L(D)$ is a k subvector-space of $\kappa(X)$. ■

Definition 2.9 We denote by $l(D)$ the dimension of $L(D)$.

Example 2.1 We have $l(0) = 1$.
Indeed, $L(0)$ is the space of functions f such that $\text{Div}(f)$ is effective. Yet, the rational functions in $\kappa(X)$ have as much zeros as poles because X is projective, so the only functions with an effective divisor are the constant ones. Therefore $L(0) \cong k$, and $l(0) = 1$.

Proposition 2.3 Let X be a projective curve over k, and D an effective divisor of X. Then $l(-D) = 0$.

Proof
Let us write $D = \sum n_P P$, with $n_P \geq 0$ for all P.
Suppose that there is a rational function f such that $\text{Div}(f) - D$ is effective. If f is constant, then $\text{Div}(f) = 0$, and $\text{Div}(f) - D$ is not effective. Thus f is not constant, so it has a pole, say Q. Since $\text{Div}(f) - D$ is effective, we must have $\nu_Q(f) - n_Q \geq 0$. But $\nu_Q(f) < 0$ because Q is a pole, and $-n_Q \leq 0$ because D is effective, so $\nu_Q(f) - n_Q < 0$, which is absurd.
Hence $L(-D) = \{0\}$ and $l(-D) = 0$. ■

Definition 2.10 Equivalence of divisors
Let X be a projective curve over k, and D, D' two divisors of X. We say that D and D' are equivalent, denoted by $D \sim D'$, when it exists $f \in \kappa(X)^*$ such that

$$D - D' = \text{Div}(f)$$

This is an equivalence relation, and we will denote by $[D]$ the class of D for this relation.

Theorem 2.2 (Riemann-Roch)
Let X be a projective smooth curve over k (always assumed to be algebraically closed), of genus g. Then for all divisor D of X, we have

$$l(D) = l(K_X - D) + \deg(D) + 1 - g$$

Where K_X is a specific divisor, called the canonical divisor of X. This canonical divisor has degree $2g-2$.

Proof admitted, see for instance [Gat18]. ■

Proposition 2.4 If k is algebraically closed, then $\mathbb{P}^1(k)$ is the only smooth projective curve of genus 0.

Proof
Let X be a smooth projective curve of genus 0. Let P and Q two distinct points on X. Then by Riemann-Roch theorem, we have $l(P-Q) = l(K-(P-Q)) + \deg(P-Q) + 1 - g = 1$, because $\deg(P-Q) = 0$ and $\deg(K-(P-Q)) = -2$ so $l(K-(P-Q)) = 0$. Then the divisor $D := P - Q$ is such that $l(D) > 0$ and $\deg(D) = 0$. As $l(D) \neq 0$, it exists $f \in k(X)$ such that $\mathrm{Div}(f) + D$ is effective. Then D is equivalent to an effective divisor. But $\deg(D) = 0$, so D is equivalent to an effective divisor of degree 0 : there is only one such divisor, it is 0. Hence $P \sim Q$, and it exists $f \in k(X)$ such that $P - Q = \mathrm{Div}(f)$.
Since X is a projective curve, we can write $X = \mathrm{Proj}(k[T_0, ..., T_n]/\overline{I})$, and $f = h_1/h_2$, with h_1, h_2 two (equivalence classes of) homogeneous polynomials of same degree. Then f is an element of the ring $O_X(D_+(h_2))$. Let us consider the following morphism of k-algebras :

$$\begin{array}{cccc} \theta : & k[T] & \longrightarrow & O_X(D_+(h_2)) \\ & T & \longmapsto & f \end{array}$$

This gives rise to a morphism of schemes

$$\phi : D_+(h_2) \longrightarrow \mathbb{A}^1(k)$$

Since ϕ is a morphism of schemes, it is a continuous map. Hence by density of $D_+(h_2)$ in X, ϕ induces a morphism of schemes

$$\overline{\phi} : X \longrightarrow \mathbb{P}^1(k)$$

We refer to [Har77], p.138, to conclude that $\overline{\phi}$ is an isomorphism. ■

3 Picard groups and relatives

To any scheme we can associate a group, the Picard group, which is invariant by fibred product with the affine line. This property will be useful to study $\mathbb{A}^1$-contractibility.

3.1 Class divisors

Here, we generalize the notion of divisor, introduced in the frame of projective curves over k in the last section. This section was widely inspired by [Bri08].

Definition 3.1 Divisors of a variety
Let X be a smooth variety over k. We call prime divisor of X a closed subvariety of X of codimension 1. More generally, a divisor of X is a formal sum $D := \sum_{i=1}^{r} n_i Z_i$, for $n_i \in \mathbb{Z}$ and Z_i prime divisor of X. Let us denote by $\mathrm{Div}(X)$ the group of divisors of X.

Remark 3.1 Any closed subset of X can be endowed with a scheme structure induced by that of X (see appendices about locally ringed spaces). Hence any integral closed subset of X is a prime divisor of X.

Example 3.1 If X is a smooth curve over k, the prime divisors are the closed points of X, which is what we studied in the last section.

Example 3.2 If $X := \mathbb{A}^n(k)$, then every curve $V(F)$, with F an irreducible polynomial in $k[T_1, ..., T_n]$, is a prime divisor of X. We will see later that the converse is true.

Proposition 3.1 Let X be a smooth variety over k, and Z a prime divisor of X. Let ξ_Z be the generic point of Z (it exists and it is unique because Z is an integral scheme). Then the ring $O_X(\xi_Z)$ is a discrete valuation ring.
Hence, it exists a valuation $\nu_Z : \kappa(X) \longrightarrow \mathbb{Z}$ such that

$$O_X(\xi_Z) = \{f \in \kappa(X) \mid \nu_Z(f) \geq 0\}$$

Proof
Since X is smooth, the stalk $O_X(\xi_Z)$ is regular, and since Z is a prime divisor of X, the dimension of $O_X(\xi_Z)$ is one. Then (see appendices, lemma 6.6) $O_X(\xi_Z)$ is a discrete valuation ring. ■

Definition 3.2 Let X be a smooth variety over k, and $f \in \kappa(X)$. The divisor of f is defined by :

$$\mathrm{Div}(f) = (f) := \sum_{Z \text{ prime divisor}} \nu_Z(f) Z$$

This makes sense because for almost all prime divisor Z, we have $\nu_Z(f) = 0$, so the sum is finite.
This kind of divisor is called a principal divisor.

Proposition 3.2 The set of all principal divisors of a smooth variety X over k, denoted by $\mathrm{Princ}(X)$, is a subgroup of $\mathrm{Div}(X)$.

Proof
• By definition, $\mathrm{Princ}(X)$ is a subset of $\mathrm{Div}(X)$.
• We have $0_{\mathrm{Div}(X)} = (1)$, with 1 the constant function equal to 1, so $0_{\mathrm{Div}(X)} \in \mathrm{Princ}(X)$.
• If $f, g \in \kappa(X)$, then $(f) + (g) = (fg)$ and $-(f) = (1/f)$, so $\mathrm{Princ}(X)$ is stable under addition and inversion.
Hence $\mathrm{Princ}(X)$ is a subgroup of $\mathrm{Div}(X)$. ■

Definition 3.3 Let X be a smooth variety over k. The class group of X is defined by the quotient

$$Cl(X) := \mathrm{Div}(X)/\,\mathrm{Princ}(X)$$

As is the previous section, for all $D \in \mathrm{Div}(X)$, we will denote by $[D]_X$, or simply $[D]$, the class of D in $Cl(X)$.

Remark 3.2 This definition makes sense because $\mathrm{Div}(X)$ is an abelian group, so we can take the quotient by $\mathrm{Princ}(X)$.

Example 3.3 For all $n \in \mathbb{N}$, $Cl(\mathbb{A}^n(k)) = \{0\}$.
Indeed, let Z be a prime divisor of $\mathbb{A}^n(k)$.
The ring $O_X(\xi_Z)$ is a local ring, so it contains a unique maximal ideal, denoted by $\mathfrak{m}_Z$. Yet $O_X(\xi_z)$ is a local regular ring because X is smooth, so it is a unique factorization domain. By the principal ideal theorem (see [Eis95] p.233), $\mathfrak{m}_Z$ is minimal over an ideal generated by one element, so $\mathfrak{m}_Z$ is minimal over a principal ideal, in a factorial domain. By lemma 6.4 (see appendices), $\mathfrak{m}_Z$ is itself principal. So we can write $\mathfrak{m}_Z = f_Z O_X(\xi_Z)$ for a certain $f_Z \in O_X(\xi_Z) \subset \kappa(\mathbb{A}^n(k)) = k(T_1, ..., T_n)$. We can write $f_Z = g/h$ with $g, h \in k[T_1, ..., T_n]$ and $h(z) \neq 0$ for all $z \in Z$. Therefore h is invertible in $O_X(\xi_Z)$, so $f_Z O_X(\xi_Z) = g O_X(\xi_Z)$, thus we can suppose that f_Z is a polynomial. Moreover, as $f_Z O_X(\xi_Z)$ is a maximal ideal, the polynomial f_Z must be irreducible. Hence :

$$Z = \mathrm{Div}(f_Z)$$

Therefore Z is a principal divisor of X. Now, if $D = \sum_{i=1}^r n_i Z_i$ is a divisor of X, we have

$$D = \mathrm{Div}\left(\prod_{i=1}^r f_{Z_i}^{n_i}\right)$$

Hence $\mathrm{Div}(X) = \mathrm{Princ}(X)$, and $Cl(\mathbb{A}^n(k)) = \{0\}$.

Proposition 3.3 Let Z be a prime divisor of a smooth variety X over k. Let $U := X \setminus Z$. Then the following sequence is exact :

$$\mathbb{Z} \xrightarrow{\phi} Cl(X) \xrightarrow{\psi} Cl(U) \longrightarrow \{0\}$$

where $\phi(1) = [Z]$ and $\psi([D]) = [D \cap U]$.

Proof

• Let us verify that the map ψ is well defined. If Z' is a prime divisor of X, then $Z' \cap U$ is a closed subvariety of U, so it is a prime divisor of U. By linearity, if $D \in Cl(X)$, then $D \cap U \in Cl(U)$. Moreover, if D and D' are in the same class in $Cl(X)$, then it exists $f \in \kappa(X)$ such that $D - D' = (f)_X$. Therefore $(D - D') \cap U = (f) \cap U$, then $D \cap U - D' \cap U = (f)_U$ (we denoted the principal ideal associated to f in X by $(f)_X$ and the one in U by $(f)_U$). Hence $\psi([D]) = \psi([D'])$, so ψ is well defined.

• Now, by definition of ϕ, we have $\phi(\mathbb{Z}) = \mathbb{Z} \cdot [Z] = \{[nZ], n \in \mathbb{Z}\}$. But for all $n \in \mathbb{Z}$, we have $nZ \cap U = \emptyset$, so $\mathrm{Im}(\phi) \subset \mathrm{Ker}(\psi)$.
Conversely, if $[D] \in \mathrm{Ker}(\psi)$, then $D \cap U = \emptyset$, so for every prime factor Z' that appears in D, we have $Z' \cap U = \emptyset$, so $Z' \subset Z$. But Z' has a codimension of 1, and $Z \neq X$ so $Z' = Z$. Finally, $D \in \mathbb{Z} \cdot Z$, and $\mathrm{Im}(\phi) = \mathrm{Ker}(\psi)$.

• We just have to show that ψ is surjective to conclude. By linearity, it suffices to show that for every prime divisor Z_U of U, there is $D \in \mathrm{Div}(X)$ such that $[D \cap U] = [Z_U]$. The topology over U is the topology induced by the topology of X : as Z_U is closed in U, there exists a closed subset Z' of X such that $Z_U = Z' \cap U$. Let us show that Z' is a prime divisor of X.
As $Z_U \neq \emptyset$, Z' is not contained in Z, so it has at least one irreducible component which is not contained in Z. Therefore we can suppose without loss of generality that none of the irreducible components of Z' are contained in Z. If Z' is not irreducible, then $Z' = (F_1 \cap Z') \cup (F_2 \cap Z')$ with F_1, F_2 two closed subsets of X. Hence $Z_U = (F_1 \cap Z_U) \cup (F_2 \cap Z_U)$, but Z_U is irreducible, so (for instance) $F_1 \cap Z_U = \emptyset$. Thus $F_1 \cap Z' \cap U = \emptyset$, so $F_1 \cap Z' \subset Z$, which is impossible, because we assumed that none of the irreducible components of Z' were contained in Z. Finally, Z' is irreducible.
Moreover, for $x \in Z$, the stalk $O_Z(x)$ is a subring of $O_X(x)$, so it is a reduced ring. Therefore Z' is an integral closed subset of X, i.e. it is a closed subvariety of X. ■

Corollary 3.1 The class group of $\mathbb{P}^n(k)$, for all $n \in \mathbb{N}^*$ is isomorphic to $\mathbb{Z}$.

Proof
Let us recall that $\mathbb{P}^n(k) \simeq \mathbb{P}^{n-1}(k) \sqcup \mathbb{A}^n(k)$, and $\mathbb{P}^{n-1}(k) = V_+((T_0))$. Hence $\mathbb{P}^{n-1}(k)$ is a closed subset of $\mathbb{P}^n(k)$. Furthermore, $\mathbb{P}^{n-1}(k)$ is a variety over k, and has a codimension of 1 in $\mathbb{P}^n(k)$. Then $\mathbb{P}^{n-1}(k)$ is a prime divisor of $\mathbb{P}^n(k)$. We can apply the previous proposition with $X = \mathbb{P}^n(k)$ and $Z = \mathbb{P}^{n-1}(k)$. By the example 3.3, we have $Cl(\mathbb{A}^n(k)) = 0$, so we have the exact sequence

$$\mathbb{Z} \xrightarrow{\phi} Cl(\mathbb{P}^n(k)) \longrightarrow \{0\}$$

which means that ϕ is surjective.

Let us show that ϕ is injective. Let $m \in \mathbb{Z}$ such that $\phi(m) = 0$. Then $[m\mathbb{P}^{n-1}(k)] = 0$, so $m = 0$ or $\mathbb{P}^{n-1}(k)$ is principal. If $\mathbb{P}^{n-1}(k)$ is principal, then it exists $f \in \kappa(\mathbb{P}^n(k))$ such that $\mathbb{P}^n(k) = \mathrm{Div}(f)$. We can write $f = g/h$ with g and h two homogeneous polynomials of identical degree. So $\mathrm{Div}(f) = \sum_Z \nu_Z(g)Z - \sum_Z \nu_Z(h)Z = \mathbb{P}^{n-1}(k)$. As $\mathbb{P}^{n-1}(k)$ is a prime divisor, we must have $\nu_{\mathbb{P}^{n-1}}(g) = 1$ and for all other prime divisor Z', $\nu_{Z'}(g) = \nu_{Z'}(h) = 0$. But then the degree of g is strictly greater than the degree of h, and this is absurd. Therefore $[\mathbb{P}^{n-1}(k)] \neq 0$, so $m = 0$ and finally ϕ is injective.

Hence the map ϕ defines an isomorphism between $\mathbb{Z}$ and $\mathbb{P}^n(k)$. ■

Proposition 3.4 The class group is $\mathbb{A}^1(k)$-invariant.

Proof
Let X be a smooth variety over k. We will denote by π the projection $X \times_k \mathbb{A}^1(k) \longrightarrow X$. The product $X \times_k \mathbb{A}^1(k)$ is still a smooth variety over k.

Let us consider the following map :

$$\begin{array}{cccc} \phi : & \mathrm{Div}(X) & \longrightarrow & Cl(X \times_k \mathbb{A}^1(k)) \\ & D = \sum n_i Z_i & \longrightarrow & \left[\sum n_i \pi^{-1}(Z_i)\right] \end{array}$$

Let us show that this map is well defined : for a prime divisor Z of X, we have to show that the set $\pi^{-1}(Z)$ is a prime divisor of $X \times_k \mathbb{A}^1(k)$.

As the inverse image of an integral closed subscheme by a morphism of schemes, $\pi^{-1}(Z)$ is an irreducible closed subscheme of $X \times_k \mathbb{A}^1(k)$. And it is reduced because $X \times_k \mathbb{A}^1(k)$ is reduced.
Furthermore, if it exists a closed subset irreducible F in $X \times_k \mathbb{A}^1(k)$ such that $\pi^{-1}(Z) \subset F \subset X \times_k \mathbb{A}^1(k)$, then $Z \subset \pi(F) \subset X$. But Z has a codimension of 1 so $\pi(F) = Z$ or $\pi(F) = X$.
If $\pi(F) = Z$, then $\pi^{-1}(Z) = \pi^{-1}(\pi(F))$ so $F \subset \pi^{-1}(Z)$, then $F = \pi^{-1}(Z)$.
If $\pi(F) = X$, then $F = X \times_k W$ with W a closed subset of $\mathbb{A}^1(k)$, but F contains $\pi^{-1}(Z)$, and $\pi^{-1}(Z) = Z \times_k \mathbb{A}^1(k)$ (see lemma 6.1 in appendices), so $F = X \times_k \mathbb{A}^1(k)$. Hence the codimension of $\pi^{-1}(Z)$ is 1.

Moreover, ϕ is a morphism of groups, and we will show that it is surjective : indeed, the prime divisors of $X \times_k \mathbb{A}^1(k)$ are the ones like $Z \times_k \mathbb{A}^1(k)$, for Z a prime divisor of X, and the ones like $X \times_k \{a\}$, where a is a closed point of $\mathbb{A}^1(k)$. For the first type of prime divisors, we have $Z \times_k \mathbb{A}^1(k) = \pi^{-1}(Z)$ so $[Z \times_k \mathbb{A}^1(k]$ is in the image of ϕ. For the second type, we have $[X \times_k \{a\}] = 0$. Indeed, the function field $\kappa(X \times_k \mathbb{A}^1(k))$ is isomorphic to $\kappa(X)(T)$, so the rational function $f_a = T - a$ can be seen as a function in $\kappa(X \times_k \mathbb{A}^1(k))$, and we have $\mathrm{Div}(f_a) = X \times_k \{a\}$.

Finally, let us compute the kernel of ϕ : let $D = \sum n_i Z_i$ be a divisor of X, with $\phi(D) = 0$. Then it exists $f \in \kappa(X \times_k \mathbb{A}^1)$ such that $\sum n_i \pi^{-1}(Z_i) = \mathrm{Div}(f)$ in $X \times_k \mathbb{A}^1$. Hence, if we see f as a fraction of T with coefficients in $\kappa(X)$, we have that for all prime divisor Z in X, $\nu_{\pi^{-1}(Z)}(f) = \nu_Z(f(0))$, with $f(0) \in \kappa(X)$. Thus $D = \mathrm{Div}(f(0))$, so $D \in \mathrm{Princ}(X)$.
Conversely, $\mathrm{Princ}(X) \subset \mathrm{Ker}(\phi)$.

By the quotient universal property, ϕ induces an isomorphism $\tilde{\phi} : Cl(X) \longrightarrow Cl(X \times_k A^1(k))$. ■

Proposition 3.5 Let X be a smooth projective curve over k, with $x \in X$ a k-rational point. Then $Cl(X) \neq 0$.

Proof
As x is a k-rational point, $\{x\}$ is a closed subset of X. Moreover, X is a curve, so $\dim(X) = 1$, and then the codimension of $\{x\}$ is 1. Therefore $\{x\}$ is a prime divisor of X. If it is principal, there exists $f \in \kappa(X)$ such that $\nu_x(f) = 1$ and for every other prime divisor Z of X, $\nu_Z(f) = 0$. So f has only one zero on x, which is absurd since f is a homogeneous fraction. So $\{x\} \notin \mathrm{Princ}(X)$, and $Cl(X) \neq \{0\}$. ■

3.2 Picard group

The class group was only defined for varieties. Now, we will introduce a more abstract group that generalize the class group.

Definition 3.4 Invertible sheaf
Let X be a scheme. A sheaf of O_X-modules on X is a sheaf $\mathcal{F}$ such that for all open set U of X, $\mathcal{F}(U)$ is an abelian group endowed with a structure of $O_X(U)$-module.
We say that $\mathcal{F}$ is invertible when X can be covered with open sets U_i such that $\mathcal{F}_{|U_i}$ is isomorphic to $O_{X|U_i}$.

Proposition 3.6 Let X be a scheme. If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves over X, then $\mathcal{L} \otimes \mathcal{M}$ is still an invertible sheaf.
Moreover, it exists an invertible sheaf over X, $\mathcal{L}'$, such that $\mathcal{L} \otimes \mathcal{L}' \cong O_X$.

Proof
As $\mathcal{L}$ is invertible, we can cover X by open sets U_i such that $\mathcal{L}_{|U_i} \cong O_{X|U_i}$ for all i. We can do the same for $\mathcal{M}$: it exists open sets V_j such that $\mathcal{M}_{|V_i} \cong O_{X|V_i}$. Then, for all i, j, we have $(\mathcal{L} \otimes \mathcal{M})_{|U_i \cap V_j} \cong O_{X|U_i \cap V_j} \otimes O_{X|U_i \cap V_j} = O_{X|U_i \cap V_j}$. Then $\mathcal{L} \otimes \mathcal{M}$ is an invertible sheaf.

Then, let $\mathcal{L}' = \mathrm{Hom}_{O_X}(\mathcal{L}, O_X)$, be the dual sheaf of $\mathcal{L}$, i.e. a sheaf of O_X-modules such that for all open set U, $\mathcal{L}'(U)$ is the set of morphisms of $O_X(U)$-modules between $\mathcal{L}(U)$ and $O_X(U)$. This set is endowed naturally with a structure of $O_X(U)$-module.
Let us prove that $\mathcal{L} \otimes \mathcal{L}' \cong O_X$. Let U be an open subset of X. Let us consider

$$\begin{array}{rccc} \Psi : & \mathcal{L}(U) \otimes \mathrm{Hom}_{O_X(U)}(\mathcal{L}(U), O_X(U)) & \longrightarrow & O_X(U) \\ & f \otimes \phi & \longmapsto & \phi(f) \end{array}$$

This is a morphism of $O_X(U)$-modules. Furthermore, we can decompose $U = \bigcup_{i\in I} U \cap U_i$, and we have an isomorphism of sheaves $(O_X)_{|U\cap U_i} \overset{\alpha_i}{\cong} \mathcal{L}_{|U\cap U_i}$ for all $i \in I$.
Let us take $f \in \mathcal{L}(U)$ and $\phi \in \mathrm{Hom}_{O_X(U)}(\mathcal{L}(U), O_X(U))$ such that $\phi(f) = 0$. Then for all $i \in I$, we have $\phi(f)_{|U\cap U_i} = 0$, so $\phi_{|U\cap U_i}(f_{|U\cap U_i}) = 0$. To simplify what follows, we take $i \in I$, and we denote by $\phi_i := \phi_{|U\cap U_i}$ and $f_i := f_{|U\cap U_i}$. We have an isomorphism between $\mathcal{L}(U\cap U_i)\otimes\mathrm{Hom}(\mathcal{L}(U\cap U_i, O_X(U\cap U_i))$ and $O_X(U\cap U_i)$, given by

$$g \otimes \psi \longmapsto \alpha_i(g)\psi(\alpha_i^{-1}(1)) = \psi(g) \qquad (*)$$

As $\phi_i(f_i) = 0$, we have $f_i \otimes \phi_i = 0$. Therefore $f_i \otimes \phi_i = 0$, for all $i \in I$. As $(U_i \cap U)$ is a cover of U, it follows that $f \otimes \phi = 0$, so Ψ is injective.

Then if $g \in O_X(U)$, we can find $f_i \otimes \phi_i \in \mathcal{L}(U \cap U_i) \otimes \mathrm{Hom}(\mathcal{L}(U \cap U_i), O_X(U \cap U_i))$ for all $i \in I$ such that $g_{|U\cap U_i} = \phi_i(f_i)$ because of the isomorphism (*). This gives rise to an element $f \otimes \phi \in \mathcal{L}(U) \otimes \mathrm{Hom}(\mathcal{L}(U), O_X(U))$ such that $\phi(f) = g$. Therefore Ψ is surjective. ∎

Definition 3.5 Let X be a scheme. The Picard group of X is the group of invertible sheaves on X, up to isomorphisms.

For "sufficiently nice" schemes, the Picard group is just the class group.

Theorem 3.1 If X is a smooth variety over k, then $\mathrm{Pic}(X) \cong Cl(X)$.

Proof see [Har77], corollary II.6.16 page 145. ∎

4 $\mathbb{A}^1$-contractibility

In this section we get closer to the goal, by defining an equivalence relation over a category of schemes, in order to give meaning to the adjective "$\mathbb{A}^1$-contractible".

In what follows, we will denote $\mathbb{A}^1(k)$ by $\mathbb{A}^1$.

4.1 Naive $\mathbb{A}^1$-homotopy equivalence

The naive definition of $\mathbb{A}^1$-homotopy equivalence is similar to the homotopy theory in algebraic topology, except that we replace the unit interval $[0, 1]$ by the affine line $\mathbb{A}^1$.

Definition 4.1 $\mathbb{A}^1$-homotopy equivalence
Let $f, g : X \longrightarrow Y$ be two morphisms of schemes over k. We say that f and g are A^1-homotopy equivalent when there exists a morphism of schemes

$$H : X \times_k \mathbb{A}^1 \longrightarrow Y$$

such that

$$\begin{cases} H_{|X\times_k\{0\}} = f \\ H_{|X\times_k\{1\}} = g \end{cases}$$

More formally, the point 0 is rational in $\mathbb{A}^1$, associated to the section $\sigma_0 : \mathrm{Spec}(k) \longrightarrow \mathbb{A}^1$ such that $\mathrm{Im}(\sigma_0) = \{0\}$. By the universal property of fibred product (see appendices), from the morphisms $\mathrm{id}_X : X \longrightarrow X$ and $\sigma_0 \circ \phi_X : X \longrightarrow \mathbb{A}^1$, we can construct a unique morphism $\Phi_0 : X \longrightarrow X \times_k \mathbb{A}^1$.
In the same way, we can construct a morphism $\Phi_1 : X \longrightarrow X \times_k \mathbb{A}^1$ from the morphisms id_X and $\sigma_1 : \mathrm{Spec}(k) \longrightarrow \mathbb{A}^1$, $\mathrm{Im}(\sigma_1) = \{1\}$.
What is required in the $\mathbb{A}^1$-homotopy equivalence is that $H \circ \Phi_0 = f$ and $H \circ \Phi_1 = g$.

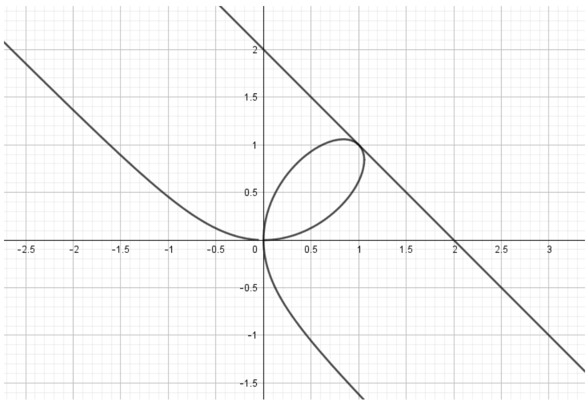
Example 4.1 The morphism identity $\mathrm{id}_{\mathbb{A}^1}$ and the constant equal to 0 are $\mathbb{A}^1$-homotopy equivalent.
First, let us recall that $\mathbb{A}^1 \times_k \mathbb{A}^1 = \mathbb{A}^2$. So let us consider the map $H : \mathbb{A}^2 \longrightarrow \mathbb{A}^1$ such that for all closed points (x, t) of $\mathbb{A}^2$, $H(x, t) = xt \in \mathbb{A}^1$. Additionally, let us notice that for $f \in k[T]$, we have $H^{-1}(D(f)) = D(\hat{f})$, where $\hat{f}$ is the image of f by the morphism $\phi : k[T] \longrightarrow k[T_1, T_2]$ such that $\phi(T) = T_1T_2$. Then, ϕ induces a morphism of rings $O_{\mathbb{A}^1}(D(f)) \longrightarrow O_{\mathbb{A}^2}(H^{-1}(D(f)))$, and it is local on stalks. Therefore H is a morphism of schemes.
Finally, we have $H_{|\mathbb{A}^1\times_k\{0\}} = 0$ and $H_{|\mathbb{A}^1\times_k\{1\}} = \mathrm{id}_{\mathbb{A}^1}$.

Proposition 4.1 The relation of $\mathbb{A}^1$-homotopy equivalence is reflexive and symmetric.

Proof
• If $f : X \longrightarrow Y$ is a morphism of schemes, then we can put $H = f \circ \pi_X : X \times_k \mathbb{A}^1 \longrightarrow Y$. Indeed, H is then a morphism of k-schemes, and $H \circ \Phi_0 = f \circ \pi_X \circ \Phi_0$, but by definition of Φ_0 we have $\pi_X \circ \Phi_0 = \mathrm{id}_X$, so $H \circ \Phi_0 = f$. It is the same for $H \circ \Phi_1$. Therefore f is $\mathbb{A}^1$-homotopy equivalent to itself.

• Then, let f, g be $\mathbb{A}^1$-homotopy equivalent by a homotopy $H : X \times_k \mathbb{A}^1 :\longrightarrow Y$ such that $H_{|X\times_k\{0\}} = f$ and $H_{|X\times_k\{1\}} = g$.
Let $e : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be a morphism of schemes such that $e(1) = 0$ and $e(0) = 1$, i.e. $e \circ \sigma_0 = \sigma_1$ and $e \circ \sigma_1 = \sigma_0$ (for example, we can take $e(t) = 1-t$ for the closed points of $\mathbb{A}^1$). Let $E : X \times_k \mathbb{A}^1 \longrightarrow X \times_k \mathbb{A}^1$ be the morphism of schemes built from e and id_X, i.e. such that $\pi_X \circ E = \pi_X$ and $\pi_{\mathbb{A}^1} \circ E = e \circ \pi_{\mathbb{A}^1}$. Then we have $\pi_X \circ (E \circ \Phi_0) = \mathrm{id}_X$ and $\pi_{\mathbb{A}^1} \circ (E \circ \Phi_0) = \sigma_1 \circ \phi_X$, so $E \circ \Phi_0$ satisfies the same property than Φ_1. By uniqueness, $E \circ \Phi_0 = \Phi_1$. By the same way, $E \circ \Phi_1 = \Phi_0$.
Now, it suffices to take $\tilde{H} := H \circ E$ to have a homotopy between f and g such that $\tilde{H} \circ \Phi_0 = g$ and $\tilde{H} \circ \Phi_1 = f$. ■

Remark 4.1 Unfortunately, this is not an equivalence relation, as it is not transitive. For instance, let us consider the affine scheme $Y = \mathrm{Spec}(\mathbb{C}[T_1, T_2]/((T_1^3 + T_2^3 - 2T_1T_2)(T_1 + T_2 - 2)))$. As $\mathbb{C}$ is algebraically closed, we will make the identification between the closed points of Y and the points in $\mathbb{C}^2$ where $(T_1^3 + T_2^3 - 2T_1T_2)(T_1 + T_2 - 2)$ vanish (see figure below).



Let us take $f, g, h : \mathbb{A}^1 \longrightarrow Y$, three morphisms of schemes such that $f(x) = (2, 0)$, $g(x) = (1, 1)$ and $h(x) = (0, 0)$ for all $x \in \mathrm{Spec}(A)$. Then f and g (resp. g and h) are $\mathbb{A}^1$-homotopy equivalent, thanks to the following homotopy :

$$H_1 : \begin{array}{ccc} \mathbb{A}^2 & \longrightarrow & Y \\ (x,t) & \longmapsto & (t, 2-t) \end{array} \quad \text{resp.} \quad H_2 : \begin{array}{ccc} \mathbb{A}^2 & \longrightarrow & Y \\ (x,t) & \longmapsto & \left(\dfrac{2(1-t)}{1+t^3}, \dfrac{2(1-t)^2}{1+t^3}\right) \end{array}$$

These two maps are morphisms of schemes, and we have $H_1(x, 0) = (0, 2)$, $H_1(x, 1) = H_2(x, 0) = (1, 1)$ and $H_2(x, 1) = (0, 0)$.
However, we can't build a homotopy between f and h. Indeed, suppose that we have a morphism of schemes $H : \mathbb{A}^2 \longrightarrow Y$ such that $H(x, 0) = (0, 2)$ and $H(x, 1) = (0, 0)$. From H we can extract a morphism of schemes $\tilde{H} : \mathbb{A}^1 \longrightarrow Y; \tilde{H}(t) = H(0, t)$. This induces a morphism of rings $\phi : k[T_1, T_2]/((T_1^3 + T_2^3 - 2T_1T_2)(T_1 + T_2 - 2)) \longrightarrow k[T]$. Let us denote by π the canonical projection $k[T_1, T_2] \longrightarrow k[T_1, T_2]/((T_1^3 + T_2^3 - 2T_1T_2)(T_1 + T_2 - 2))$.

Then, by morphism properties, $\phi(\pi(T_1^3 + T_2^3 - 2T_1T_2))\phi(\pi(T_2 + T_1 - 2)) = 0$. But $k[T]$ is an integral domain, so we must have $\phi(\pi(T_2 + T_1 - 2)) = 0$ (for instance, cause the proof is symmetric for $\phi(\pi(T_2^3 + T_1^3 - 2T_1T_2)) = 0$).

Let us show that $\mathrm{Im}(\tilde{H})$ is contained in $\mathrm{Spec}(k[T_1, T_2]/(T_1 + T_2 - 2))$ (corresponding to the line $T_1 + T_2 = 2$, see figure). Let $\mathfrak{p} \in \mathrm{Im}(\tilde{H})$. We have $\mathrm{Im}(\tilde{H}) \subset V(\mathrm{Ker}(\phi))$ (indeed, see the proof of proposition 1.7), so $\mathfrak{p}$ contains $\mathrm{Ker}(\phi)$. But we saw that $\pi(T_1 + T_2 - 2) \in \mathrm{Ker}(\phi)$, so $\pi(T_1 + T_2 - 2)$ is in $\mathfrak{p}$, and then $(T_1 + T_2 - 2) \subset \mathfrak{p}$, which means that $\mathfrak{p}$ is (or can be seen as) an ideal of $\mathrm{Spec}(k[T_1, T_2]/(T_1 + T_2 - 2))$.

However, by construction we have $\tilde{H}(1) = (0,0)$, that is not on the line $T_1 + T_2 = 2$. Therefore f and h are not $\mathbb{A}^1$-homotopy equivalent. (There is other examples in [Aso16]).

Definition 4.2 Naive equivalence of morphisms
We define a naive $\mathbb{A}^1$-homotopy equivalence relation by taking the relation generated by $\mathbb{A}^1$-homotopy equivalence defined above.
More precisely, we will say that two morphisms of schemes $f, g : X \longrightarrow Y$ are naively $\mathbb{A}^1$-homotopy equivalent when it exists finitely many morphisms $h_1, ..., h_n : X \longrightarrow Y$ such that $h_0 = f$, $h_n = g$ and h_i is $\mathbb{A}^1$-homotopy equivalent to h_{i+1} for all i.
For two schemes X and Y, we denote by $[X,Y]_{\mathcal{N}(k)}$ the set of naive equivalence classes of morphisms from X to Y.

Definition 4.3 Naive equivalence of schemes
Let X, Y be two schemes over k. We say that X and Y are naively $\mathbb{A}^1$-homotopy equivalent when there exists a morphism $f : X \longrightarrow Y$ and a morphism $g : Y \longrightarrow X$ such that $f \circ g$ (resp. $g \circ f$) is naively $\mathbb{A}^1$-homotopy equivalent to id_Y (resp. id_X).
We denote by $\mathcal{N}(k)$ the category of schemes over k where maps are naive $\mathbb{A}^1$-homotopy equivalence classes of morphisms.

Example 4.2 Let Z be a proper closed subset of $\mathbb{A}^1$, i.e. a finite number of points in $\mathbb{A}^1$. Let $U := \mathbb{A}^1 \setminus Z$. Then

$$[\mathrm{Spec}(k), U]_{\mathcal{N}(k)} = U(k)$$

Indeed, let s_1 and s_2 be two sections $\mathrm{Spec}(k) \longrightarrow U$. If there was a homotopy H between s_1 and s_2, then H would be a morphism of schemes from $\mathbb{A}^1$ to U. Let us show that H must be constant.
As Z is a closed subset of $\mathbb{A}^1$, there is a polynomial $f \in k[T]$, such that $Z = V(f)$ and $\deg(f) \geq 1$. Then

$$U = D_f = \mathrm{Spec}\left(k\left[T, \frac{1}{f}\right]\right)$$

is the localisation of $k[T]$ by the powers of f. Then H induces a morphism of rings $\phi : k\left[T, \frac{1}{f}\right] \longrightarrow k[T]$.
We will show that the image of ϕ is in k.
We must have $\phi(f)\phi(1/f) = 1$, so f is mapped on a unit of $k[T]$, i.e a nonzero element of $c \in k$. Let us write $f = \sum_{j=0}^{d} a_j T^j$, with $a_j \in k$ and $a_d \neq 0$. Then $c = \phi(f) = \sum_{j=0}^{d} \phi(a_j)\phi(T)^j$. If $\phi(T) = 0$, then the image of ϕ is in k. Otherwise, the degree of $\phi(f)$ is 0 because it is in k, and it is also equal to $d \times \deg(\phi(T))$. As $d \neq 0$, we must have $\deg(\phi(T)) = 0$ which means that $\phi(T) \in k$. Then the image of ϕ is in k.
Therefore for all prime ideal $\mathfrak{p}$ in $\mathrm{Spec}(k[T])$, we have $H(\mathfrak{p}) = \phi^{-1}(\mathfrak{p}) = \mathrm{Ker}(\phi)$. Then H is constant. But H is supposed to make a link between s_1 and s_2, so $s_1 = s_2$.
Finally all the sections of U are in two by two distinct classes of naive $\mathbb{A}^1$-homotopy classes.

Definition 4.4 Naive $\mathbb{A}^1$-contractibility
We say that a scheme X over k is naively $\mathbb{A}^1$-contractible when it is naively $\mathbb{A}^1$-homotopy equivalent to $\mathrm{Spec}(k)$.

Proposition 4.2 When k is infinite, the only open subscheme of $\mathbb{A}^1(k)$ that is naively $\mathbb{A}^1(k)$-contractible is $\mathbb{A}^1(k)$.

Proof
Indeed, let $U = \mathbb{A}^1(k) \setminus Z$ be an open subscheme of $\mathbb{A}^1(k)$. If U is naively $\mathbb{A}^1$-contractible and is not $\mathbb{A}^1(k)$, then Z is a proper closed subset of $\mathbb{A}^1(k)$, so by the previous example (4.2), we have $[\mathrm{Spec}(k), U]_{\mathcal{N}(k)} = U(k)$. But as U is naively $\mathbb{A}^1$-contractible, we have $[\mathrm{Spec}(k), U]_{\mathcal{N}(k)} = \{*\}$ a point. Therefore there is only one rational point in $U(k)$.
Yet, as k is infinite, there is an infinity of ideals in $k[T]$ of the form $(T - a)$ with $a \in k$, and all these ideals are rational points of $\mathbb{A}^1(k)$ (see section 1.5). As Z is only a finite set of closed points, U must have an infinity of rational points. Then $Z = \emptyset$ and $U = \mathbb{A}^1(k)$. ■

Example 4.3 Special linear group

Let n be a positive integer. We define the ring

$$R := k[t_{ij}, 1 \leq i, j \leq n]/(\det -1)$$

where det is the determinant polynomial for the variables t_{ij}. Then $\mathrm{SL}_n(k) := \mathrm{Spec}(R)$ is an affine scheme endowed with a group structure. It is a subscheme of the scheme $\mathbb{A}^{n^2}(k)$ of $n \times n$-matrices over k. We will compute here the set (it is actually a group) $[\mathrm{Spec}(k), \mathrm{SL}_n(k)]_{\mathcal{N}(k)}$.

First, a morphism of schemes from $\mathrm{Spec}(k)$ to $\mathrm{SL}_n(k)$ is just the choice of a rational point of $\mathrm{SL}_n(k)$. As the rational points of $\mathrm{SL}_n(k)$ are the usual matrices of the special linear group over k, we will work in k^{n^2} instead of $\mathbb{A}^{n^2}(k)$.
Furthermore, every matrix in k^{n^2} can be written as a product of elementary matrices (see appendices), i.e. matrices like $E_{ij}(\alpha) := I_n + \alpha e_{ij}$, with $i \neq j$, where e_{ij} is the matrix with zeros except that the coefficient i, j is a 1, and $\alpha \in k$. Let us show that these elementary matrices are $\mathbb{A}^1$-homotopy equivalent to the identity matrix I_n. For $i, j \in [\![1, n]\!]$, $i \neq j$, let us consider the map

$$H : \begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \mathrm{SL}_n(k) \\ t & \longmapsto & I_n + t\alpha e_{ij} \end{array}$$

Then H is a homotopy between $E_{ij}(\alpha)$ and I_n. Hence every matrix in $\mathrm{SL}_n(k)$ is $\mathbb{A}^1$-homotopy equivalent to I_n, so $[\mathrm{Spec}(k), \mathrm{SL}_n(k)]_{\mathcal{N}(k)} = \{I_n\}$.

4.2 $\mathbb{A}^1$-invariance

Before building the "good" homotopy category for schemes, we present some examples of objects that are invariant by the naive equivalence relation. As always, what is interesting with these invariant objects is that they separate schemes that are not naively $\mathbb{A}^1$-homotopy equivalent.

Definition 4.5 A contravariant functor $\mathcal{F} : \mathrm{Sm}_k \longrightarrow \mathrm{Set}$ is said $\mathbb{A}^1$-invariant when for all $X \in \mathrm{Sm}_k$, the map $\mathcal{F}(X) \longrightarrow \mathcal{F}(X \times_k \mathbb{A}^1)$, given by the projection $X \times_k \mathbb{A}^1 \longrightarrow X$, is a bijection.

Proposition 4.3 Let $\mathcal{F}$ be a contravariant functor $\mathbb{A}^1$-invariant. Let X, Y be two schemes over k, and let us suppose that they are naively $\mathbb{A}^1$-homotopy equivalent. Then $\mathcal{F}(X) \cong \mathcal{F}(Y)$.

Proof
Since X and Y are naively $\mathbb{A}^1$-homotopy equivalent, it exists two morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g$ is naively homotopy equivalent to id_Y and $g \circ f$ to id_X. Then it exists a homotopy $H_X : X \times_k \mathbb{A}^1 \longrightarrow X$ such that $H_X \circ \Phi_0 = g \circ f$ and $H_X \circ \Phi_1 = \mathrm{id}_X$. Therefore we have a morphism $\mathcal{F}(g \circ f) : \mathcal{F}(X) \longrightarrow \mathcal{F}(X)$ that is equal to $\mathcal{F}(\Phi_0) \circ \mathcal{F}(H_X)$.
Now, by hypothesis, $\mathcal{F}(\pi_X) : \mathcal{F}(X \times_k \mathbb{A}^1) \longrightarrow \mathcal{F}(X)$ is a bijection. Moreover, we know that $\pi_X \circ \Phi_0 = \pi_X \circ \Phi_1 = \mathrm{id}_X$. Hence $\mathcal{F}(\Phi_0)$ and $\mathcal{F}(\Phi_1)$ both are left inverse of $\mathcal{F}(\pi_X)$. By unicity of the inverse function, we have $\mathcal{F}(\Phi_0) = \mathcal{F}(\Phi_1)$.
Then we can write $\mathcal{F}(g \circ f) = \mathcal{F}(\Phi_1) \circ \mathcal{F}(H_X) = \mathrm{id}_{\mathcal{F}(X)}$. Since f and g have symmetric roles, we also have $\mathcal{F}(f \circ g) = \mathrm{id}_{\mathcal{F}(Y)}$, so $\mathcal{F}(f)$ defines a bijection between $\mathcal{F}(X)$ and $\mathcal{F}(Y)$. ■

Example 4.4 The class group is $\mathbb{A}^1$-invariant. Hence the affine space $\mathbb{A}^n(k)$ and the projective space $\mathbb{P}^n(k)$ are not naively $\mathbb{A}^1$-equivalent.

Example 4.5 We can associate to a scheme X the group of units of its global sections $O_X(X)^*$. This defines a contravariant functor $\mathrm{Sm}_k \longrightarrow \mathrm{Grp}$. Let us show that it is $\mathbb{A}^1$-invariant on affine schemes.

Let $X = \mathrm{Spec}(R)$ an affine scheme over k. We have $X \times_k \mathbb{A}^1 = \mathrm{Spec}(R \otimes_k k[T]) = \mathrm{Spec}(R[T])$, so $O_{X \times_k \mathbb{A}^1}(X \times_k \mathbb{A}^1) = R[T]$. But the group of units of $R[T]$ is R^*, so

$$O_{X \times_k \mathbb{A}^1}(X \times_k \mathbb{A}^1)^* = O_X(X)^*$$

Theorem 4.1 Representable functors in the category $\mathcal{N}(k)$ are $\mathbb{A}^1$-invariant on affine schemes.

Proof Let Y be an affine scheme over k, and $\mathcal{F}$ be the functor represented by Y on the affine schemes, i.e. for all affine scheme $X = \mathrm{Spec}(R)$ over k, $\mathcal{F}(X) = [X, Y]_{\mathcal{N}(k)}$.
Let us denote by π the projection $X \times_k \mathbb{A}^1 \longrightarrow X$. As the morphism of rings $\phi : R \otimes_k k[T] \longrightarrow R$ that sends $r \otimes P$ on r is surjective, the associated morphism of schemes $i : X \longrightarrow X \times_k \mathbb{A}^1$ is injective, and we have $\pi \circ i = \mathrm{id}_X$. Then, if $[f], [g] \in [X, Y]_{\mathcal{N}(k)}$ are such that $[f \circ \pi] = [g \circ \pi]$ in $[X \times_k \mathbb{A}^1, Y]_{\mathcal{N}(k)}$, we can compose by i to get that $[f] = [g]$. Therefore $\mathcal{F}(X) \longrightarrow \mathcal{F}(X \times_k \mathbb{A}^1)$ is injective.

Now, let $[\tilde{f}] \in [X \times_k \mathbb{A}^1, Y]_{\mathcal{N}(k)}$, and let us consider $f : X \longrightarrow Y$ such that $f(x) = \tilde{f}(x, 0)$. We want to show that $f \circ \pi$ is naively $\mathbb{A}^1$-homotopy equivalent to $\tilde{f}$. We can take

$$\begin{array}{rccc} H: & X \times_k \mathbb{A}^1 \times_k \mathbb{A}^1 & \longrightarrow & Y \\ & (x, u, t) & \longmapsto & \tilde{f}(x, ut) \end{array}$$

H is a morphism of schemes, and $H(x, u, 0) = \tilde{f}(x, 0) = f(\pi(x, u))$, $H(x, u, 1) = \tilde{f}(x, u)$, so $f \circ \pi$ and $\tilde{f}$ are naively $\mathbb{A}^1$-homotopy equivalent, and $\mathcal{F}(X) \longrightarrow \mathcal{F}(X \times_k \mathbb{A}^1)$ is surjective. ■

Proposition 4.4 The multiplicative group defines a functor $\mathrm{Aff}_k \longrightarrow \mathrm{Grp}$ (from affine schemes over k to groups), by setting $\mathbb{G}_m(X) = \mathrm{Hom}(X, \mathbb{G}_m)$. This functor is the one of example 4.5 above.

Remark 4.2 Therefore this functor of example 4.5 is representable, and by the previous theorem, we thus have immediately that it is $\mathbb{A}^1$-invariant on affine schemes.

Proof
Let $X = \mathrm{Spec}(R)$ be an affine scheme. Let us show that $O_X(X)^* \cong \mathrm{Hom}(X, \mathbb{G}_m)$.
Let $f : X \longrightarrow \mathbb{G}_m$ be a morphism of schemes. Since X is affine, the morphism $\hat{f} : O_{\mathbb{G}_m}(\mathbb{G}_m) \longrightarrow O_X(X)$ is actually a morphism $\hat{f} : k[T, T^{-1}] \longrightarrow R$. But then $\hat{f}(T)$ is invertible in R, so the following map is well defined :

$$\begin{array}{rccc} \Psi: & \mathrm{Hom}(X, \mathbb{G}_m) & \longrightarrow & R^* \\ & f & \longrightarrow & \hat{f}(T) \end{array}$$

Furthermore, we have $\hat{fg} = \hat{f}\hat{g}$, so Ψ is a morphism of groups.
Finally, if we have $u \in R^*$, we can associate to u the morphism of rings $\phi_u : k[T, T^{-1}] \longrightarrow R$ such that $\phi_u(T) = u$, and this induces a morphism of schemes $f_u : X \longrightarrow \mathbb{G}_m$, satisfying $\Psi(f_u) = u$. Hence Ψ is an isomorphism of groups. ■

4.3 $\mathbb{A}^1$-weak equivalence

Now, we will refine the notion of $\mathbb{A}^1$-equivalence, by building a homotopy category that has "best properties" than the naive one. To have more details, see [Mor02], [Dug98], [Dug00], [AE16] or [Sev06].

Definition 4.6 Simplicial sets
The category of simplical sets, denoted by $sSET$, is the category where objects are contravariant functors $\Delta^{op} \longrightarrow SET$ and maps are natural transformations between these functors.

Definition 4.7 Simplicial presheaves over a category.
Let C be a small category. A simplicial presheaf over C is a contravariant functor $C^{op} \longrightarrow sSET$. The category of simplicial presheaves over C is then the category where objects are simplicial presheaves over C and maps are natural transformations.

We will denote by Spc_k (resp. Spc'_k) the category of simplicial presheaves over Sm_k (resp. Sch_k).

Example 4.6 Let X be a scheme over k. Then we can build a simplicial presheaf associated to X :

$$\begin{array}{rccc} r(X): & \mathrm{Sm}_k & \longrightarrow & sSET \\ & Z & \longmapsto & \left(\begin{array}{ccc} \Delta & \longrightarrow & SET \\ [n] & \longmapsto & [Z, X] \end{array} \right) \end{array}$$

For all morphism of scheme $f : X \longrightarrow Y$, there is a natural transformation between the simplicial presheaves associated $r(f) : r(X) \longrightarrow r(Y)$ defined by : for all scheme Z, for all $n \in \mathbb{N}$ and for all $\phi : Z \longrightarrow X$,

$$r(f)_{Z,[n]}(\phi) = f \circ \phi$$

Definition 4.8 Nisnevich distinguished squares
Let X be a scheme over k and U, V two open subsets of X. Let $j : U \longrightarrow X$ an open immersion of schemes, and $\phi : V \longrightarrow X$ an étale morphism, such that $\phi^{-1}(X \setminus U) \longrightarrow X \setminus U$ is an isomorphism. Then a Nisnevich diagram is the following pullback diagram :

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \phi \\ U & \xrightarrow{j} & X \end{array}$$

We will denote by $U \xrightarrow{j} X \xleftarrow{\phi} V$ such a diagram.

Definition 4.9 Nisnevich topology
The Nisnevich topology is the Grothendieck topology (see appendices) on Sm_k such that a family of morphisms $\{p_i : U_i \longrightarrow X\}_{i \in I}$ is a Nisnevich cover when for all $i \in I$, p_i is étale, and for all $x \in X$, it exists $i \in I$ and $y \in U_i$ such that $f(y) = x$ and the morphism $k(x) \longrightarrow k(y)$ is an isomorphism.

Remark 4.3 We can check that the three axioms of Grothendieck topology are satisfied with this definition.

Proposition 4.5 A Nisnevich distinguished square $U \xrightarrow{j} X \xleftarrow{\phi} V$ defines a Nisnevich covering of X.

Proof
Let $U \xrightarrow{j} X \xleftarrow{\phi} V$ be a Nisnevich square. By definition, ϕ is étale and j is an open immersion, so j is étale too (see proposition 1.11).
Moreover, let $x \in X$. If x is in $j(U)$, then it exists $y \in U$ such that $j(y) = x$, and the induced morphism $k(x) \longrightarrow k(y)$ is an isomorphism, because $\hat{j} : O_X(x) \longrightarrow O_U(y)$ is an isomorphism. If x is not in U, then $x \in X \setminus U$. Since ϕ is an isomorphism between $\phi^{-1}(X \setminus U)$ and $X \setminus U$, it exists (a unique) $y \in \phi^{-1}(X \setminus U)$ such that $\phi(y) = x$, and $k(x) \cong k(y)$ via ϕ.
Therefore the family $\{\phi, j\}$ is a Nisnevich cover. ■

Let X be a scheme over k. Let $\{U_i \longrightarrow X\}_{i \in I}$ be a Nisnevich covering of X. Then the family $\{U_i \times_X U_j \longrightarrow X\}_{i \in I}$ is a Nisnevich covering of X for all $j \in I$. This gives a family $\{U_i \times_X U_j \longrightarrow X\}_{i,j \in I}$ that is a nisnevich covering, and we can keep on going this, to have coverings like $\{U_{i_1} \times_X \cdots \times_X U_{i_n} \longrightarrow X\}$, etc... Finally, we have a simplicial presheaf on Sm_k, denoted by $\check{C}(\{U_i\})$:

$$\begin{array}{rccc} \check{C}(\{U_i\}) : & \Delta & \longrightarrow & \mathrm{Spc}_k \\ & [n] & \longmapsto & \coprod_{i_0, \dots, i_n} r\left(U_{i_0} \times_X \dots \times_X U_{i_n}\right) \end{array}$$

where $r(U_{i_0} \times_X \cdots \times_X U_{i_n})$ is the simplicial presheaf associated to $U_{i_0} \times_X \cdots \times_X U_{i_n}$ as in example 4.6. Hence we get a natural transformation $\check{C}(\{U_i\}) \longrightarrow r(X)$, given by the maps $U_{i_1} \times_X \cdots \times_X U_{i_n} \longrightarrow X$.

Definition 4.10 Homotopy category
The homotopy category, denoted by $H_{\mathbb{A}^1}(k)$, will be the category where objects are simplicial presheaves and maps are morphisms of simplicial presheaves, but where we impose that the maps $\check{C}(U) \longrightarrow r(X)$ are isomorphisms, and projections $r(X \times_k \mathbb{A}^1) \longrightarrow r(X)$ are isomorphisms too. This process is called localization, in analogy with the ring localization where we arbitrarily invert some elements. To have more details, see the appendix A of [DDC03].

Remark 4.4 As we impose some morphisms to be isomorphisms, other morphisms has to match in the homotopy category. For example, in the homotopy category, the projection $\pi_X : X \times_k \mathbb{A}^1(k) \longrightarrow X$ is an isomorphism. Yet, we can build several converse morphisms $\Phi_t : X \longrightarrow X \times_k \mathbb{A}^1(k)$, from the sections over the rational point $t \in \mathbb{A}^1(k)$. These morphisms satisfy $\pi_X \circ \Phi_t = \mathrm{id}_X$ (see definition 4.1). By unicity of the inverse function of a bijection, all the Φ_t must be the same in homotopy category.

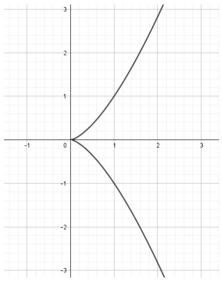
Therefore there is a surjection from the set of morphisms between two schemes X and Y in the set $[X, Y]_{H_{\mathbb{A}^1}(k)}$ of morphisms between $r(X)$ and $r(Y)$ in the homotopy category.

Definition 4.11 A $\mathbb{A}^1$-weak equivalence between two schemes will be an isomorphism in $H_{\mathbb{A}^1}(k)$.
We will say that a scheme over k is $\mathbb{A}^1$-contractible if the morphism $X \longrightarrow \mathrm{Spec}(k)$ is a $\mathbb{A}^1$-weak equivalence.

Remark 4.5 When two schemes X, Y are naively $\mathbb{A}^1$-homotopy equivalent, it means that there is an isomorphism of simplicial presheaves between the two simplicial presheaves associated to X and Y. Therefore, if two schemes are naively $\mathbb{A}^1$-homotopy equivalent, then they are $\mathbb{A}^1$-weakly equivalent.

Remark 4.6 By construction of $H_{\mathbb{A}^1}(k)$, for all scheme X, the projection $X \times_k \mathbb{A}^1 \longrightarrow X$ is a $\mathbb{A}^1$-weak equivalence. In particular, $\mathbb{A}^1 \longrightarrow \mathrm{Spec}(k)$ is a $\mathbb{A}^1$-weak equivalence, so $\mathbb{A}^1$ is $\mathbb{A}^1$-contractible. Likewise, the affine space $\mathbb{A}^n = \mathbb{A}^1 \times_k ...\mathbb{A}^1 \times_k \mathbb{A}^1$ is $\mathbb{A}^1$-contractible, by induction.

Example 4.7 The cupsidal curve
Let X be the curve $\mathrm{Spec}(k[T_1, T_2]/(T_1^3 - T_2^2))$.



Then X is $\mathbb{A}^1$-contractible. Indeed, we have an $\mathbb{A}^1$-homotopy equivalence between $\mathbb{A}^1$ and X, given by $x \mapsto (x^2, x^3)$. As $\mathbb{A}^1$ is $\mathbb{A}^1$-contractible, the curve X is also.
Here, we found a $\mathbb{A}^1$-contractible curve that is not $\mathbb{A}^1$. The hypothesis of smoothness is then necessary to caracterize $\mathbb{A}^1$ as the only smooth curve $\mathbb{A}^1$-contractible.

Example 4.8 The motivic sphere (in dimension 1)
Let us suppose that k is algebraically closed.

We will show that $\Sigma\mathbb{G}_m$ is A^1-weakly equivalent to $\mathbb{P}^1$. (see appendices for the definition of the suspension $\Sigma\mathbb{G}_m$).

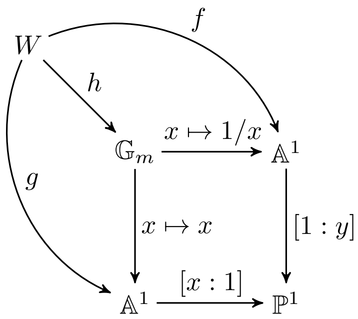
We can cover $\mathbb{P}^1$ with the two charts $U_1 := \{[1 : y], y \in k\}$ and $U_2 := \{[x : 1], x \in k\}$, and these charts are isomorphic to $\mathbb{A}^1$. Furthermore, we saw that $\mathbb{G}_m$ is isomorphic to $\mathbb{A}^1 \setminus \{0\}$, so we can consider the following commutative diagram :

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{x \mapsto 1/x} & \mathbb{A}^1 \\ {\scriptstyle x \mapsto x}\downarrow & & \downarrow{\scriptstyle y \mapsto [1:y]} \\ \mathbb{A}^1 & \xrightarrow[x \mapsto [x:1]]{} & \mathbb{P}^1 \end{array}$$

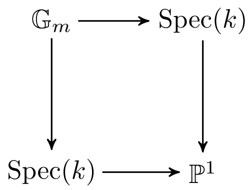
Let us show that this is a Nisnevich diagram. First, it is a pullback diagram. Indeed, if we take a scheme W together with morphisms $f, g : W \longrightarrow \mathbb{A}^1$ such that

$$\begin{array}{ccc} W & \xrightarrow{f} & \mathbb{A}^1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle y \mapsto [1:y]} \\ \mathbb{A}^1 & \xrightarrow[x \mapsto [x:1]]{} & \mathbb{P}^1 \end{array}$$

commutes, then we have, for all $w \in W$, $[g(w) : 1] = [1 : f(w)]$, so $f(w) \neq 0$, $g(w) \neq 0$ and $g(w) = 1/f(w)$. Then the map g arrive actually in $\mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$, and the following diagram is commutative :



Furthermore, the morphism $x \longmapsto [x:1]$ is an open immersion as $\mathbb{A}^1 \simeq U_2 \subset \mathbb{P}^1$, and the morphism $y \longmapsto [1:y]$ is also an open immersion, so it is an étale morphism. Finally, $\mathbb{P}^1 \setminus U_2 = \{[1:0]\}$ and if we denote by i the map $y \mapsto [1:y]$, we have $i^{-1}(\mathbb{P}^1 \setminus U_2) = \{0\}$, so it induces an isomorphism between $\{0\}$ and $\{[1:0]\}$.
As $\mathbb{A}^1$ is $\mathbb{A}^1$-contractible, this gives rise to a commutative diagram :



Then $\mathbb{P}^1$ is $\mathbb{A}^1$-homotopy equivalent to $\Sigma\mathbb{G}_m$. Actually, the space $\Sigma\mathbb{G}_m$ is the fisrt bigraded motivic sphere, $S^{1,1}$. See [Aso16], page 13 for more details.

4.4 $\mathbb{A}^1$-rigidity

Definition 4.12 Let X be a scheme over k. We say that X is $\mathbb{A}^1$-rigid when for all scheme U over k, the following map, induced vu the projection $U \times_k \mathbb{A}^1 \longrightarrow U$,

$$\mathrm{Hom}_{\mathrm{Spec}(k)}(U, X) \longrightarrow \mathrm{Hom}_{\mathrm{Spec}(k)}(U \times_k \mathbb{A}^1, X)$$

is a bijection.
The category of smooth, separated, $\mathbb{A}^1$-rigid schemes over k of finite type will be denoted by Sm_k^{rig}.

Proposition 4.6 The functor $\mathrm{Sm}_k^{rig} \longrightarrow H_{\mathbb{A}^1}(k)$ is fully faithful.

Proof see [MV99], example 2.4 page 106. ■

Proposition 4.7 If X is a smooth projective curve over k, of positive genus, then X is $\mathbb{A}^1$-rigid.

Proof
Let U be a scheme over k. Let $\psi \in \mathrm{Hom}_{\mathrm{Spec}(k)}(U \times_k \mathbb{A}^1, X)$. Let $\sigma_0 : \mathrm{Spec}(k) \longrightarrow \mathbb{A}^1(k)$ be the section associated to the rational point 0 in $\mathbb{A}^1(k)$. We set $\phi : U \longrightarrow X$ such that $\phi(u) = \psi(u, 0)$ for all $u \in U$.
Let us show that for all $u \in U$, for all $t \in \mathbb{A}^1(k)$, we have $\psi(u,t) = \psi(u,0)$.
Let $u \in U$. The map $f : \mathbb{A}^1(k) \longrightarrow X$ defined by $f(t) = \psi(u,t)$ for all $t \in \mathbb{A}^1(k)$ is a morphism of curves. Then by proposition II.6.8 of [Har77] (p.137), f is constant or surjective. But if f is surjective, then we can find a surjective morphism of curves $\mathbb{P}^1(k) \longrightarrow X$, that is finite by the lemma 7.3.10 of [Liu02]. Hence by theorem 7.4.16 of [Liu02], we must have $0 \geq g$, which is wrong. Thus f is constant, and we have what we wanted.
Hence $\phi \circ \pi_U = \psi$, so the map $\mathrm{Hom}_{\mathrm{Spec}(k)}(U, X) \longrightarrow \mathrm{Hom}_{\mathrm{Spec}(k)}(U \times_k \mathbb{A}^1, X)$ is surjective. It is also always injective, as π_U is surjective, so X is $\mathbb{A}^1$-rigid. ■

5 $\mathbb{A}^1$-contractible smooth curves

Finally, we are able to prove that the only $\mathbb{A}^1$-contractible smooth curve is the affine line.

5.1 When k is algebraically closed

For the moment, we assume that k is algebraically closed.

Theorem 5.1 Let X be a smooth curve over k. If X is $\mathbb{A}^1$-contractible, then X is isomorphic to $\mathbb{A}^1(k)$.

Proof
First, we can suppose that X has k-rational points. Indeed, the map $X(k) \longrightarrow [\mathrm{Spec}(k), X]_{H_{\mathbb{A}^1}(k)}$ is surjective (see remark 4.2), so if $X(k)$ was empty we would have $[\mathrm{Spec}(k), X]_{H_{\mathbb{A}^1}(k)} = \emptyset$. But X is weakly equivalent to $\mathrm{Spec}(k)$, so $[\mathrm{Spec}(k), X]_{H_{\mathbb{A}^1}} = [\mathrm{Spec}(k), \mathrm{Spec}(k)]_{H_{\mathbb{A}^1}} \neq \emptyset$. Therefore we will assume that $X(k)$ is not empty.

By proposition 2.1, X is either affine or projective.

Case 1 : X is projective.
As $X(k) \neq \emptyset$, we can apply proposition 3.3. For some rational point $z \in X$, there is an exact sequence

$$\mathbb{Z} \xrightarrow{\phi} Cl(X) \longrightarrow Cl(X \setminus \{z\}) \longrightarrow \{0\}$$

Let us show that the map ϕ is injective. Let $m \in \mathbb{Z}$ such that $\phi(m) = 0$. Then $m = 0$ or $\{z\} = \mathrm{Div}(f)$ for some $f \in \kappa(X)$. So f has only one zero and no pole. It is impossible in a projective space, so m must be zero, and ϕ is injective.
Therefore $Cl(X) \neq \{0\}$. But X is $\mathbb{A}^1$-contractible so its class group must be zero : this gives an absurdity. Finally, X can't be projective.

Case 2 : X is affine. Let us write $X := \mathrm{Spec}(k[T_1, ..., T_n]/I)$, with I an ideal of $k[T_1, ..., T_n]$. By proposition 1.20, we can consider its projectivization $\overline{X}$, which is a smooth projective curve over k such that $\overline{X} = X \sqcup \{x_1, ..., x_m\}$, with x_i some closed points, the "points at infinity". Let us show that $\overline{X}$ has genus 0.
As X is $\mathbb{A}^1$-contractible, in $H_{\mathbb{A}^1}(k)$, we have $\overline{X} \cong \{x_1, ..., x_m\} \cup \{x_{m+1}\}$. Yet, the discrete scheme $\{x_1, ..., x_{m+1}\}$ is $\mathbb{A}^1$-rigid, so by proposition 4.6, it is the only $\mathbb{A}^1$-rigid scheme that is sent on $\overline{X}$ in the homotopy category $H_{\mathbb{A}^1}(k)$. Now, if the genus of $\overline{X}$ is positive, then $\overline{X}$ is $\mathbb{A}^1$-rigid by proposition 4.7, so by faithfulness, we must have $\overline{X} = \{x_1, ..., x_{m+1}\}$, which means that X is discrete. This is absurd, and thus $\overline{X}$ has a zero genus.
Therefore $\overline{X} \cong \mathbb{P}^1(k)$, because k is algebraically closed (proposition 2.4). We have then $X \cong \mathbb{P}^1(k) \setminus \{y_1, ..., y_m\}$ with y_i some closed points of $\mathbb{P}^1(k)$, and $m \geq 1$. But we saw that $\mathbb{A}^1(k)$ is isomorphic to the complement of one point in $\mathbb{P}^1(k)$, so $X \cong \mathbb{A}^1(k) \setminus \{y_2, ..., y_m\}$. Yet if $m \geq 2$, then $\mathbb{A}^1(k) \setminus \{y_2, ..., y_m\}$ is not $\mathbb{A}^1$-contractible, by proposition 4.2. Therefore $m = 1$ and $X \cong \mathbb{A}^1(k)$. ■

Remark 5.1 Here, if X was not a curve but just a smooth scheme of finite type over k of dimension 1, the proof would have been the same.

5.2 When k is any field

Even when we don't suppose that k is algebraically closed, the theorem still holds : the only $\mathbb{A}^1$-contractible smooth curve over k is $\mathbb{A}^1(k)$. To prove this, we will use base change.
In all what follows, we take a field k and we denote by $\bar{k}$ an algebraic closure of k. For any curve X over k, we will denote by $X_{\bar{k}}$ the base change of X over $\bar{k}$.

Proposition 5.1 Let X be an $\mathbb{A}^1$-contractible, smooth curve over k. Then $X \cong \mathbb{A}^1(k)$.

Proof
We would like to apply the previous case to the scheme $X_{\bar{k}}$. But we saw that the base change of a variety is not always a variety, so $X_{\bar{k}}$ is not necessarily a curve. However, by proposition 1.15, $X_{\bar{k}}$ is a smooth finite type scheme over $\bar{k}$, of dimension 1. Moreover, by the remark before proposition 2.8, page 108 of [MV99], the morphism $\mathrm{Spec}(\bar{k}) \longrightarrow \mathrm{Spec}(k)$ induces a functor $H_{\mathbb{A}^1(k)} \longrightarrow H_{\mathbb{A}^1(\bar{k})}$. Hence if X is $\mathbb{A}^1(k)$-contractible, then the base change $X_{\bar{k}}$ is $\mathbb{A}^1(\bar{k})$-contractible. We can thus apply the remark that follows the previous theorem to get $X_{\bar{k}} \cong \mathbb{A}^1(\bar{k})$.

Hence $\overline{X}_{\bar{k}} \cong \mathbb{P}^1(\bar{k})$. This implies that $\overline{X}$ is geometrically integral (the base change $\overline{X}_{\bar{k}}$ is integral), and that $\overline{X}$ has genus 0 (by lemma 53.8.2 of [SP]). Moreover, as in the previous proof, we can assume that X has rational points (indeed, we didn't use the fact that k was algebraically closed in this part of the proof). Therefore $\overline{X}$ is a geometrically integral smooth projective curve of genus 0 that has rational points. By proposition 7.4.1 of [Liu02], we get $\overline{X} \cong \mathbb{P}^1(k)$.

Then X is an open subscheme of $\mathbb{P}^1(k)$, so $X = \mathbb{P}^1(k) \setminus Z$ with a closed subset Z in $\mathbb{P}^1(k)$. Furthermore, the base change over $\bar{k}$ of $\mathbb{P}^1(k) \setminus Z$ is $\mathbb{P}^1(\bar{k}) \setminus Z$, so $\mathbb{P}^1(\bar{k}) \setminus Z \cong \mathbb{A}^1(\bar{k})$. Yet $\mathbb{A}^1(\bar{k})$ is isomorphic to the complement of a closed point in $\mathbb{P}^1(\bar{k})$, so Z is actually a closed point in $\mathbb{P}^1(k)$. Finally $X \cong \mathbb{A}^1(k)$. ■

6 Appendices

6.1 Presheaves and sheaves

Definition 6.1 Presheaf of rings
Let X be a topological space. To all open set U of X, we associate a ring $\mathcal{F}(U)$, such that for every inclusion of open sets $V \subset U$, there is a morphism of rings

$$\mathrm{Res}^{\mathcal{F}}_{U,V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

Moreover, let us suppose that, for all open sets U, V, W,

(i) $\mathrm{Res}^{\mathcal{F}}_{U,U} = \mathrm{id}_U$;

(ii) If $W \subset V \subset U$ then $\mathrm{Res}^{\mathcal{F}}_{U,W} = \mathrm{Res}^{\mathcal{F}}_{V,W} \circ \mathrm{Res}^{\mathcal{F}}_{U,V}$.

Then $\mathcal{F}$ is a presheaf of rings over X, and the rings $\mathcal{F}(U)$ are called the rings of sections over U.

Remark 6.1 Let U, V be two open sets of X, such that $V \subset U$, and let $s \in \mathcal{F}(U)$. Sometimes, we will write $s_{|V}$ instead of $\mathrm{Res}^{\mathcal{F}}_{U,V}(s)$.

Definition 6.2 Stalks
Let X be a topological space, and $\mathcal{F}$ be a presheaf over X. Let $x \in X$. The stalk on x of the presheaf $\mathcal{F}$ is the inductive limit

$$\mathcal{F}_x := \lim_{\rightarrow} \mathcal{F}(U)$$

the limit being taken over the open sets U that contains x, with the relation of inclusion $V \subset U$ and the restriction morphisms $\mathrm{Res}^{\mathcal{F}}_{U,V}$.

Definition 6.3 Sheaf of rings
Let X be a topological space, and $\mathcal{F}$ be a presheaf of rings over X.
We say that $\mathcal{F}$ is a sheaf of rings when for all open set U, for all open covering $U = \bigcup_{i \in I} U_i$, for all family of sections $\{s_i\}_{i \in I}$, such that $s_i \in \mathcal{F}(U_i)$, if we have

$$\mathrm{Res}^{\mathcal{F}}_{U_i, U_i \cap U_j}(s_i) = \mathrm{Res}^{\mathcal{F}}_{U_j, U_i \cap U_j}(s_j) \quad \text{for all } i, j \in I$$

then it exists a unique section $s \in \mathcal{F}(U)$ such that for all $i \in I$, $\mathrm{Res}^{\mathcal{F}}_{U,U_i}(s) = s_i$.

Remark 6.2 This additional condition means that it suffices to know the sections locally to know them globally.

Definition 6.4 Morphism of sheaves
Let X be a topological space, and $\mathcal{F}$, $\mathcal{G}$ two presheaves of rings over X. A morphism between $\mathcal{F}$ and $\mathcal{G}$ is a family $\{\Phi(U)\}_{U \subset X, \text{ open set}}$ of maps such that for all open set $U \subset X$, $\Phi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is a morphism of rings, and for all inclusion $V \subset U$, the following diagram commutes :

$$\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\Phi(U)} & \mathcal{G}(U) \\
{\scriptstyle \mathrm{Res}^{\mathcal{F}}_{U,V}} \downarrow & & \downarrow {\scriptstyle \mathrm{Res}^{\mathcal{G}}_{U,V}} \\
\mathcal{F}(V) & \xrightarrow{\Phi(V)} & \mathcal{G}(V)
\end{array}$$

Furthermore, we say that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic when for all open set U, the map $\Phi(U)$ is an isomorphism of rings.

Definition 6.5 Locally ringed space
Let X be a topological space, and $\mathcal{F}$ be a sheaf of rings over X.
We say that $(X, \mathcal{F})$ is a locally ringed space when for every point $x \in X$, the stalk $\mathcal{F}_x$ is a local ring. We will denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{F}_x$.

Definition 6.6 Subspaces
Let $(X, \mathcal{F})$ be a locally ringed space.
• Let U be an open subset of X. Then U can be endowed with a locally ringed space structure : for any open set V of X, we put

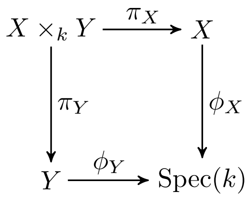
$$\mathcal{F}_U(U \cap V) = \mathcal{F}(U \cap V)$$

• Likewise, let Z be a closed subset of X. Then Z can also be endowed with a structure of locally ringed space : for any open set V of X, we put

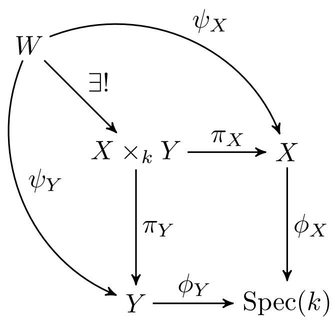
$$\mathcal{F}_Z(Z \cap V) = \{s \in \mathcal{F}(V) \mid \forall x \in Z^c, s_{|x} = 0\}$$

6.2 Fibred product

Definition 6.7 Let X and Y be two schemes over k, with the maps $X \xrightarrow{\phi_X} \text{Spec}(k)$ and $Y \xrightarrow{\phi_Y} \text{Spec}(k)$. The fibred product of X and Y over k is a scheme over k, denoted by $X \times_k Y$ and two maps called projections π_X, π_Y such that the following diagram is commutative, and which satisfy the universal property below :



Proposition 6.1 Universal property of fibred product (for schemes)
For all scheme W with morphisms $\psi_X : W \longrightarrow X$ and $\psi_Y : W \longrightarrow Y$ such that $\phi_X \circ \psi_X = \phi_Y \circ \psi_Y$, there exists a unique morphism $W \longrightarrow X \times_k Y$, such that the diagram commutes :



In the category of schemes, the fibred product always exists :

Proposition 6.2 Let X and Y be schemes over k. Then the fibred product $X \times_k Y$ exists and is unique up to isomorphism.
Moreover, if X and Y are affine, say $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$, then

$$X \times_k Y = \text{Spec}(A \otimes_k B)$$

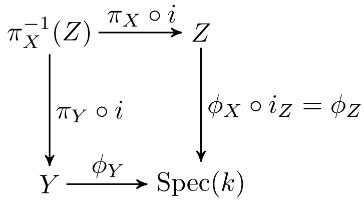
Finally, if X and Y are projective, say $X = \text{Proj}(R)$ and $Y = \text{Proj}(S)$, then

$$X \times_k Y = \text{Proj}(R \otimes S)$$

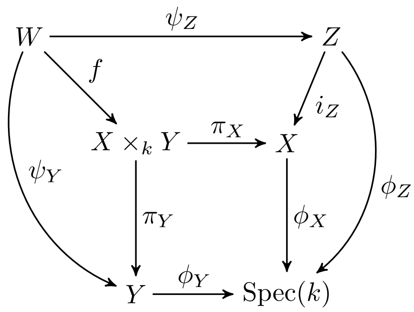
Proof See [Liu02], page 79. ■

Lemma 6.1
Let X and Y be schemes over k. Let Z be a closed subset of X. Then $\pi_X^{-1}(Z) = Z \times_k Y$.

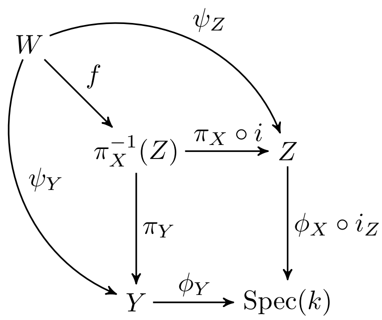
Proof Let us denote by i the inclusion $\pi_X^{-1}(Z) \longrightarrow X \times_k Y$ and by i_Z the inclusion $Z \longrightarrow X$.
First, we have maps $\pi_X \circ i : \pi_X^{-1}(Z) \longrightarrow Z$ and $\pi_Y \circ i : \pi_X^{-1}(Z) \longrightarrow Y$, and these maps make the diagram commutative :



Moreover, let W be a scheme over k, and $\psi_Z : W \longrightarrow Z$, $\psi_Y : W \longrightarrow Y$ two morphisms such that $\phi_Y \circ \psi_Y = \phi_Z \circ \psi_Z$. Then $\phi_Y \circ \psi_Y = \phi_X \circ (i_Z \circ \psi_Z)$, so by universal property of the fibred product $X \times_k Y$, it exists a unique morphism $f : W \longrightarrow X \times_k Y$ such that $\pi_X \circ f = \pi_Y \circ f = \psi_Y = i_Z \circ \psi_Z$ (see the following diagram) :



But as $i_Z \circ \psi_Z = \pi_X \circ f$, the image of f is in $\pi_X^{-1}(Z)$, so f induces a morphism $W \longrightarrow \pi^{-1}(Z)$ such that the following diagram commutes :



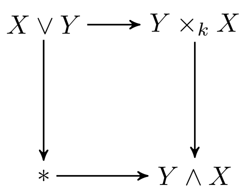
By universal property of fibred product, we have then $\pi_X^{-1}(Z) = Z \times_k Y$ and $\pi_X \circ i = \pi_Z$. ■

6.3 Smash product and suspension

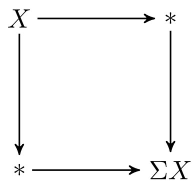
Here we will denote by $*$ the scheme Spec(k).

Definition 6.8 Let X be a scheme over k. A pointed space is the given of the simplicial presheaf associated to X, denoted by $r(X)$ (see example 4.6), and of a morphism $\sigma : * \longrightarrow r(X)$.

Definition 6.9 Wedge sum and smash product
Let X, σ_X and Y, σ_Y be two pointed spaces.
• The wedge sum $X \vee Y$ is the pushout of the diagram $X \longleftarrow * \longrightarrow Y$.
• The smash product $X \wedge Y$ is then the limit of the diagram



• The suspension of X is the colimit of the following diagram :



Proposition 6.3 It exists a pointed space S^1, called the motivic sphere of dimension 1, such that for any pointed space X, we have $X \wedge S^1 \cong \Sigma X$.

6.4 Grothendieck topologies

Let C a category. Here we define a notion of topology over C, inspired by the case where C is the category of open sets of a topological space.

Definition 6.10 Sieves
For an object U of C, a sieve on U is a family of morphisms $\{U_i \longrightarrow U\}_{i\in I}$.

Example 6.1 If X is a topological space and C the category of open sets of X, then we can define a sieve on an open set U to be a family of inclusion $\{U_i \hookrightarrow U\}$ such that $U = \bigcup_{i\in I} U_i$.

Definition 6.11 Site
Let C be a category. A Grothendieck topology on C is a family of sieves $J(U)$ for every object $U \in C$, such that :

(i) If $V \longrightarrow U$ is an isomorphism, then $\{V \longrightarrow U\} \in J(U)$;

(ii) If $\{U_i \longrightarrow U\}_{i\in I} \in J(U)$ and for all $i \in I$, $\{V_{ij} \longrightarrow U_i\}_{j\in J_i} \in J(U_i)$, then $\{V_{ij} \longrightarrow U\}_{i,j} \in J(U)$;

(iii) If $\{U_i \longrightarrow U\}_{i\in I} \in J(U)$ and $V \longrightarrow U$ is a map, then $U_i \times_U V$ exists for all $i \in I$, and $\{U_i \times_U V \longrightarrow V\}_{i\in I} \in J(V)$.

A site is the given of a category endowed with a Grothendieck topology.

6.5 Little lemmas of algebra and arithmetic

These results are basic propositions that can be found in [W], in [Eis95] or in our course of algebraic geometry [Stu21].

Definition 6.12 Reduced ring
Let R be a ring. We say that R is reduced when $\sqrt{\{0\}} = \{0\}$ in R, i.e. the only nilpotent element of R is zero.

Lemma 6.2 Let A be a ring and S a multiplicative subset of A. If A is an integral domain and S does not contain zero, then the localization $S^{-1}A$ is an integral domain.

Lemma 6.3 Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Then the localization of A by $S := A \setminus \mathfrak{p}$ is a local ring, with maximal ideal $\overline{\mathfrak{p}}$, the image of $\mathfrak{p}$ in $S^{-1}A$. Moreover, we have

$$S^{-1}A/\overline{\mathfrak{p}} \cong \mathrm{Frac}(A/\mathfrak{p})$$

Lemma 6.4 Let A be a unique factorization domain, and I a principal ideal of A. If $\mathfrak{p}$ is a prime ideal minimal among the prime ideals containing I, then $\mathfrak{p}$ is itself principal.

Lemma 6.5 Let A be a ring and I an ideal of A. Then

$$\bigcap_{\substack{\mathfrak{p}\text{ prime}\\ I\subset\mathfrak{p}}} \mathfrak{p} = \sqrt{I}$$

Lemma 6.6 Let A be a regular ring of dimension one. Then A is a discrete valuation ring.

Lemma 6.7 Every matrix in $M_n(k)$ can be written as a product of elementary matrices.

Definition 6.13 Let A be a ring. Let M be a A-module. We say that M is flat as a A-module when for every exact sequence f A-modules

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

The image sequence by tensor product by M is still exact :

$$0 \longrightarrow N_1 \otimes_A M \longrightarrow N_2 \otimes_A M \longrightarrow N_3 \otimes_A M \longrightarrow 0$$

Proposition 6.4 Let A be a ring. Let B be a ring that is also a A-module, and C be a ring that is a B-module. Let us suppose that B is flat as a A-module, and that C is flat as a B-module. Then C is flat as a A-module.

Proof
Let $0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$ be an exact sequence of A-modules. Since B is flat as a A-module, the following sequence is exact

$$0 \longrightarrow N_1 \otimes_A B \longrightarrow N_2 \otimes_A B \longrightarrow N_3 \otimes_A B \longrightarrow 0$$

Moreover, $N_i \otimes_A B$ is a B-module, and we have $(N_i \otimes_A B) \otimes_B C = N_i \otimes_A C$. Since C is flat as a B-module, the following sequence is exact :

$$0 \longrightarrow (N_1 \otimes_A B) \otimes_B C \longrightarrow (N_2 \otimes_A B) \otimes_B C \longrightarrow (N_3 \otimes_A B) \otimes_B C \longrightarrow 0$$

Then the sequence $0 \to N_1 \otimes_A C \to N_2 \otimes_A C \to N_3 \otimes_A C \to 0$ is exact, so C is flat as a A-module. ■

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