# Modular deformation of rational numbers and plane geometry

## Clusters, Deformations and Positivity, ICJ Lyon

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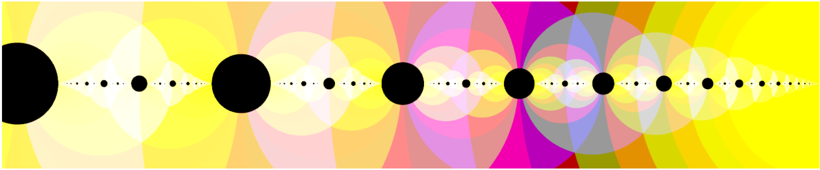

# Producing $q$-analogues of rationals

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## Example: $q$-integers

For $n \in \mathbb{N}$,

$$[n]_q := \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.$$

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- there is no recursive relation $[x+1]_q \neq q[x]_q + 1$ ;
- the order is not preserved: if $x < y$, in general $[x]_q \not< [y]_q$ (for $q \in \mathbb{R}_+$).

Therefore we will define $\left[\frac{a}{b}\right]_q$ in another way.

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- the shift $x \mapsto x+1$ is realized by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,
- the negation-inversion $x \mapsto -\frac{1}{x}$ is realized by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

# Producing $q$-analogues of rationals

## Deformation of matrices [MGO20, LMG21]

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$$\begin{array}{ccc} \mathrm{PSL}_2(\mathbb{Z}) & \xrightarrow{\simeq} & \langle T_q, S_q \rangle \subset \mathrm{PGL}_2(\mathbb{Z}[q, q^{-1}]). \\ M & \longmapsto & M_q \end{array}$$

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This is the unique way of deforming matrices of $\mathrm{PSL}_2(\mathbb{Z})$, up to conjugacy [JPRT26a].

## $q$-rationals [MGO20]

Let $x \in \mathbb{Q}$. Let $M \in \mathrm{PSL}_2(\mathbb{Z})$ be such that $M \cdot \frac{1}{0} = x$. Then define

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## $q$-rationals [MGO20] (right version)

Let $x \in \mathbb{Q}$. Let $M \in \mathrm{PSL}_2(\mathbb{Z})$ be such that $M \cdot \frac{1}{0} = x$. Then define

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## $q$-rationals [BBL23] (left version)

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$$\left[\frac{5}{2}\right]_q^\flat = \frac{1+q+q^2+q^3+q^4}{1+q^2} \text{ and } \left[\frac{5}{2}\right]_q^\sharp = \frac{1+2q+q^2+q^3}{1+q}.$$

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These polynomials have many properties [AL25, BOSZ24, FQ23, OR23, KW19, LMG23, LMGOV24, MSS21, OP25, Ove23, Ovs25, Ren24, Sik24, Tho24]...

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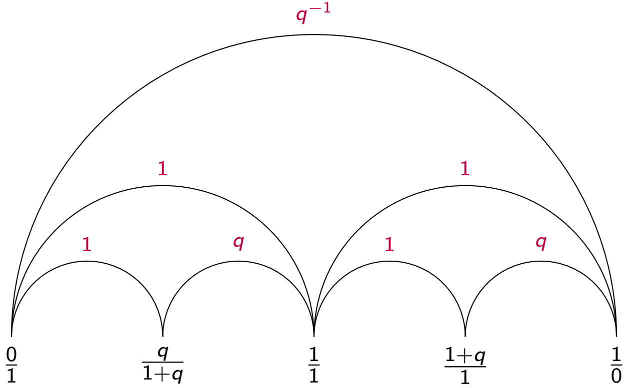

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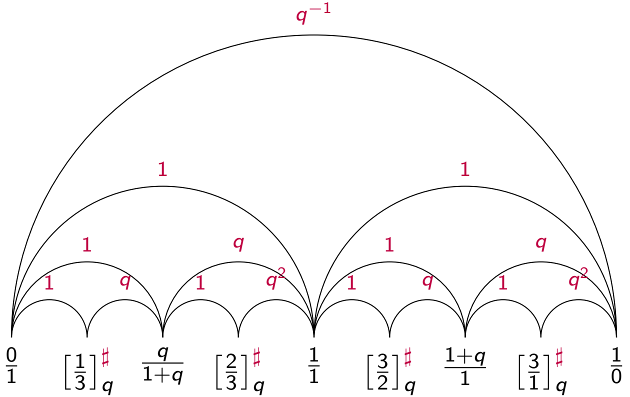

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This equivariance extends to $\mathrm{PGL}_2(\mathbb{Z})$, by

$$N = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{q} N_q := \begin{pmatrix} -1 & 1 - q^{-1} \\ q - 1 & 1 \end{pmatrix} .$$

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Then

$$\mathrm{PGL}_2(\mathbb{Z}) \xrightarrow{\simeq} \langle T_q, S_q, N_q \rangle,$$

and for any $M \in \mathrm{PGL}_2(\mathbb{Z})$ with $\det(M) = -1$,

$$[M \cdot x]_q^{\sharp} = M_q \cdot [x]_q^{\flat} \quad , \quad [M \cdot x]_q^{\flat} = M_q \cdot [x]_q^{\sharp},$$

with $x \in \mathbb{P}^1(\mathbb{Q})$.

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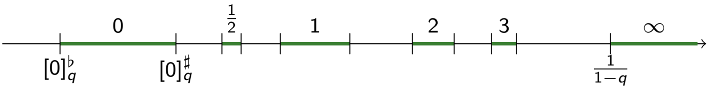

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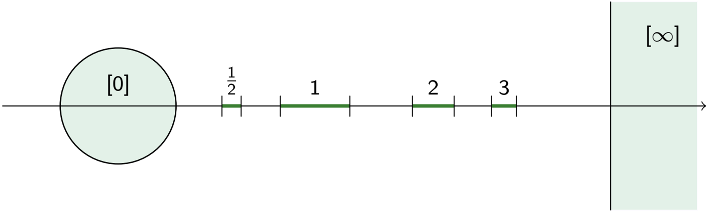
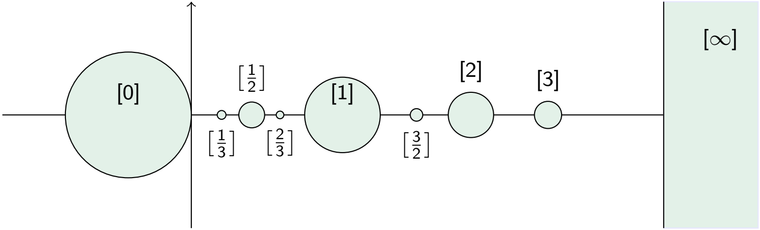



Notation: $[x]$ is the disk passing by $[x]_q^\sharp$ and $[x]_q^\flat$, orthogonal to the real axis.

The union of all the disks $[x]$, for $x \in \mathbb{P}^1(\mathbb{Q})$, is denoted by $\mathcal{Q}$.



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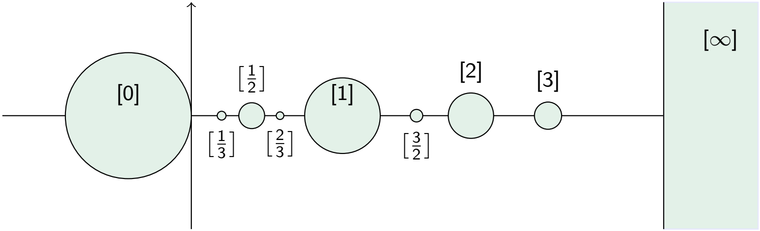
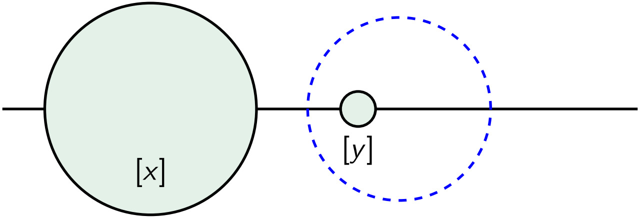



The symmetries of $\mathcal{Q}$ are elements of $\mathrm{PGL}_{2,q}(\mathbb{Z})$, acting by Möbius transformations.

## Geometric $q$-Farey rule

Let $x, y$ be two rationals with Farey determinant 1. Consider the unique inversion in circle exchanging $[x]$ and $[y]$.



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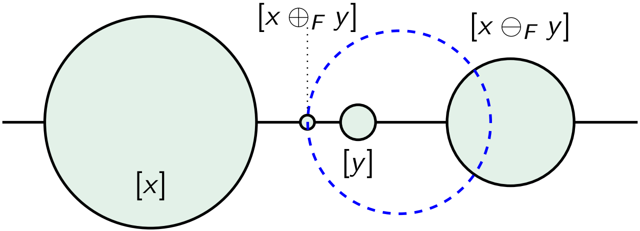


Then $[x \oplus_F y]$ and $[x \ominus_F y]$ are stabilized by this inversion [JPRT26b].

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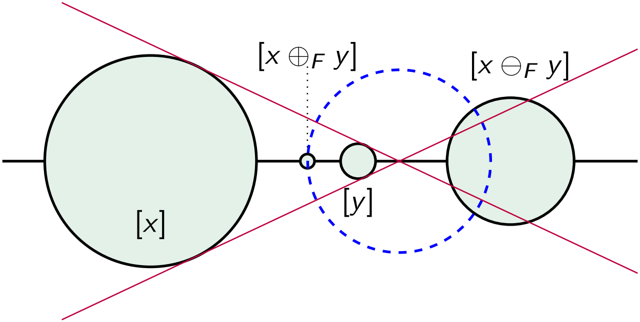


# Homothety centers

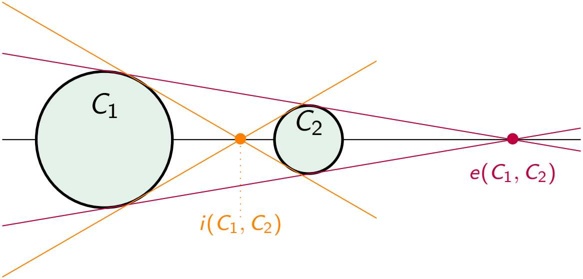


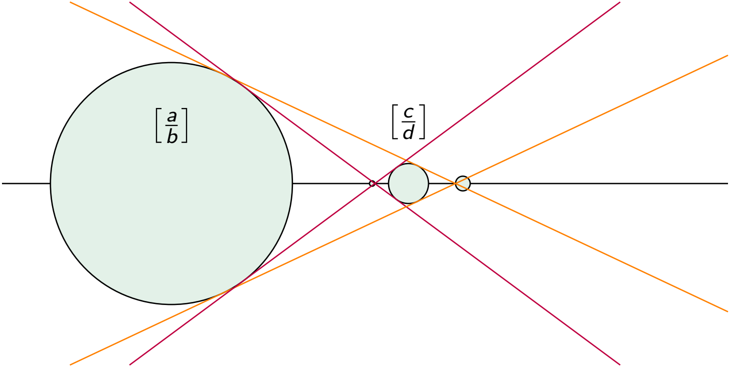

Figure: Inner (in orange) and outer (in purple) homothety centers of $C_1$ and $C_2$.

Suppose $C_i$ has center $z_i \in \mathbb{C}$ and radius $r_i$. Then

$$i(C_1, C_2) = \frac{r_1 z_2 + r_2 z_1}{r_1 + r_2} \text{ and } e(C_1, C_2) = \frac{r_1 z_2 - r_2 z_1}{r_1 - r_2}.$$

# Observations [JPRT26b]

Suppose $x = \frac{a}{b}$ and $y = \frac{c}{d}$ have Farey determinant 1 or 2. Then we observe



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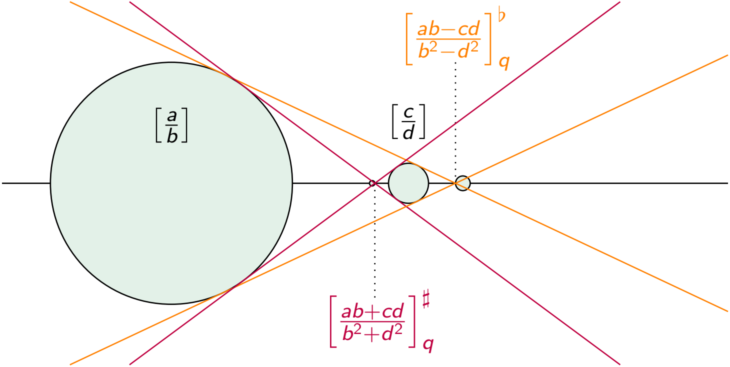

# Springborn operations [Spr24, JPRT26b]

Let $\frac{a}{b}$ and $\frac{c}{d}$ be two rationals in reduced form. Define

$$\frac{a}{b} \oplus_S \frac{c}{d} = \frac{ab+cd}{b^2+d^2},$$

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These expressions may not be reduced. And

$$\gcd(ab + cd, b^2 + d^2) \mid d_F\left(\tfrac{a}{b}, \tfrac{c}{d}\right),$$

$$\gcd(ab - cd, b^2 - d^2) \mid d_F\left(\tfrac{a}{b}, \tfrac{c}{d}\right).$$

# Main result [JPRT26b]

Let $\frac{a}{b}$ and $\frac{c}{d}$ be two rationals in reduced form. Denote by $d_F = d_F(\frac{a}{b}, \frac{c}{d}) = |bc - ad|$.

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## Inner version

Suppose that $\gcd(ab + cd, b^2 + d^2, a^2 + c^2) = d_F$. Then

$$i\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \left[\frac{a}{b} \oplus_S \frac{c}{d}\right]_q^\sharp.$$

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## Outer version

Suppose that $\gcd(ab - cd, b^2 - d^2, a^2 - c^2) = d_F$. Then

$$e\left(\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]\right) = \left[\frac{a}{b} \ominus_S \frac{c}{d}\right]_q^{\flat}.$$

## Regular pairs

We say that a pair $(\frac{a}{b}, \frac{c}{d})$ is inner regular when

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## Example

Any pair with Farey determinant 1 or 2 is both inner and outer regular.

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### Modular invariance

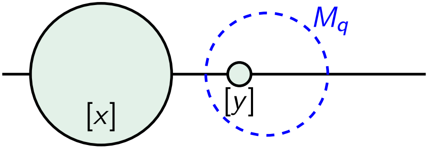
The set of outer-regular pairs is invariant under the action of $\mathrm{PSL}_2(\mathbb{Z})$.

### Characterisation

A pair $(x, y)$ is outer regular iff there is an orientation-reversing involution in $\mathrm{PGL}_2(\mathbb{Z})$ exchanging $x$ and $y$.

# Sketch of proof of the main result (outer version)

If $(x, y)$ is outer regular, let $M \in \mathrm{PGL}_2(\mathbb{Z})$ be the involution exchanging $x$ and $y$. Then $M_q \in \mathrm{PGL}_{2,q}(\mathbb{Z})$ is a symmetry of $\mathcal{Q}$.



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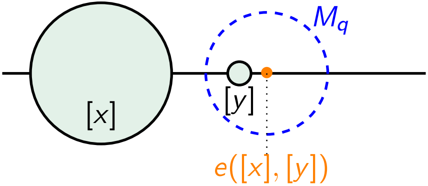


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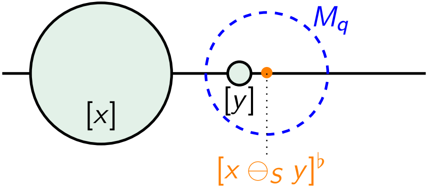


On one hand, $M_q \cdot \infty = e([x], [y])$.

On the other hand, $\exists z \in \mathbb{Q}$ s.t. $M_q \cdot [\infty]_q^\sharp = [z]_q^\flat$. Taking $q = 1$, we get $z = x \ominus_S y$.

Hence $e([x], [y]) = [x \ominus_S y]_q^\flat$.

# Consequence: $q$-Springborn operations

Let $(\frac{a}{b}, \frac{c}{d})$ be an outer regular pair. Then there are explicit integers $\varepsilon_1$ and $\varepsilon_2$ such that

$$\left[\frac{a}{b} \ominus_S \frac{c}{d}\right]_q^\flat = \frac{q^{\varepsilon_2} A^\sharp B^\flat - q^{\varepsilon_1} C^\sharp D^\flat}{q^{\varepsilon_2} B^\sharp B^\flat - q^{\varepsilon_1} D^\sharp D^\flat}.$$

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Similarly, if the pair is inner regular,

$$\left[\frac{a}{b} \oplus_S \frac{c}{d}\right]_q^\sharp = \frac{q^{\varepsilon_2} A^\sharp B^\flat + q^{\varepsilon_1} C^\flat D^\sharp}{q^{\varepsilon_2} B^\sharp B^\flat + q^{\varepsilon_1} D^\sharp D^\flat}.$$

# Application: midpoint formula

Let $0 < \frac{a}{b} < \frac{c}{d}$ with Farey determinant 1. Then there is an explicit integer $\varepsilon$ such that

$$\left[\frac{1}{2}\left(\frac{a}{b}+\frac{c}{d}\right)\right]_q^\flat = \frac{A^\sharp D^\flat + q^\varepsilon C^\sharp B^\flat}{B^\sharp D^\flat + q^\varepsilon B^\flat D^\sharp}.$$

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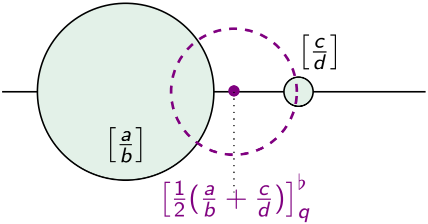
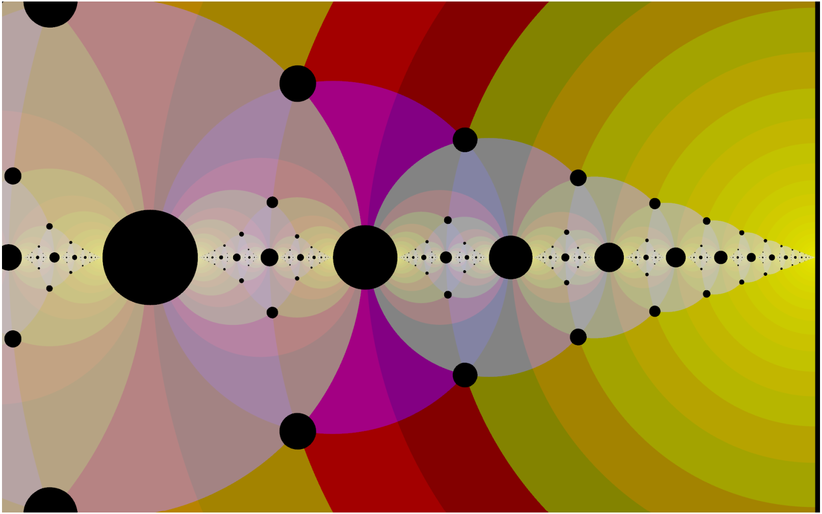
This is because the inversion stabilizing $\left[\frac{a}{b}\right]$ and $\left[\frac{c}{d}\right]$ has center $\left[\frac{1}{2}(\frac{a}{b}+\frac{c}{d})\right]_q^\flat$.



# Thank you!



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# Examples of a non regular pairs

The pair $\left(\frac{1}{3}, \frac{2}{9}\right)$ is neither inner nor outer regular. Yet, the right $q$-version of its Springborn sum is the inner homothety center.

The pair $\left(\frac{2}{7}, \frac{3}{7}\right)$ is neither inner nor outer regular. Yet, the left $q$-version of its Springborn difference is the outer homothety center.