Symmetries of the q-deformed real projective line

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Background on q-real numbers [3]

Define

$$S_q := \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} \text{ and } T_q := \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}.$$

Then $S_q^2 = 1, (T_q S_q)^3 = 1$ (up to $\pm q^k$), so

$$\mathrm{PSL}_{2,q}(\mathbb{Z}) := \langle T_q, S_q \rangle \simeq \mathrm{PSL}_2(\mathbb{Z}).$$

Quantized action of the modular group :

$$\begin{array}{ccc} \mathrm{PSL}_{2,q}(\mathbb{Z}) \times \mathbb{P}^1(\mathbb{Z}(q)) & \longrightarrow & \mathbb{P}^1(\mathbb{Z}(q)) \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot f(q) & = & \frac{Af(q)+B}{Cf(q)+D} \end{array}.$$

Left and right q-rationals [1]

- Right version $[\cdot]^\sharp$ = orbit of $\frac{0}{1}$ under the quantized action above,
- Left version $[\cdot]^\flat$ = orbit of $\frac{q-1}{q}$.

Example : q-integers

For $n \in \mathbb{Z}_{\geq 0}$,

$$[n]^\sharp = T_q^n \cdot \frac{0}{1} = \frac{q^n - 1}{q-1} = 1 + q + \cdots + q^{n-1}.$$

$$[n]^\flat = T_q^n \cdot \frac{q-1}{q} = \frac{q^{n+1} - q^n + q^{n-1} - 1}{q-1} = 1 + q + \cdots + q^{n-2} + q^n.$$

Quantized irrationals [4]

Let $x \in \mathbb{R} \setminus \mathbb{Q}$, and $(x_n)_n \in \mathbb{Q}^{\mathbb{N}}$ such that $x_n \to x$.
Then $[x]_q$ is a formal Laurent series in q, limit of both $([x_n]^\sharp)_n$ and $([x_n]^\flat)_n$.

$$[x]_q \in \mathbb{Z}[[q]][q^{-1}] = \left\{ \sum_{k \geq \nu} x_k q^k \mid x_k \in \mathbb{Z},\ \nu \in \mathbb{Z} \right\}.$$

And the quantized action of $\mathrm{PSL}_2(\mathbb{Z})$ commutes with the quantization of irrational real numbers. For all $x \in \mathbb{R} \setminus \mathbb{Q}$,

$$[x+1]_q = q[x]_q + 1 \text{ and } \left[\frac{-1}{x}\right]_q = \frac{-1}{q[x]_q}.$$

First extension of symmetries

Define $N_q = \begin{pmatrix} -1 & 1-q^{-1} \\ q-1 & 1 \end{pmatrix}$, and a quantized version of $\mathrm{PGL}_2(\mathbb{Z})$,

$$\mathrm{PGL}_{2,q}(\mathbb{Z}) := \langle T_q, S_q, N_q \rangle.$$

Theorem [2]

1. $\mathrm{PGL}_{2,q}(\mathbb{Z}) \simeq \mathrm{PGL}_2(\mathbb{Z})$ acts on $\mathbb{P}^1(\mathbb{Z}(q))$.
2. For all $M \in \mathrm{PGL}_2(\mathbb{Z})$ with $\det(M) = -1$,
$$\forall x \in \mathbb{Q}, \quad \begin{cases} M_q \cdot [x]^\flat = [M \cdot x]^\sharp, \\ M_q \cdot [x]^\sharp = [M \cdot x]^\flat. \end{cases}$$
3. $\mathrm{PGL}_{2,q}(\mathbb{Z})$ acts on q-irrationals, and for all $M \in \mathrm{PGL}_2(\mathbb{Z})$,
$$\forall x \in \mathbb{R} \setminus \mathbb{Q}, \ M_q \cdot [x]_q = [M \cdot x]_q.$$

Corollary

For all $x \in \mathbb{R} \setminus \mathbb{Q}$,

$$[-x]_q = \frac{-[x]_q + 1 - q^{-1}}{(q-1)[x]_q + 1} \text{ and } \left[\frac{1}{x}\right]_q = \frac{(q-1)[x]_q + 1}{q[x]_q + 1 - q}.$$

Application to algebraic numbers [2]

Let $b > 2$. The equation $x^4 - bx^2 + 1 = 0$ $(\star)$ is quantized as

$$X^4 - \Sigma_1 X^3 + \left(\frac{q + q^{-1} - 3}{q-1} \Sigma_1 + 2q^{-1} \right) X^2 + q^{-1} \Sigma_1 X + q^{-2} = 0,$$

where $\Sigma_1 = [x_1]_q + [x_2]_q + [x_3]_q + [x_4]_q$, if the x_i's are the roots of $(\star)$.

Second extension of symmetries

Define a twist operator

$$\begin{array}{cccc} \tau : & \mathbb{P}^1(\mathbb{Z}(q)) & \longrightarrow & \mathbb{P}^1(\mathbb{Z}(q)) \\ & f & \longrightarrow & f(q^{-1}) \end{array}.$$

Let $I_q = \begin{pmatrix} 1 & q-1 \\ 1-q & q \end{pmatrix}$, and $\overline{I}_q := I_q \cdot \tau$.

For any $M \in \mathrm{PGL}_2(\mathbb{Z})$, the **twisted q-deformation** of M is the operator

$$\overline{M}_q := M_q \cdot \overline{I}_q : f(q) \longmapsto M_q I_q \cdot f(q^{-1}).$$

Theorem [2]

1. We get an extended action of $\mathrm{PGL}_2(\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{P}^1(\mathbb{Z}(q))$.
2. Let $M \in \mathrm{PGL}_2(\mathbb{Z})$. For all $x \in \mathbb{Q}$,
   - if $\det(M) = 1$, $\begin{cases} \overline{M}_q \cdot [x]^\sharp = [M \cdot x]^\flat, \\ \overline{M}_q \cdot [x]^\flat = [M \cdot x]^\sharp. \end{cases}$
   - if $\det(M) = -1$, $\begin{cases} \overline{M}_q \cdot [x]^\flat = [M \cdot x]^\flat, \\ \overline{M}_q \cdot [x]^\sharp = [M \cdot x]^\sharp. \end{cases}$

Corollary

For all $x \in \mathbb{Q}$,

$$[x]_q^\sharp = \frac{[x]^\flat_{q^{-1}} + q - 1}{(1-q)[x]^\flat_{q^{-1}} + q} \text{ and } [x]_q^\flat = \frac{[x]^\sharp_{q^{-1}} + q - 1}{(1-q)[x]^\sharp_{q^{-1}} + q}. \text{ [5]}$$

and

$$[-x]_q = -q^{-1}[x]_{q^{-1}} \text{ and } \left[\frac{1}{x}\right]_q = \frac{1}{[x]_{q^{-1}}}.$$

References

[1] A. Bapat, L. Becker, and A. M. Licata. q-deformed rational numbers and the 2-Calabi–Yau category of type A_2. *Forum of Mathematics, Sigma*, 11, 2023.

[2] P. Jouteur. Symmetries of the q-deformed real projective line. *ArXiv*, 2503.02122, 2025.

[3] S. Morier-Genoud and V. Ovsienko. q-deformed rationals and q-continued fractions. *Forum of Mathematics, Sigma*, 8, 2020.

[4] S. Morier-Genoud and V. Ovsienko. On q-deformed real numbers. *Experimental Mathematics*, 31(2):652–660, Apr. 2022.

[5] A. Thomas. Infinitesimal Modular Group: q-Deformed $\mathfrak{sl}_2$ and Witt Algebra. *SIGMA*, June 2024.