Burau representation of B_4 and q-deformed rational projective plane

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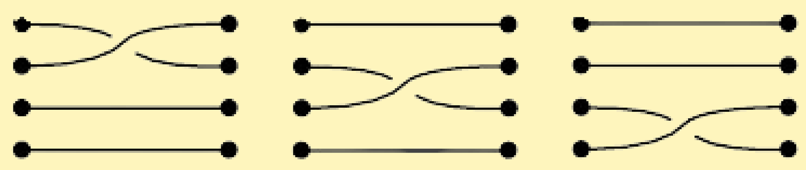
Integral Burau representation of B_4

Braid group B_4 $\xrightarrow[\text{Integral Burau } \rho]{}$ **Special linear group** $\mathrm{SL}_3(\mathbb{Z})$... ... **acting on** $\mathbb{P}^2(\mathbb{Q})$.

σ_1 σ_2 σ_3



$\rho(\sigma_1)$ $\rho(\sigma_2)$ $\rho(\sigma_3)$

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\-1&1&1\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&-1&1\end{pmatrix}$$

$$[r:s:t] \overset{\sigma_1}{\longmapsto} [r+s:s:t]$$
$$[r:s:t] \overset{\sigma_2}{\longmapsto} [r:s+t-r:t]$$
$$[r:s:t] \overset{\sigma_3}{\longmapsto} [r:s:t-s].$$

Theorem : classification of orbits [2]

$$\mathbb{P}^2(\mathbb{Q}) = \{[1:0:1]\} \sqcup \mathrm{Orb}_{B_4}([0:1:0]) \sqcup \bigsqcup_{\substack{n\geq 2\\ 0<m<n/2\\ m\wedge n=1}} \mathrm{Orb}_{B_4}([m:n:m]).$$

For every couple (m,n) of coprime integers,

$$\mathrm{Orb}_{B_4}([m:n:m]) = \left\{ [r:s:t] \;\middle|\; \begin{cases}\gcd(r-t,s)=n;\\ r,t\equiv \pm m \mod (n).\end{cases} \right\}.$$

Example : the orbit $\mathcal{O}_1$ of $[0:1:0]$ contains the points $[r:s:t]$ such that $\gcd(r-t,s)=1$. In particular, the **projective line** is entirely in $\mathcal{O}_1$, through the embedding

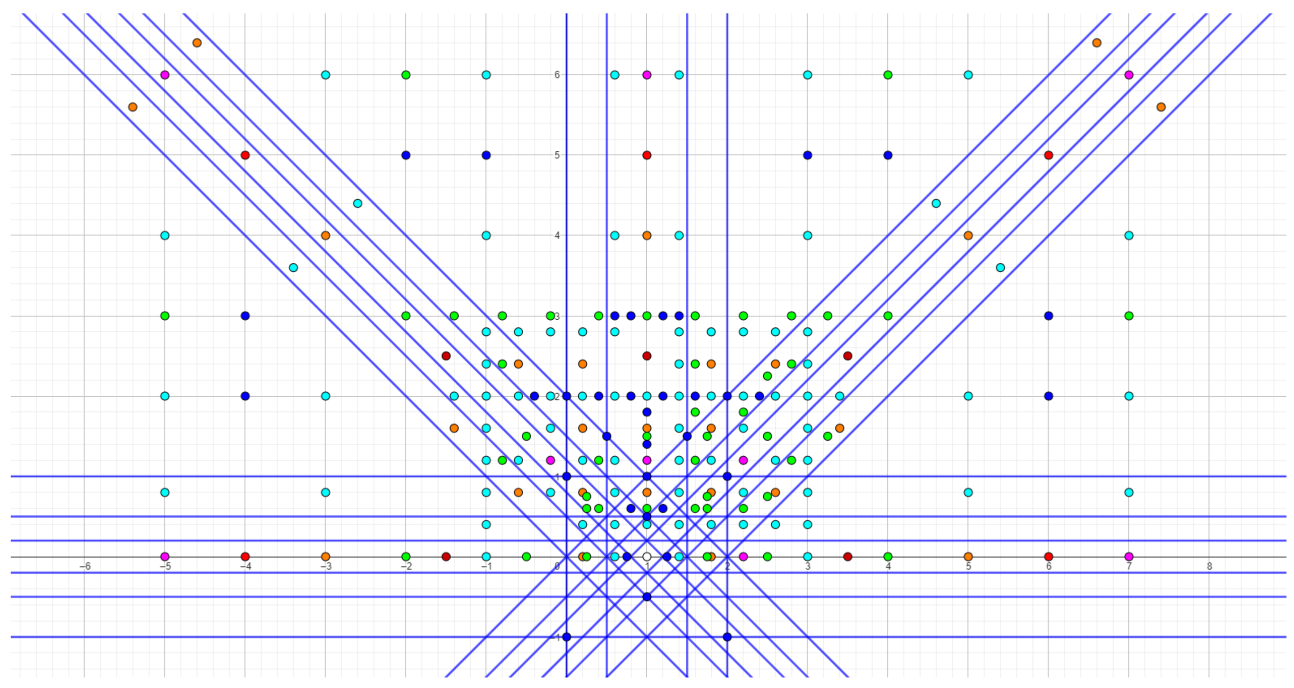
$$\mathbb{P}^1(\mathbb{Q}) \overset{\iota}{\hookrightarrow} \mathcal{O}_1 \subset \mathbb{P}^2(\mathbb{Q}), \qquad [r:s]\mapsto[r:s:0].$$

Stabilizers

Let $\mathcal{BI}_4 := \ker(\rho)$ be the **Braid Torelli group** [1].
Let $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$, $\tau_i = (\sigma_i\sigma_2\sigma_i)^2$, $i\in\{1,3\}$.

$$\mathrm{Stab}_{[m:n:m]}/\mathcal{BI}_4 = \begin{cases} \langle\, \tau_1\Delta,\ \sigma_2 \,\rangle & \text{if } n\geq 3,\\ \langle\, \tau_1\Delta,\ \sigma_2,\ \sigma_1\sigma_2^2\sigma_3 \,\rangle & \text{if } n=2,\\ \langle\, \tau_1,\ \Delta,\ \sigma_2 \,\rangle & \text{if } n=1.\end{cases}$$

Sketch of orbits

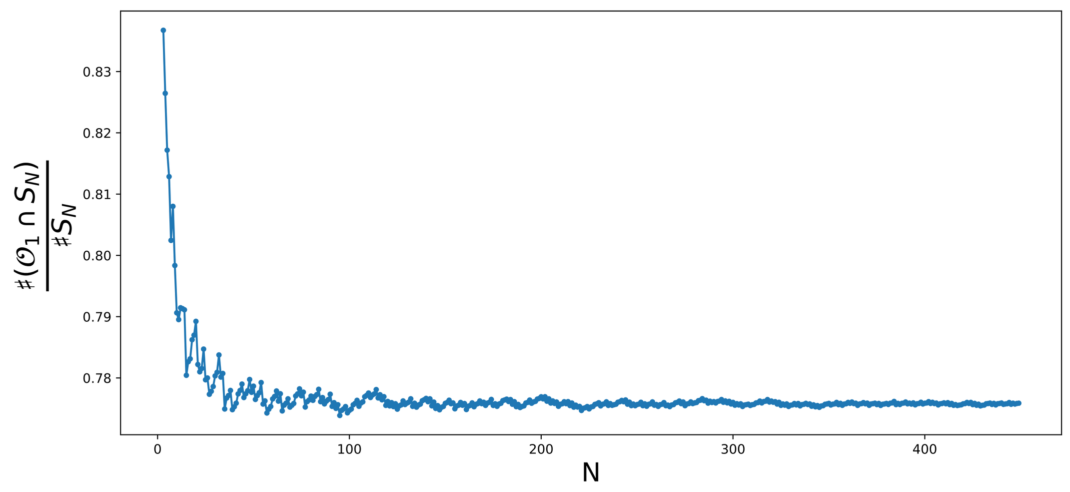


Colors meanings

- dark blue : Orb([0 : 1 : 0])
- light blue : Orb([1 : 2 : 1])
- green : Orb([1 : 3 : 1])
- orange : Orb([1 : 4 : 1])
- red : Orb([1 : 5 : 1])
- pink : Orb([1 : 6 : 1])

Principal orbit $\mathcal{O}_1 := \mathrm{Orb}([0:1:0])$

- $\mathcal{O}_1$ is the only orbit containing **affine lines**.
- Conjecture : For every $N\in\mathbb{N}$, the orbit $\mathcal{O}_1$ contains at least three times **more points** in the subset $S_N := \{[1:a/b:c/d] \mid |a|,|b|,|c|,|d|\leq N\}$ than the union of the other orbits.



Quantization of $\mathcal{O}_1$ via the Burau representation

B_4 **acting on** $\mathbb{P}^2(\mathbb{Z}(q))$ **via the Burau representation**

$$\rho_q(\sigma_1)=\begin{pmatrix}q&1&0\\0&1&0\\0&0&1\end{pmatrix},\rho_q(\sigma_2)=\begin{pmatrix}1&0&0\\-q&q&1\\0&0&1\end{pmatrix},\rho_q(\sigma_3)=\begin{pmatrix}1&0&0\\0&1&0\\0&-q&q\end{pmatrix}.$$

The quantization map is a function $\mathcal{Q}:\mathcal{O}_1\to\mathcal{P}(\mathbb{P}^2(\overline{\Lambda}))$,

$$\forall p\in\mathcal{O}_1,\ \mathcal{Q}(p) := \{\rho_q(\beta)([0:1:0]) \mid \beta \text{ s.t. } \rho(\beta)([0:1:0])=p\}.$$

An example

Let us quantize $[4:3:-1]\in\mathcal{O}_1$, via $\beta=\sigma_1\sigma_3\sigma_2^2\sigma_1\sigma_3^{-2}$:

$$\rho_q(\beta)([0:1:0]) = [q^2+2q+1 : 2q+1 : -q^2].$$

Now with $\beta' = \sigma_3\sigma_2^{-1}\sigma_1^4\sigma_3^{-2}$, we get another deformation :

$$[q^5+q^4+q^3+q^2 : q^5+q^4+q^3+q^2-1 : -q^6-q^5-q^4+q^2+q].$$

Remark : the braid β can be computed via a Jacobi-Perron type multidimensional continued fractions algorithm.

Theorem : link with q-rationals

Let $\frac{r}{s}\in\mathbb{Q}$.
Let $[x]_q = \frac{R(q)}{S(q)}$ be the q-**deformed number** in the sense of Morier-Genoud and Ovsienko [3],[4].

Then $[R(q):S(q):0]$ is in $\mathcal{Q}([r:s:0])$ and it is the minimal deformation of the point $[r:s:0]$.

Unicity of the deformation

Conjecture : There is a **unique** deformation $[R:S:T]$ of p in $\mathcal{Q}(p)$ such that $\deg(R)$, $\deg(S)$, and $\deg(T)$ are **together minimal**.

Definition : Assuming this, we can define **the quantization** of a point $p\in\mathcal{O}_1$ to be the minimal deformation of p.

References

[1] T. Brendle, D. Margalit, and A. Putman. Generators for the hyperelliptic Torelli group and the kernel of the Burau representation at $t=-1$. *Inventiones mathematicae*, 200(1):263–310, July 2014.

[2] P. Jouteur. Burau representation of B_4 and quantization of the rational projective plane. *https://arxiv.org/abs/2407.20645*, 2024.

[3] S. Morier-Genoud and V. Ovsienko. q-deformed rationals and q-continued fractions. *Forum of Mathematics, Sigma*, 8, 2020.

[4] S. Morier-Genoud, V. Ovsienko, and A. P. Veselov. Burau representation of braid groups and q-rationals. *International Mathematics Research Notices*, page 318, January 2024.