2019/2020

Master 2 Recherche fondamentale en mathématiques

Numerical methods for kinetic transport equations

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Collisional Invariants

Statement 1. We consider the following elastic collision dynamics: two particles of velocities $v \in \mathbb{R}^3$ and $v_* \in \mathbb{R}^3$ may collide, and their velocities $v' \in \mathbb{R}^3$ and $v'_* \in \mathbb{R}^3$ after the collision are given in terms of (v, v_*) by the relations

$$v + v_* = v' + v'_* \tag{1a}$$
$$|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2 \tag{1b}$$

The goal here is to prove the following result: Any function $\phi \in L^1_{loc}(\mathbb{R}^3)$ satisfying

$$\phi(v) + \phi(v_*) = \phi(v') + \phi(v'_*) \tag{2}$$

for all v, v_* in $\mathbb{R}^3$ and for all $(v', v'_*) \in (\mathbb{R}^3)^2$ linked to (v, v_*) by the relations (1), is of the form

$$\phi(v) = A|v|^2 + B.v + C, \quad \text{for almost all } v \in \mathbb{R}^3, \tag{3}$$

where $A \in \mathbb{R}$, $B \in \mathbb{R}^3$, $C \in \mathbb{R}$ are constants.

Consider then a function $\phi \in L^1_{loc}(\mathbb{R}^3)$ satisfying (2).

Question 1. Let $(v, v_*, v', v'_*) \in (\mathbb{R}^3)^4$ satisfying (1), and let $u = v' - v$, $u_* = v'_* - v$. Show that $u.u_* = 0$.

Solution 1. We have $u = v' - v$ and $u_* = v'_* - v = v_* - v'$ by (1a).

Recall that $v_* - v'_* = v' - v$ by (1a) and, since $|v + v_*|^2 = |v' + v'_*|^2$ (1a) and $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$ by (1b), we have $v.v_* = v'.v'_*$.

Therefore

$$\begin{aligned} u.u_* &= (v' - v).(v_* - v') \\ &= v'.v_* - \underbrace{v.v_*}_{=v'.v'_*} - (v' - v).v' \\ &= v'.\underbrace{(v_* - v'_*)}_{=v'-v} - (v' - v).v' \\ &= v'.(v' - v) - v'.(v' - v) = 0 \end{aligned}$$

Therefore we have the expected result: $u.u' = 0$.

Question 2. Let $v \in \mathbb{R}^3$ be given, and set

$$\psi(u) = \phi(v + u) - \phi(v), \quad \forall u \in \mathbb{R}^3.$$

Prove that

$$\psi(u) + \psi(u_*) = \psi(u + u_*),$$

for all $u, u_* \in \mathbb{R}^3$ such that $u.u_* = 0$.

Solution 2. Let take $V = v + u$ and $V_* = v + u_*$. We use the expression (2) for V, V_* and $V' = v + u + u_*$, $V'_* = v$.

$$\phi(V) + \phi(V_*) = \phi(V') + \phi(V'_*)$$

Indeed, we have $V + V_* = v + u + v + u_* = v + u + u_* + v = V' + V'_*$ and

$$\begin{aligned}|v+u|^2 + |v+u_*|^2 &= |v|^2 + |u|^2 + |u_*|^2 + |v|^2 + 2v.u + 2v.u_* \\ |v+u+u_*|^2 + |v|^2 &= |v|^2 + |u|^2 + |u_*|^2 + |v|^2 + 2v.u + 2v.u_* + 2u.u_*\end{aligned}$$

There is an equality because $u.u_* = 0$ by assumption.

Therefore we are allowed to write

$$\begin{aligned}\phi(v+u) + \phi(v+u_*) &= \phi(v+u+u_*) + \phi(v) \\ \phi(v+u) - \phi(v) + \phi(v+u_*) - \phi(v) &= \phi(v+u+u_*) - \phi(v) \text{ while subtracting } 2\phi(v) \\ \psi(u) + \psi(u_*) &= \psi(u+u_*) \text{ by definition of } \psi\end{aligned}$$

Therefore we have the expected result.

Question 3. We now decompose ψ as the sum of its even part ψ_p and its odd part ψ_i :

$$\psi_p(u) = \frac{\psi(u) + \psi(-u)}{2}, \quad \psi_i(u) = \frac{\psi(u) - \psi(-u)}{2}.$$

Prove that $\psi_p(u)$ depends only on $|u|$.

Solution 3. Let take two vectors u and v with the same norm $|u|^2 = |v|^2$. We want to prove that $\psi_p(u) = \psi_p(v)$, then we will have that ψ_p depends only on the norm of u.

We use the Question 2, with $U = \pm\frac{u-v}{2}$ and $U_* = \pm\frac{u+v}{2}$. Indeed, we have

$$U.U_* = \pm\frac{1}{4}(u-v).(u+v) = \pm\frac{1}{4}(|u|^2 - |v|^2) = 0.$$

Then we have

$$\begin{aligned}\psi(u) &= \psi\left(\frac{u-v}{2}\right) + \psi\left(\frac{u+v}{2}\right) \\ \psi(-u) &= \psi\left(\frac{v-u}{2}\right) + \psi\left(-\frac{u+v}{2}\right) \\ \psi(v) &= \psi\left(\frac{v-u}{2}\right) + \psi\left(\frac{u+v}{2}\right) \\ \psi(-v) &= \psi\left(\frac{u-v}{2}\right) + \psi\left(-\frac{u+v}{2}\right)\end{aligned}$$

It follows that

$$2\psi_p(u) = \psi(u) + \psi(-u) = \psi(v) + \psi(-v) = 2\psi_p(v)$$

Therefore we have the expected result.

Question 4. Set

$$\psi_p(u) = \theta(|u|^2).$$

Prove that

$$\theta(x+y) = \theta(x) + \theta(y), \quad \text{for all } x \geqslant 0, y \geqslant 0.$$

Solution 4. Let take $x, y \geqslant 0$ and two vectors u and v such that $|u|^2 = x$, $|v|^2 = y$ and $u.v = 0$. We have

$$\begin{aligned} 2(\theta(x) + \theta(y)) &= 2(\psi_p(u) + \psi_p(v)) \\ &= \psi(u) + \psi(-u) + \psi(v) + \psi(-v) \\ &= \psi(u+v) + \psi(-u-v) \text{ by Question 2 and } u.v = 0 \end{aligned}$$

and

$$\begin{aligned} 2\theta(x+y) &= 2\theta(|u|^2 + |v|^2) \\ &= 2\theta(|u+v|^2) \text{ because } u.v = 0 \\ &= 2\psi_p(u+v) \\ &= \psi(u+v) + \psi(-u-v) \end{aligned}$$

Then we have the equality and while dividing by 2, we obtain the expected result

$$\theta(x+y) = \theta(x) + \theta(y).$$

Question 5. Check that $s \mapsto \sqrt{s}\theta(s)$ is locally integrable on $\mathbb{R}_+$, and deduce that the function ω defined by

$$\omega(x) = \int_0^1 \theta(tx)\sqrt{t}dt$$

is continuous on $\mathbb{R}_*^+$.

Solution 5. **Step 1 :** We want to show that $s \mapsto \sqrt{s}\theta(s) \in L^1_{loc}(\mathbb{R}_+)$.

- We know that $\phi \in L^1_{loc}(\mathbb{R}^3)$, then $\psi \in L^1_{loc}(\mathbb{R}^3)$.

  Indeed, for all K compact of $\mathbb{R}^3$, we have

  $$\begin{aligned} \int_K |\psi(u)|du &\leqslant \int_K (|\phi(v+u)| + |\phi(v)|)du \\ &\leqslant \int_{K-v} |\phi(\sigma)|d\sigma + |K||\phi(v)| < +\infty \end{aligned}$$

  because $K - v := \{x - v, x \in K\}$ is compact.[1]

[1] we used the change of variables : $\sigma = v + u$

- We know that $\psi \in L^1_{loc}(\mathbb{R}^3)$, then $\psi_p \in L^1_{loc}(\mathbb{R}^3)$.
  Indeed for all K compact of $\mathbb{R}^3$, we have

$$\begin{aligned}\int_K |\psi_p(u)|du &\leqslant \frac{1}{2}\int_K (|\phi(u)| + |\phi(-u)|)du \\ &\leqslant \frac{1}{2}\int_K (|\phi(u)|du + \int_{-K} |\phi(\sigma|)d\sigma < +\infty\end{aligned}$$

  because $-K := \{-x, x \in K\}$ is compact.[2]

- We know that $\psi_p \in L^1_{loc}(\mathbb{R}^3)$, then $s \mapsto \sqrt{s}\theta(s) \in L^1_{loc}(\mathbb{R}_+)$.
  Indeed for all K compact of $\mathbb{R}_+$, we have

$$\begin{aligned}\int_K \sqrt{s}|\theta(s)|ds &= \int_{\tilde{K}} v|\theta(v^2)|2vdv \text{ while changing } s \text{ into } v^2 \\ &= 2\int_{\tilde{K}} |\theta(v^2)|v^2dv \\ &= 2C\int_{\hat{K}} |\theta(|u|^2)|du \\ &= 2C\int_{\hat{K}} |\psi_p(u)|du < +\infty\end{aligned}$$

  because $\tilde{K}$ and $\hat{K}$ are compact. We use a spherical change of coordinates, the constant C represents the integrals on φ and ϕ, the angles in the change of coordinates.

Step 2 : We want to show that $\omega(x) = \displaystyle\int_0^1 \theta(tx)\sqrt{t}dt$ is continuous on $\mathbb{R}_*^+$.

We have

$$\begin{aligned}\omega(x) &= \int_0^1 \theta(tx)\sqrt{t}dt \\ &= \frac{1}{x^{3/2}}\int_0^x \theta(s)\sqrt{s}ds \text{ while using the change of variables : } u = tx\end{aligned}$$

We use the dominated convergence theorem :

- $s \mapsto \sqrt{s}\theta(s)$ is measurable for all $x \in \mathbb{R}_*^+$ by assumption on θ.
- $x \mapsto \dfrac{1}{x^{3/2}}\mathbb{1}_{[0,x]}(s)\sqrt{s}\theta(s)$ is continuous for almost all $s \in]0, +\infty[$.
- For all compact $[a, b] \subset]0, +\infty[$, we have

$$\forall x \in [a,b], \forall s \in]0,+\infty[, \quad \left|\mathbb{1}_{[0,x]}\frac{\theta(s)\sqrt{s}}{x^{3/2}}\right| \leqslant \left|\underbrace{\mathbb{1}_{[0,b]}\frac{\theta(s)\sqrt{s}}{a^{3/2}}}_{\in L^1([0,b])}\right|$$

It follows that $\omega : x \mapsto \displaystyle\int_0^1 \theta(tx)\sqrt{t}dt$ is continuous on $\mathbb{R}_*^+$.

[2] we used the change of variables : $\sigma = -u$

Question 6. Prove that $\omega(x+y) = \omega(x) + \omega(y), \forall x > 0, \forall y > 0$, and that there exists a constant C such that $\omega(w) = Cx$ for all $x \geqslant 0$.

Solution 6. **Step 1 :** Prove that $\omega(x+y) = \omega(x) + \omega(y), \forall x > 0, \forall y > 0$.

For all $x > 0$ and $y > 0$, we have

$$\begin{aligned}\omega(x+y) &= \int_0^1 \theta(t(x+y))\sqrt{t}dt \\ &= \int_0^1 (\theta(tx) + \theta(ty))\sqrt{t}dt \text{ while using Question 4} \\ &= \omega(x) + \omega(y)\end{aligned}$$

Step 2 : Prove that there exists a constant C such that $\omega(w) = Cx$ for all $x \geqslant 0$.

We have $\omega(0) = 0$. Indeed, $\omega(0) = \displaystyle\int_0^1 \theta(0)\sqrt{t}dt = 0$ because $\theta(0) = 0$, since $\theta(0+0) = \theta(0)+\theta(0)$.

By induction, we have that $\omega(n) = \omega(1+1+\cdots+1) = \omega(1) + \omega(1) + \cdots + \omega(1) = n\omega(1)$ for all $n \in \mathbb{N}$.

Then, we have $q\omega(1) = \omega(q) = \omega\left(p\dfrac{q}{p}\right) = p\omega\left(\dfrac{p}{q}\right)$ for all $p \in \mathbb{N}^*, q \in \mathbb{N}$. It follows that

$$\omega\left(\frac{p}{q}\right) = \frac{p}{q}\omega(1)$$

Since ω is continuous and $\mathbb{Q}$ is dense in $\mathbb{R}$, we have

$$\omega(x) = x\omega(1), \quad \forall x \in \mathbb{R}^+$$

We have the expected result, with $C = \omega(1)$.

Question 7. Deduce that there exists a constant A such that $\psi_p(u) = A|u|^2$ for almost all $u \in \mathbb{R}^3$.

Solution 7. By Question 6, since $\omega(x) = Cx$, we know that ω is differentiable.

$$\omega'(x) = C = \int_0^1 \theta'(tx)t\sqrt{t}dt = \int_0^1 \theta'(tx)t^{3/2}dt$$

By integration by parts with $u(t) = \dfrac{\theta(tx)}{x}$ and $v(t) = t^{3/2}$, we have

$$C = \int_0^1 \theta'(tx)t^{3/2}dt = \left[\frac{\theta(tx)}{x}t^{3/2}\right]_0^1 - \int_0^1 \frac{\theta(tx)}{x}\frac{3}{2}\sqrt{t}dt$$

$$C = \frac{\theta(x)}{x} - 0 - \frac{3}{2x}\omega(x) = \frac{\theta(x)}{x} - \frac{3}{2}C$$

$$\theta(x) = \frac{5}{2}Cx$$

It follows that

$$\psi_p(u) = \theta(|u|^2) = A|u|^2, \quad \text{for almost all } u \in \mathbb{R}^3$$

where $A = \dfrac{5}{2}C$.

Question 8. Our aim is now to determine the odd part ψ_i. To this purpose, let e be a unit vector in $\mathbb{R}^3$ annd set $\theta_i(x) = \psi_i(xe), \forall x > 0$. Show that θ_i satisfies

$$\theta_i(x+y) = \theta_i(x) + \theta_i(y), \quad \forall x > 0, \forall y > 0.$$

(write $\psi_i((x+y)e) = \psi_i(xe + ze_1 + ye - ze_1)$ where e_1 is unitary and orthogonal to e, and where z is suitably choosen in terms of x, y and e.)

Solution 8. Let take $x, y \in \mathbb{R}_*^+$. Let take e_1 a unit vector such that $e.e_1 = 0$ and $z := \sqrt{xy}$. We have

$$\begin{aligned}
2\theta_i(x+y) &= 2\psi_i((x+y)e) \\
&= 2\psi_i(xe + ye) \\
&= 2\psi_i(xe + ze_1 + ye - ze_1) \\
&= \psi(xe + ze_1 + ye - ze_1) - \psi(-xe - ze_1 - ye + ze_1) \\
&= \psi(xe + ze_1) + \psi(ye - ze_1) - \psi(-xe - ze_1) - \psi(-ye + ze_1) \\
&= \psi(xe) + \psi(ze_1) + \psi(ye) + \psi(-ze_1) - \psi(-xe) - \psi(-ze_1) - \psi(-ye) - \psi(ze_1) \\
&= \psi(xe) + \psi(ye) - \psi(-xe) - \psi(-ye) \\
&= 2\psi_i(xe) + 2\psi_i(ye) \\
&= 2(\theta_i(x) + \theta_i(y))
\end{aligned}$$

We use Question 2 twice, because $(xe+ze_1).(ye-ze_1) = xy|e|^2 + zye_1.e - xze.e_1 - z^2|e_1|^2 = xy - z^2 = 0$ and $(xe).(ze_1) = xz(e.e_1) = 0$. It follows that we have $\theta_i(x+y) = \theta_i(x) + \theta_i(y), \quad \forall x > 0, \forall y > 0.$

Question 9. Deduce that there exists a constant $B \in \mathbb{R}^3$ such that $\psi_i(u) = B.u$ for almost all $u \in \mathbb{R}^3$.

Solution 9. We use the same arguments that for Questions 4, 5, 6, 7, then there exists a constant B_e such that for all $x \geqslant 0$, $\theta_i(x) = B_e x$ (The constant depends only on the unit vector e).

Let take $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$. We have $e_1.e_2 = e_1.e_3 = e_2.e_3 = 0$.

$$\begin{aligned}
\psi_i(u) &= \psi_i\left(\sum_{j=1}^{3} u_j e_j\right) \\
&= \sum_{j=1}^{3} \psi_i(u_j e_j) \text{ by Question 2} \\
&= \sum_{j=1}^{3} \theta_{i,e_j}(u_j) \\
&= \sum_{j=1}^{3} B_{e_j} u_j \\
&= B.u
\end{aligned}$$

where $B = (B_{e_1}, B_{e_2}, B_{e_3})$.

Therefore we have the expected result.

Question 10. Conclude.

Solution 10. By definition of ψ_i and ψ_p, and Questions 7 and 9, we have

$$\psi(u) = \psi_p(u) + \psi_i(u) \ = A|u|^2 + B.u$$

But by definition of ψ, we have

$$\begin{aligned}\phi(u) &= \psi(u) + \phi(0) \text{ with } v = 0\\ &= A|u|^2 + B.u + C\end{aligned}$$

because we use a ϕ well-defined everywhere.