2019/2020

Master 2 Recherche fondamentale en mathématiques

Sobolev spaces's exercises

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1 Sobolev spaces

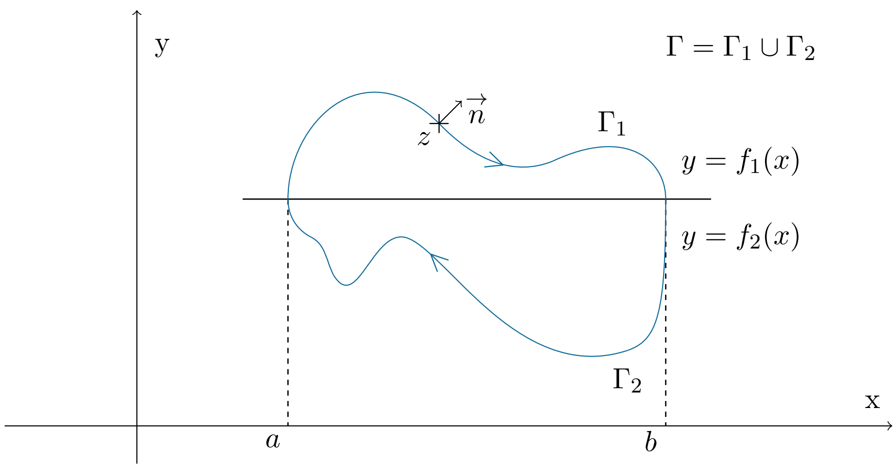
1.1 General framework

Exercise 1. $N = 2$

1) Define the outward unit vector at any point x

2) Define $\int_{\Gamma_i} g(x) d\sigma(x)$ for g smooth

3) Prove (IP) by using $1D$ integration by part and Fubini's theorem.



Solution. 1) The tangent of Γ_1 on x has $(1, f_1'(x))$ as director vector. Then, the normal vector is $(-f_1'(x), 1)$ because the inner product has to be zero. We want to have a outward unit vector, so we divide by the norm and find

$$\vec{n}^{(1)} = \left(\frac{-f_1'(x)}{\sqrt{(f_1'(x))^2 + 1}}, \frac{1}{\sqrt{(f_1'(x))^2 + 1}}\right)$$

Same method for Γ_2 and we find :

$$\vec{n}^{(2)} = \left(\frac{-f_2'(x)}{\sqrt{(f_2'(x))^2 + 1}}, \frac{1}{\sqrt{(f_2'(x))^2 + 1}}\right)$$

2) We parameterize Γ_1 with $(x, f_1(x))$ for $x \in [a, b]$. Then we use line integral.

$$\begin{aligned}\int_{\Gamma_1} g(x)d\sigma(x) &= \int_a^b g(f_1(x))||f_1'(x)||dx \\ &= \int_a^b g(f_1(x))\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{df_1(x)}{dx}\right)^2}dx \\ &= \int_a^b g(f_1(x))\sqrt{1 + (f_1'(x))^2}dx\end{aligned}$$

Same method for Γ_2 :

$$\int_{\Gamma_2} g(x)d\sigma(x) = \int_b^a g(f_2(x))\sqrt{1 + (f_2'(x))^2}dx$$

We change the bounds order because we go from b to a on the boundary Γ_2.

3) We use the Fubini theorem to split the open Ω

$$\int_\Omega u(x, y)\frac{\partial v}{\partial y}(x, y)dxdy = \int_{x=a}^b \left(\int_{y=f_2(x)}^{f_1(x)} u(x, y)\frac{\partial v}{\partial y}(x, y)dy\right) dx$$

Then we use integration by parts :

$$= \int_{x=a}^b \left([u(x, y)v(x, y)]_{f_2(x)}^{f_1(x)} - \int_{f_2(x)}^{f_1(x)} \frac{\partial u}{\partial y}(x, y)v(x, y)dy\right) dx$$

$$= \int_{x=a}^b \left(u(x, f_1(x))v(x, f_1(x)) - u(x, f_2(x))v(x, f_2(x)) - \int_{f_2(x)}^{f_1(x)} \frac{\partial u}{\partial y}(x, y)v(x, y)dy\right) dx$$

$$\begin{aligned}&= \int_{x=a}^b u(x, f_1(x))v(x, f_1(x))\frac{1}{\sqrt{1 + (f_1'(x))^2}}\sqrt{1 + (f_1'(x))^2}dx \\ &\quad - \int_{x=a}^b u(x, f_2(x))v(x, f_2(x))\frac{1}{\sqrt{1 + (f_2'(x))^2}}\sqrt{1 + (f_2'(x))^2}dx \\ &\quad - \int_{x=a}^b \int_{f_2(x)}^{f_1(x)} \frac{\partial u}{\partial y}(x, y)v(x, y)dydx \\ &= \int_{x=a}^b uvn_2^{(1)}\sqrt{1 + (f_1'(x))^2}dx \\ &\quad + \int_{x=b}^a uvn_2^{(2)}\sqrt{1 + (f_2'(x))^2}dx \\ &\quad - \int_{x=a}^b \int_{f_2(x)}^{f_1(x)} \frac{\partial u}{\partial y}(x, y)v(x, y)dydx \\ &= \int_{\Gamma_1} uvn_2^{(1)}d\sigma + \int_{\Gamma_2} uvn_2^{(2)}d\sigma \ - \int_\Omega \frac{\partial u}{\partial y}(x, y)v(x, y)dydx \\ &= \int_\Gamma uvn_2 d\sigma \ - \int_\Omega \frac{\partial u}{\partial y}(x, y)v(x, y)dydx\end{aligned}$$

We finally find the (IP) formula. (We justify the Fubini Theorem because $u, v \in \mathcal{C}^1$, then they are continuous on Ω compact, then $u\dfrac{\partial v}{\partial y} \in L^1(\Omega)$)

□

Exercise 2. Let $\Omega =]-1;1[\subset \mathbb{R}$.

$$\operatorname{sgn}(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \in L^\infty(\Omega)$$

Show that $x \mapsto \operatorname{sgn}(x)$ has no weak derivative.

Solution. We work in the distributions. For all $\varphi \in \mathcal{D}(]-1,1[)$,

$$\begin{aligned} \langle \operatorname{sgn}', \varphi \rangle &= -\langle \operatorname{sgn}, \varphi' \rangle \\ &= +\int_{-1}^{0} \varphi' - \int_{0}^{1} \varphi' \\ &= [\varphi]_{-1}^{0} - [\varphi]_{0}^{1} \\ &= 2\varphi(0) \\ &= \langle 2\delta_0, \varphi \rangle \end{aligned}$$

Then $\operatorname{sgn}' = 2\delta_0$. But this distribution isn't in L^1_{loc}, so $x \mapsto \operatorname{sgn}(x)$ has no weak derivative.

To prove that the dirac δ_0 is not in L^1_{loc}, we work for a contradiction. Let $f \in L^1_{loc}$ such that

$$\forall \varphi \in \mathcal{D}(]-1,1[), \quad \varphi(0) = \int_{-1}^{1} f\varphi$$

We use a sequence $\varphi_n \in \mathcal{D}(]-1,1[)$ such that $\operatorname{supp}(\varphi_n) \subset [-\frac{1}{n}, \frac{1}{n}]$, $\varphi_n(0) = 1$ and the supremum of each φ_n equal to 1.

Then we have by the dominated convergence theorem that $\int_{-1}^{1} f\varphi_n$ goes to 0 because we can dominate $|\mathbb{1}_{[-\frac{1}{n},\frac{1}{n}]} f\varphi_n|$ by $|f|\mathbb{1}_{[-1,1]}$ which is in L^1.

We have a contradiction because $\varphi_n(0) = 1$ for all $n \in \mathbb{N}$. □

Exercise 3. Let U_1 and U_2 be two opensets in $\mathbb{R}^n$ such that $U_1 \cap U_2 \neq \emptyset$. Let $\Omega = U_1 \cup U_2$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multiindex, and $u \in L^1_{loc}(\Omega)$

Assume that u has a weak derivative $v_1 = D^\alpha u$ in U_1 and $v_2 = D^\alpha u$ in U_2.

1) Prove that $v_1 = v_2$ in $U_1 \cap U_2$

2) Let $v = \begin{cases} v_1 \text{ in } U_1 \\ v_2 \text{ in } U_2 \end{cases}$.

   Prove that $D^\alpha u$ exists in Ω and $D^\alpha u = v$.

Solution. 1) Soit $\varphi \in \mathcal{D}(U_1 \cap U_2)$,

$$\langle u, D^\alpha \varphi \rangle = \int_{U_1 \cap U_2} u D^\alpha \varphi = -\int_{U_1 \cap U_2} v_1 \varphi$$

and

$$\langle u, D^\alpha \varphi \rangle = \int_{U_1 \cap U_2} u D^\alpha \varphi = -\int_{U_1 \cap U_2} v_2 \varphi$$

Then $\int_{U_1 \cap U_2} (v_2 - v_1)\varphi = 0$ for all $\varphi \in \mathcal{D}(U_1 \cap U_2)$. Hence, we have $v_1 = v_2$ in $U_1 \cap U_2$. We just use the injectivity of

$$\begin{cases} L^1_{loc}(\Omega) & \to \mathcal{D}'(\Omega) \\ f & \mapsto T_f \end{cases}$$

2) Let $\varphi \in \mathcal{D}(\Omega)$. We take a unit partition of $\Omega = U_1 \cup U_2$ relatively to $\operatorname{supp}(\varphi)$. Then we have $\theta_1, \theta_2 \in \mathcal{D}(\Omega)$ with $\theta_1 + \theta_2 = 1$, $0 \leqslant \theta_1, \theta_2 \leqslant 1$, $\operatorname{supp}(\theta_1) \subset U_1 \cap \operatorname{supp}(\varphi)$ and $\operatorname{supp}(\theta_2) \subset U_2 \cap \operatorname{supp}(\varphi)$

We have $\varphi = \theta_1 \varphi + \theta_2 \varphi$

$$\begin{aligned} \langle D^\alpha u, \varphi \rangle &= \langle D^\alpha u, \theta_1 \varphi \rangle + \langle D^\alpha u, \theta_2 \varphi \rangle \\ &= \int_{U_1} v_1 \theta_1 \varphi + \int_{U_2} v_2 \theta_2 \varphi \text{ because } \operatorname{supp}\theta_{1,2} \subset U_{1,2} \\ &= \int_{U_1} v \theta_1 \varphi + \int_{U_2} v \theta_2 \varphi \\ &= \int_{U_1 \cup U_2} v \theta_1 \varphi + \int_{U_1 \cup U_2} v \theta_2 \varphi \text{ by definition of } \theta_1 \text{ and } \theta_2 \text{ supports} \\ &= \int_{U_1 \cup U_2} v (\theta_1 + \theta_2) \varphi \\ &= \langle v, \varphi \rangle \end{aligned}$$

Hence $D^\alpha u$ exists in Ω and we have $D^\alpha u = v$ in Ω.

□

1.2 Definition and basic properties of $W^{m,p}(\Omega)$

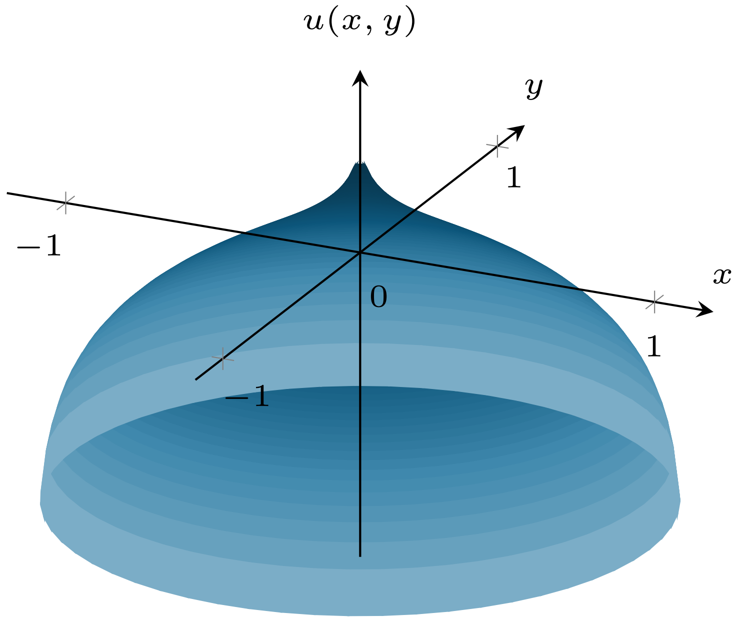
Exercise 4. Let $\Omega = B(0,1) \subset \mathbb{R}^2$.

$$u : \begin{cases} \Omega & \to \mathbb{R} \\ x & \mapsto \ln\left(\left| \ln \frac{1}{|x|} \right| \right) \end{cases}$$

Prove that $u \in H^1(\Omega)$ (but $u \notin \mathcal{C}^0(\Omega)$)

Remark :

I just show the result for $H^1(B(0,\frac{1}{2}))$ because I didn't succeed for $u' \in L^2(B(0,1))$ (since it's not true), but it keeps the spirit of the exercise because we have a function in $H^1(B(0,\frac{1}{2}))$ which is not in $\mathcal{C}^0(B(0,\frac{1}{2}))$.



Solution. The function u is not continuous in $(0,0)$, so $u \notin \mathcal{C}^0(\Omega)$. And we can't find a continuous representative of u (somebody continuous equal to u almost everywhere).

We want to prove that $u \in L^2(\Omega)$ and $\dfrac{\partial}{\partial x}u, \dfrac{\partial}{\partial y}u \in L^2(\Omega)$.

Step 1 : Let start with $u \in L^2(\Omega)$.

$$\begin{aligned}\int_\Omega |u(x)|^2 dx &= \int_{\rho=0}^{1}\int_{\theta=0}^{2\pi} \ln\left(\left|\ln\frac{1}{\rho}\right|\right)^2 \rho d\theta d\rho\\ &= 2\pi\int_0^1 \left(\ln\left(\ln\frac{1}{\rho}\right)\right)^2 \rho d\rho\end{aligned}$$

while doing a change of coordinates and seeing that $\ln\frac{1}{\rho}$ is non negative for $\rho \in]0,1[$.

We use the inequality $\ln(x) \leqslant x-1$ for $x \geqslant 0$.

$$\begin{aligned}\int_\Omega |u(x)|^2 dx &= 2\pi\int_0^1 \left(\ln\left(\frac{1}{\rho}\right)-1\right)^2 \rho d\rho\\ &= 2\pi\int_0^1 (-\ln(\rho)-1)^2\rho d\rho\\ &= 2\pi\int_0^1 (\ln(\rho)+1)^2\rho d\rho\end{aligned}$$

We use an other change of coordinates : $\rho = \exp(-x)$

$$\int_\Omega |u(x)|^2 dx = 2\pi \int_0^{+\infty} (1-x)^2 \mathrm{e}^{-2x} dx < +\infty$$

We know that $x \mapsto (1-x)^2\mathrm{e}^{-2x}$ is continuous on $\mathbb{R}^+$ and summable in $+\infty$ and 0. Hence,

$$u \in L^2(\Omega)$$

Step 2 : We want to show that $\nabla u \in L^2(B(0,\frac{1}{2}))$ $(B(0,\frac{1}{2})$ as I said in the previous remark).

Let $u(x,y) = \ln\left(\ln\left(\frac{1}{\sqrt{x^2+y^2}}\right)\right)$ for all $(x,y) \in B(0,\frac{1}{2})$.

Assume for the moment that the weak derivative is equal to the classical derivative.

We have

$$\frac{\partial}{\partial x}u(x,y) = \frac{-2x}{\frac{1}{\sqrt{x^2+y^2}}2(\sqrt{x^2+y^2})^{3/2}} . \frac{1}{\ln\left(\frac{1}{\sqrt{x^2+y^2}}\right)} = \frac{x}{(x^2+y^2)\ln(\sqrt{x^2+y^2})}$$

and

$$\frac{\partial}{\partial y}u(x,y) = \frac{y}{(x^2+y^2)\ln(\sqrt{x^2+y^2})}$$

Let compute the L^2 norm.

$$\begin{aligned}
&\int_{B(0,1/2)} \frac{\partial}{\partial x}u(x,y)^2 dxdy + \int_{B(0,1/2)} \frac{\partial}{\partial y}u(x,y)^2 dxdy \\
=&\int_{B(0,1/2)} \frac{x^2}{(x^2+y^2)^2\ln^2(\sqrt{x^2+y^2})} + \frac{y^2}{(x^2+y^2)^2\ln^2(\sqrt{x^2+y^2})} dxdy \\
=&\int_{B(0,1/2)} \frac{1}{(x^2+y^2)\ln^2(\sqrt{x^2+y^2})} dxdy \\
=&\int_{\theta=0}^{2\pi}\int_{\rho=0}^{1/2} \frac{1}{\rho^2\ln^2\rho}\rho d\rho d\theta \\
=&2\pi\int_{\rho=0}^{1/2} \frac{1}{\rho\ln^2\rho} d\rho \quad \text{we use the change of variables } \rho = \mathrm{e}^x \\
=&2\pi\int_{x=-\infty}^{-\ln(2)} \frac{\mathrm{e}^x}{\mathrm{e}^x x^2} dx \quad \text{because } d\rho = \mathrm{e}^x dx \\
=&2\pi\int_{x=-\infty}^{-\ln(2)} \frac{1}{x^2} dx < +\infty
\end{aligned}$$

We see in the last line why I work in $B(0,\frac{1}{2})$.

It proves that $\nabla u \in L^2(B(0,\frac{1}{2}))$ and then $u \in H^1(B(0,\frac{1}{2}))$.

Step 3 : We have to justify that the weak derivative is equal to the classical derivative.

Let take $\varphi \in \mathcal{D}(B(0,1))$,

$$\begin{aligned}
\langle \frac{\partial}{\partial x}u, \varphi\rangle &= -\langle u, \frac{\partial}{\partial x}\varphi\rangle \\
&= -\int_{B(0,1)} u\frac{\partial}{\partial x}\varphi \\
&= -\lim_{\epsilon\to 0}\int_{B(0,1)\backslash B(0,\epsilon)} u\frac{\partial}{\partial x}\varphi
\end{aligned}$$

We can apply the Green Formula

$$\int_{B(0,1)\backslash B(0,\epsilon)} u\frac{\partial}{\partial x}\varphi = -\int_{B(0,1)\backslash B(0,\epsilon)} \frac{\partial}{\partial x}u\varphi + \int_{S(0,\epsilon)} u\varphi n_x d\sigma + \int_{S(0,1)} u\varphi n_x d\sigma$$

But $\varphi \in \mathcal{D}(B(0,1))$, so φ vanishes on $S(0,1)$[1].

And $\left|\int_{S(0,\epsilon)} u\varphi n_x d\sigma\right| \leqslant \|u\|_{\infty,S(0,\epsilon)}\|\varphi\|_{\infty,S(0,\epsilon)}2\pi\epsilon$ because $2\pi\epsilon$ is the perimeter of a cercle with radius 1. This quantity goes to zero when ϵ goes to zero.

It follows that

$$\begin{aligned}\langle\frac{\partial}{\partial x}u,\varphi\rangle &= -\lim_{\epsilon\to 0} -\int_{B(0,1)\backslash B(0,\epsilon)} \frac{\partial}{\partial x}u\varphi \\ &= \lim_{\epsilon\to 0}\int_{B(0,1)\backslash B(0,\epsilon)} \frac{\partial}{\partial x}u\varphi \\ &= \lim_{\epsilon\to 0}\int_{B(0,1)\backslash B(0,\epsilon)} \frac{x}{(x^2+y^2)\ln(\sqrt{x^2+y^2})}\varphi(x,y)dxdy \\ &= \int_{B(0,1)} \frac{x}{(x^2+y^2)\ln(\sqrt{x^2+y^2})}\varphi(x,y)dxdy\end{aligned}$$

while using the dominated convergence theorem.

We can do the same method for $\frac{\partial}{\partial y}u$, then the weak derivative is equal to the classical one. □

Exercise 5. Let $(r_i)_i$ be a countable and dense set in $B(0,1) \subset \mathbb{R}^N$, $\alpha > 0$.

$$u(x) = \sum_{i=0}^{+\infty} \frac{1}{2^i}|x - r_i|^{-\alpha}$$

For $p > 1$, prove that $u \in W^{1,p}(B(0,1))$ for all $\alpha < \alpha_0$ with $\alpha_0 = \alpha_0(N,p)$ to be calculated.

(*Hint : start by studying* $x \mapsto \frac{1}{|x|^\alpha}$)

Remark :

I will prove the result for $\alpha_0 = \frac{N}{p} - 1$ and that is a sufficient condition as the exercise asks. But it's a necessary and sufficient condition.

Solution.

Step 1 : We want to show that $\int_{B(0,1)} \frac{1}{|x|^\alpha}dx < \infty$ if and only if $\alpha < N$. To compute $\int_{B(0,1)} \frac{1}{|x|^\alpha}dx$, we use a spherical coordinates change in dimension N.

[1] $S(0,1)$ represents the sphere of radius 1

$$
\begin{aligned}
x_1 &= \rho\cos(\theta_1)\\
x_2 &= \rho\sin(\theta_1)\cos(\theta_2)\\
x_3 &= \rho\sin(\theta_1)\sin(\theta_2)\cos(\theta_3)\\
\vdots &\qquad \vdots\\
x_{N-1} &= \rho\sin(\theta_1)\dots\sin(\theta_{N-2})\cos(\theta_{N-1})\\
x_N &= \rho\sin(\theta_1)\dots\sin(\theta_{N-2})\sin(\theta_{N-1})
\end{aligned}
$$

The spherical volume element is

$$dV = \rho^{N-1}\sin(\theta_1)^{N-2}\sin(\theta_2)^{N-3}\dots\sin(\theta_{N-3})^2\sin(\theta_{N-2})d\rho d\theta_1\dots d\theta_{N-1}$$

$$
\begin{aligned}
&\int_{B(0,1)}\frac{1}{|x|^\alpha}dx\\
=&\int_{\rho=0}^{1}\int_{\theta_{1,\dots,N-2}=0}^{\pi}\int_{\theta_{N-1}=0}^{2\pi}\frac{1}{\rho^\alpha}dV
\end{aligned}
$$

But the following quantity is finite, let call it K.

$$K = \int_{\theta_{1,\dots,N-2}=0}^{\pi}\int_{\theta_{N-1}=0}^{2\pi}\sin(\theta_1)^{N-2}\sin(\theta_2)^{N-3}\dots\sin(\theta_{N-3})^2\sin(\theta_{N-2})d\theta_1\dots d\theta_{N-1}$$

Then the computation gives

$$
\begin{aligned}
&\int_{B(0,1)}\frac{1}{|x|^\alpha}dx\\
=&K\int_{\rho=0}^{1}\frac{1}{\rho^\alpha}\rho^{N-1}d\rho\\
=&K\int_{\rho=0}^{1}\frac{1}{\rho^{\alpha-N+1}}d\rho
\end{aligned}
$$

The integral converges if and only if $\alpha - N + 1 < 1$. Then

$$\int_{B(0,1)}\frac{1}{|x|^\alpha}dx < \infty \text{ if and only if } \alpha < N$$

Step 2 : Let introduce $f_\alpha : \mathbb{R}^N \to \mathbb{R}$ such that $f(x) = \dfrac{1}{|x|^\alpha}$. Thank to the step 1, we have $f_\alpha \in L^p(B(0,1))$ if and only if $p\alpha < N$.

So we have $f_\alpha \in L^p(B(0,1))$ if and only if $\alpha < \dfrac{N}{p}$.

Step 3 : We want to show that $u \in L^p(B(0,1))$ if $\alpha < \frac{N}{p}$.

We start by using the Fatou's Lemma with the functions

$$h_n(x) = \left(\sum_{i=0}^{n} \frac{1}{2^i} |x - r_i|^{-\alpha} \right)^p$$

We have

$$\int \liminf h_n \leqslant \liminf \int h_n$$

While passing to the power $\frac{1}{p}$, we have

$$\left(\int \liminf h_n \right)^{\frac{1}{p}} \leqslant \liminf \left(\int h_n \right)^{\frac{1}{p}}$$

$$\left(\int \liminf \left(\sum_{i=0}^{n} \frac{1}{2^i} |x - r_i|^{-\alpha} \right)^p dx \right)^{\frac{1}{p}} \leqslant \liminf \left(\int \left(\sum_{i=0}^{n} \frac{1}{2^i} |x - r_i|^{-\alpha} \right)^p dx \right)^{\frac{1}{p}}$$

$$\left(\int \left(\sum_{i=0}^{+\infty} \frac{1}{2^i} |x - r_i|^{-\alpha} \right)^p dx \right)^{\frac{1}{p}} \leqslant \liminf \left\| \sum_{i=0}^{n} \frac{1}{2^i} f_\alpha(. - r_i) \right\|_p$$

$$\|u\|_p \leqslant \liminf \sum_{i=0}^{n} \frac{1}{2^i} \|f_\alpha(. - r_i)\|_p$$

$$\|u\|_p \leqslant \sum_{i=0}^{+\infty} \frac{1}{2^i} \|f_\alpha(. - r_i)\|_p$$

We used the Minkowski inequality to get out the sum of the norm L^p.

But thanks to step 2, while using a translation of r_i, we have

$$\|f_\alpha(. - r_i)\|_p < +\infty \text{ if and only if } \alpha < \frac{N}{p}$$

And all those quantities are dominated by the same function $g = f_\alpha$ on $B(0,2)$, so

$$\sum_{i=0}^{+\infty} \frac{1}{2^i} \|f_\alpha(. - r_i)\|_p < +\infty$$

Then $u \in L^p$ if $\alpha < \dfrac{N}{p}$.

Step 4 : We want to show that $u' \in L^p(B(0,1))$.

We start to find the derivative of $f_\alpha(. - r)$ for $r = (r_1, \ldots, r_N) \in B(0,1)$. For the same reasons of the Exercise 4 Step 3, the weak derivative is equal to the classical derivative.

$$\begin{aligned} \frac{\partial}{\partial x_k} f_\alpha(. - r) &= \frac{\partial}{\partial x_k} \frac{1}{(\sum (x_j - r_j)^2)^{\alpha/2}} \\ &= \frac{-\frac{\alpha}{2} 2(x_k - r_k) \left(\sum (x_j - r_j)^2\right)^{\frac{\alpha}{2} - 1}}{|x - r|^{2\alpha}} \\ &= \frac{-\alpha(x_k - r_k)}{|x - r|^{\alpha + 2}} \end{aligned}$$

Then we have $\left|\dfrac{\partial}{\partial x_k} f_\alpha(. - r)\right| < \dfrac{\alpha}{|x-r|^{\alpha+1}}$. And it's in L^p if $\alpha+1 < \frac{N}{p}$. (because of the Step 2)

We want to show that

$$\frac{\partial}{\partial x_k}\left(\sum_{i=0}^{+\infty}\frac{1}{2^i} f_\alpha(. - r_i)\right) = \sum_{i=0}^{+\infty}\frac{1}{2^i}\frac{\partial}{\partial x_k} f_\alpha(. - r_i) \tag{1}$$

For all $\varphi \in \mathcal{D}(B(0,1))$,

$$\begin{aligned}
\left\langle \frac{\partial}{\partial x_k}\left(\sum_{i=0}^{+\infty}\frac{1}{2^i} f_\alpha(. - r_i)\right), \varphi \right\rangle &= \left\langle \sum_{i=0}^{+\infty}\frac{1}{2^i} f_\alpha(. - r_i), \frac{\partial}{\partial x_k}\varphi \right\rangle \\
&= \int \sum_{i=0}^{+\infty}\frac{1}{2^i} f_\alpha(. - r_i)\frac{\partial}{\partial x_k}\varphi \\
&= \sum_{i=0}^{+\infty}\frac{1}{2^i}\int f_\alpha(. - r_i)\frac{\partial}{\partial x_k}\varphi \\
&= \sum_{i=0}^{+\infty}\frac{1}{2^i}\left\langle f_\alpha(. - r_i), \frac{\partial}{\partial x_k}\varphi \right\rangle \\
&= \sum_{i=0}^{+\infty}\frac{1}{2^i}\left\langle \frac{\partial}{\partial x_k} f_\alpha(. - r_i), \varphi \right\rangle
\end{aligned}$$

The justification of the change of $\int$ and $\sum$ is because of the dominated convergence theorem.

$$\sum_{i=0}^{n}\frac{1}{2^i} f_\alpha(. - r_i)\frac{\partial}{\partial x_k}\varphi \xrightarrow[n\to\infty]{} \sum_{i=0}^{+\infty}\frac{1}{2^i} f_\alpha(. - r_i)\frac{\partial}{\partial x_k}\varphi \quad a.e.$$

and

$$\left|\sum_{i=0}^{n}\frac{1}{2^i} f_\alpha(. - r_i)\frac{\partial}{\partial x_k}\varphi\right| \leqslant \left\|\frac{\partial}{\partial x_k}\varphi\right\|_\infty \underbrace{\mathbb{1}_{\text{Supp }\varphi}\underbrace{\sum_{i=0}^{\infty}\frac{1}{2^i} f_\alpha(. - r_i)}_{\in L^p \text{ (Step 3)}}}_{\in L^1(\text{Supp }\varphi)}$$

Then it justifies the egality (1) and so, thanks the same methods that Step 3 and the beginning of Step 4, we have

$$\left\|\frac{\partial}{\partial x_k}\left(\sum_{i=0}^{+\infty}\frac{1}{2^i} f_\alpha(. - r_i)\right)\right\|_p < +\infty \text{ if and only if } \alpha < \frac{N}{p} - 1$$

Step 5 : Conclusion

Thanks to Step 3 and 4, we have $u \in W^{1,p}(B(0,1))$ for all $\alpha < \alpha_0$ with $\alpha_0 = \dfrac{N}{p} - 1$.

□

1.3 Duality spaces $W^{-m,p}$

Exercise 6. Let $\alpha = (\alpha_1, \dots, \alpha_N), x = (x_1, \dots, x_N)$. Case of $\Omega = \mathbb{R}^N$.

$$\mathcal{S} := \{u \in L^2(\mathbb{R}^N) \text{ such that } x^\alpha D^\beta u \in L^2(\mathbb{R}^N) \forall \alpha, \beta \text{ multiindex}\}$$

Show that

$$H^m(\mathbb{R}^N) = \{u \in \mathcal{S}', \text{ such that } (1+|\xi|^2)^{m/2}\hat{u} \in L^2(\mathbb{R}^N)\}$$

where $\hat{u}$ is the Fourier transform of u.

Solution. We recall that

$$H^m(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N), D^\alpha u \in L^2(\mathbb{R}^N), \forall |\alpha| \leqslant m\}$$

Step 1 : We want to remark that $\mathcal{F}(D^\alpha u)(\xi) = i^{|\alpha|}\xi^\alpha \mathcal{F}(u)(\xi)$ for $u \in \mathcal{S}$, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, $\xi \in \mathbb{R}^N$ with $|\alpha| = \sum_i \alpha_i$ and $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_N^{\alpha_N}$.

For $\varphi \in \mathcal{S}$,

$$\begin{aligned}
\langle \mathcal{F}(D^\alpha u), \varphi\rangle &= \langle D^\alpha u, \varphi\rangle \\
&= (-1)^{|\alpha|}\langle u, D^\alpha(\mathcal{F}\varphi)\rangle \\
&= (-1)^{|\alpha|}\langle u, (-i)^{|\alpha|}\xi^\alpha \mathcal{F}\varphi\rangle \\
&= i^{|\alpha|}\xi^\alpha\langle u, \mathcal{F}\varphi\rangle \\
&= i^{|\alpha|}\xi^\alpha\langle \mathcal{F}u, \varphi\rangle
\end{aligned}$$

Then we have $\mathcal{F}(D^\alpha u)(\xi) = i^{|\alpha|}\xi^\alpha \mathcal{F}(u)(\xi)$.

Step 2 : We want to prove the direct inclusion.

Let $u \in H^m(\mathbb{R}^N)$, then we have by Fourier–Plancherel, $\mathcal{F}(D^\alpha u) \in L^2$ for all α such that $|\alpha| \leqslant m$. But $D^\alpha u \in \mathcal{S}'$, so $i^{|\alpha|}\xi^\alpha \hat{u} \in L^2$ (by Step 1).

$$\begin{aligned}
\int_{\mathbb{R}^N} \left((1+|\xi|^2)^{\frac{m}{2}}\hat{u}(\xi)\right)^2 d\xi &= \int_{\mathbb{R}^N} (1+|\xi|^2)^m \hat{u}^2(\xi) d\xi \\
&= \int_{\mathbb{R}^N} \sum_{k=0}^{m} \binom{m}{k} |\xi|^{2k}\hat{u}^2(\xi) d\xi \\
&= \sum_{k=0}^{m} \binom{m}{k} \underbrace{\int_{\mathbb{R}^N} |\xi|^{2k}\hat{u}^2(\xi) d\xi}_{<\infty} \text{ because } k \leqslant m
\end{aligned}$$

Since u is in $\mathcal{S}'$ (because $L^2 \subset \mathcal{S}'$), then

$$u \in \{u \in \mathcal{S}', \text{ such that } (1+|\xi|^2)^{m/2}\hat{u} \in L^2(\mathbb{R}^N)\}$$

Step 3 : We want to prove the other inclusion.

Let $u \in \mathcal{S}'$ such that $(1+|\xi|^2)^{m/2}\hat{u} \in L^2(\mathbb{R}^N)$.

Let $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leqslant m$.

We know that

$$(1+|\xi|^2)^m \hat{u}^2(\xi) \in L^1(\mathbb{R}^N),$$

then

$$(1+|\xi|^2)^\alpha \hat{u}^2(\xi) \in L^1(\mathbb{R}^N).$$

It follows that

$$|\xi|^{2\alpha} \hat{u}^2(\xi) \in L^1(\mathbb{R}^N),$$

then

$$i^{|\alpha|}\xi^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^N).$$

We recognize the Fourier Transform of $D^\alpha u$, and, by Fourier–Plancherel, we conclude that

$$D^\alpha u \in L^2(\mathbb{R}^N).$$

Therefore $u \in H^m(\mathbb{R}^N)$. □

1.4 Study of $W^{1,p}(\Omega)$

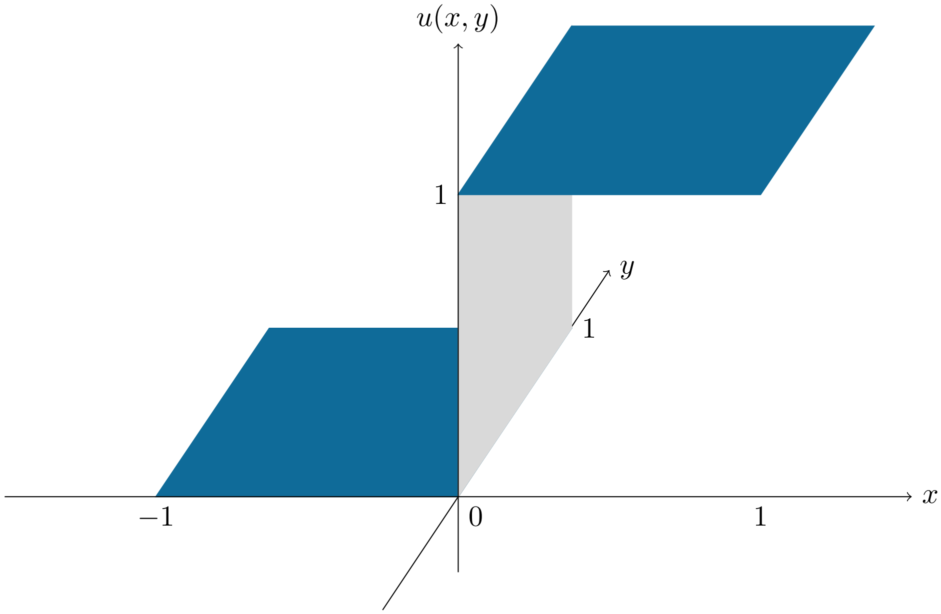
Exercise 7. Let $\Omega = \{(x,y), 0 < |x| < 1, 0 < y < 1\} \subset \mathbb{R}^2$.

Let $u(x,y) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

1) Show that $u \in W^{1,p}(\Omega)$, $\forall p \geqslant 1$

2) Show there is $\epsilon > 0$, such that there is no function $\phi \in \mathcal{C}^1(\overline{\Omega})$ such that $\|u - \phi\|_{1,p} < \epsilon$.

3) What's up ?



Solution. 1) **Step 1 :** We want to show that $u \in L^p(\Omega)$.

$$\int_{x=-1}^{0}\int_{y=0}^{1} \underbrace{|u(x,y)|^p}_{=0}\, dydx + \int_{x=0}^{1}\int_{y=0}^{1} \underbrace{|u(x,y)|^p}_{=1}\, dydx = \int_{x=0}^{1}\int_{y=0}^{1} 1dydx = 1$$

Then $u \in L^p(\Omega)$.

Step 2 : We want to show that $Du \in L^p(\Omega)$.

Let $\varphi \in \mathcal{D}(\Omega)$, then since the support of φ is compact in Ω, we have $\varphi(0,y) = \varphi(1,y) = 0$ for all $y \in [0,1]$ and $\varphi(x,1) = \varphi(x,0) = 0$ for all $x \in [0,1]$.

$$\begin{aligned}
\langle \partial_y u, \varphi \rangle &= -\langle u, \partial_y \varphi \rangle \\
&= -\int_\Omega u \partial_y \varphi \\
&= -\int_{x=0}^{1}\int_{y=0}^{1} \partial_y \varphi(x,y) dydx \\
&= -\int_{x=0}^{1} (\varphi(x,1) - \varphi(x,0)) dx \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle \partial_x u, \varphi \rangle &= -\langle u, \partial_x \varphi \rangle \\
&= -\int_\Omega u \partial_x \varphi \\
&= -\int_{y=0}^{1}\int_{x=0}^{1} \partial_x \varphi(x,y) dxdy \\
&= -\int_{y=0}^{1} (\varphi(1,y) - \varphi(0,y)) dy \\
&= 0
\end{aligned}$$

Then $\partial_x u(x,y) = \partial_y u(x,y) = 0$ for all $(x,y) \in \Omega$.
So $\partial_x u, \partial_y u \in L^p(\Omega)$ and therefore $u \in W^{1,p}(\Omega)$.

2) We want to show that there exists ε such that for all $\phi \in \mathcal{C}^1(\overline{\Omega})$,

$$\|u - \phi\|_{1,p,\Omega} \geqslant \varepsilon.$$

We work for a contradiction. We suppose that

$$\forall n > 0, \exists \phi_n \in \mathcal{C}^1(\overline{\Omega}) \;\; s.t. \;\; \|u - \phi_n\|_{1,p,\Omega} \leqslant \frac{1}{n}$$

So we have a sequence $(\phi_n)_n$ such that

$$\phi_n \xrightarrow[n\to\infty]{W^{1,p}(\Omega)} u$$

It means that

$$\phi_n \xrightarrow[n\to\infty]{L^p(\Omega)} u \quad \text{and} \quad \nabla\phi_n \xrightarrow[n\to\infty]{L^p(\Omega)} 0 \tag{2}$$

Because $\partial_x u(x,y) = \partial_y u(x,y) = 0$ for all $(x,y) \in \Omega$.

By the Lebesgue Inverse Theorem, we have a subsequence, still called ϕ_n, that converges to u almost everywhere in Ω.

We have for almost everywhere $y \in]0,1[, x_1 < 0, x_2 > 0$

$$\begin{cases} \phi_n(x_1,y) \xrightarrow[n\to\infty]{} u(x_1,y) = 0 \\ \phi_n(x_2,y) \xrightarrow[n\to\infty]{} u(x_2,y) = 1 \end{cases}$$

Then

$$|\phi_n(x_1,y) - \phi_n(x_2,y)| \xrightarrow[n\to\infty]{} |u(x_1,y) - u(x_2,y)| = 1 \tag{3}$$

On the other hand,

$$\begin{aligned} |\phi_n(x_1,y) - \phi_n(x_2,y)| &\leqslant \int_0^1 |D\phi_n(tx_1 + (1-t)x_2, y).(x_1 - x_2, 0)|dt \\ &\leqslant \int_0^1 \|D\phi_n(tx_1 + (1-t)x_2, y)\|\|(x_1 - x_2, 0)\|dt \text{ by Cauchy–Schwarz} \\ &\leqslant \|(x_1 - x_2, 0)\| \left(\int_0^1 \|D\phi_n(tx_1 + (1-t)x_2, y)\|^p dt\right)^{1/p} \text{ by Hölder} \end{aligned}$$

We use a convexity inequality

$$\left(\frac{a^2 + b^2}{2}\right)^{p/2} \leqslant \frac{x^p + y^p}{2}$$

In other words,

$$(a^2 + b^2)^{p/2} \leqslant 2^{\frac{p}{2}-1}(a^p + b^p)$$

$$\int_0^1 \|D\phi_n(tx_2 + (1-t)x_2, y)\|^p dt \tag{4}$$

$$= \int_0^1 (\partial_x \phi_n^2(tx_1 + (1-t)x_2, y) + \partial_y \phi_n^2(tx_1 + (1-t)x_2, y))^{p/2} dt \tag{5}$$

$$\leqslant C \int_0^1 \partial_x \phi_n^p(tx_1 + (1-t)x_2, y) + \partial_y \phi_n^p(tx_1 + (1-t)x_2, y) dt \tag{6}$$

We use Fatou to have

$$\underbrace{\int_0^1 \liminf |\phi_n(x_1,y)-\phi_n(x_2,y)|dy}_{=1 \text{ by } (3)} \leqslant \liminf \int_0^1 |\phi_n(x_1,y)-\phi_n(x_2,y)|dy \tag{7}$$

By (6),

$$\int_0^1 |\phi_n(x_1,y)-\phi_n(x_2,y)|dy \leqslant C\int_0^1\int_0^1 \partial_x\phi_n^p(tx_1+(1-t)x_2,y)+\partial_y\phi_n^p(tx_1+(1-t)x_2,y)dtdy$$
$$\leqslant C\int_A \|\nabla\phi_n\|p \xrightarrow[n\to+\infty]{} 0 \text{ because of (2)}$$

where $A=\{(x,y), x_1<x<x_2, 0<y<1\}$

Therefore, by (7), we have a contradiction.

To conclude, we have that there exists ε such that for all $\phi\in\mathcal{C}^1(\overline{\Omega})$,

$$\|u-\phi\|_{1,p,\Omega}\geqslant\varepsilon.$$

3) We have a theorem that says : If Ω is of class $\mathcal{C}^1$ and $u\in W^{1,p}(\Omega)$, then there exists a sequence $(u_n)_n$ of functions in $\mathcal{D}(\mathbb{R}^N)$ such that

$$u_{n|\Omega} \xrightarrow[n\to\infty]{W^{1,p}(\Omega)} u$$

The problem is that, in our exercise, Ω is not of class $\mathcal{C}^1$ and then we can't use that theorem.

□

1.4.1 Density results

Exercise 8. 1) Show that $\forall v\in H^m(\mathbb{R}^N), \forall m\geqslant 1$,

$$\|v\star\rho_\epsilon - v\|_{m-1,2}\leqslant C\epsilon\|v\|_{m,2}$$

2) Show that $\forall v\in H^m(\mathbb{R}^N), \forall k>0$,

$$\|v\star\rho_\epsilon\|_{m+k,2}\leqslant\frac{C_{m,k}}{\epsilon^k}\|v\|_{m,2}$$

3) Show that $\forall v\in H^m(\mathbb{R}^N), \forall k>0, \forall|\alpha|\leqslant k$

$$\|\rho_\epsilon\star D^\alpha v\|_{0,\infty}\leqslant\frac{C_k}{\epsilon^{N/2+k}}\|v\|_{0,2}$$

Remark :

In this exercise, for the purposes of notation, I will write sometimes $\|f(x)\|_{L^2}$ instead of $\|f\|_{L^2}$ or $\|f(\cdot)\|_{L^2}$. It is just because I want to mention the variables sometimes.

Moreover, there is a constant while using Fourier Plancherel due to my definition of the Fourier transform, but I will never put it (as it was 1). It doesn't matter because it will be in the constant C of the exercise.

Solution. 1) $\forall |\alpha| \leqslant m-1$, $\forall v \in H^m(\mathbb{R}^N)$

$$\begin{aligned}\|D^\alpha(v \star \rho_\epsilon - v)\|_{L^2} &= \|\mathcal{F}(D^\alpha(v \star \rho_\epsilon - v))\|_{L^2} \text{ with Fourier Plancherel}\\ &= \|\xi^\alpha \mathcal{F}(v \star \rho_\epsilon - v)\|_{L^2} \text{ by properties of } \mathcal{F}\\ &= \|\xi^\alpha \mathcal{F}(v)(\mathcal{F}(\rho_\epsilon) - 1)\|_{L^2} \text{ because of the convolution product}\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\rho_\epsilon)(\xi) &= \int_{\mathbb{R}^N} \rho_\epsilon(x) \mathrm{e}^{-i\xi x} dx\\ &= \frac{1}{\epsilon^N} \int_{\mathbb{R}^N} \rho\left(\frac{x}{\epsilon}\right) \mathrm{e}^{-i\xi x} dx\\ &= \frac{1}{\epsilon^N} \int_{\mathbb{R}^N} \rho(u) \mathrm{e}^{-i\xi\epsilon u} \epsilon^N du \text{ change of variable } x = \epsilon u\\ &= \mathcal{F}(\rho)(\epsilon\xi)\end{aligned}$$

So we have

$$\mathcal{F}(\rho_\epsilon)(\xi) = \mathcal{F}(\rho)(\epsilon\xi) \tag{8}$$

We remark that $1 = \mathcal{F}(\rho)(0)$ because $\int_{\mathbb{R}^N} \rho(x)dx = 1$.

Now we use the mean value theorem in $\mathbb{R}^N$, we have

$$\exists \theta \in]0,1[,\ s.t.\ \ \mathcal{F}(\rho_\epsilon) - 1 = \mathcal{F}(\rho)(\epsilon\xi) - \mathcal{F}(\rho)(0) = D\mathcal{F}(\rho)(\epsilon\xi\theta).(\epsilon\xi)$$

But $D\mathcal{F}(\rho)(\epsilon\xi\theta).(\epsilon\xi)$ is bounded by $C|\epsilon|\xi$ where C is a constant.[2]

To conclude we have

$$\begin{aligned}\|D^\alpha(v \star \rho_\epsilon - v)\|_{L^2} &\leqslant C\epsilon \|\xi^\alpha \xi \mathcal{F}(v)\|_{L^2}\\ &\leqslant C\epsilon \|\xi^{\alpha+1} \mathcal{F}(v)\|_{L^2}\\ &\leqslant C\epsilon \|\mathcal{F}(D^{\alpha+1} v)\|_{L^2}\\ &\leqslant C\epsilon \|D^{\alpha+1} v\|_{L^2}\\ &\leqslant C\epsilon \|v\|_{m,2} \text{ because } |\alpha| + 1 \leqslant m\end{aligned}$$

So we have the result. We just used the notation $\alpha + 1$, we have to understand that for instance we write the mean value theorem for the first derivative, then the "+1" is on the first component of α.

[2]Indeed, $D\mathcal{F}(\rho) = \mathcal{F}(x\rho)$ is bounded, because it's the Fourier transform of a $\mathcal{C}_0^\infty$ function, so we can bound it by the L^1-norm.

2) $\forall |\alpha| \leqslant m$, $\forall v \in H^m(\mathbb{R}^N)$, $\forall |\beta| \leqslant k$

$$\begin{aligned}
\|D^{\alpha+\beta}(v\star\rho_\epsilon)\|_{L^2} &= \|\mathcal{F}(D^{\alpha+\beta}(v\star\rho_\epsilon))\|_{L^2} \text{ with Fourier Plancherel}\\
&= \|\xi^{\alpha+\beta}\mathcal{F}(v\star\rho_\epsilon)\|_{L^2} \text{ by properties of } \mathcal{F}\\
&= \|\xi^{\alpha+\beta}\mathcal{F}(v)\mathcal{F}(\rho_\epsilon)\|_{L^2} \text{ by properties of } \mathcal{F} \text{ and convolution product}\\
&= \|\xi^{\alpha}\mathcal{F}(v)\frac{(\epsilon\xi)^\beta}{\epsilon^{|\beta|}}\mathcal{F}(\rho)(\epsilon\xi)\|_{L^2} \text{ because (8)}\\
&= \frac{1}{\epsilon^k}\|\mathcal{F}(D^\alpha v)\mathcal{F}(D^\beta\rho)(\epsilon\xi)\|_{L^2} \text{ because } |\beta|\leqslant k\\
&\leqslant \frac{C}{\epsilon^k}\|\mathcal{F}(D^\alpha v)\|_{L^2}\\
&\leqslant \frac{C}{\epsilon^k}\|D^\alpha v\|_{L^2}\\
&\leqslant \frac{C}{\epsilon^k}\|v\|_{m,2} \text{ because } |\alpha|\leqslant m
\end{aligned}$$

So we have the result. We just used C the bound of $\mathcal{F}(D^\beta\rho)$ that exists since $\rho\in\mathcal{C}^\infty$.

3) First we will use the following inequality

$$\forall f,g\in L^2,\quad |f\star g(x)|\leqslant \|f\|_{L^2}\|g\|_{L^2}$$

To prove that inequality, we use the definition of the convolution product and the Cauchy–Schwarz inequality.

So we have, for $|\alpha|\leqslant k$,

$$\begin{aligned}
|\rho_\epsilon\star D^\alpha v(x)| &= |D^\alpha(\rho_\epsilon)\star v(x)| && (9)\\
&\leqslant \|D^\alpha\rho_\epsilon\|_{L^2}\|v\|_{0,2} && (10)
\end{aligned}$$

We want to prove that $\|D^\alpha\rho_\epsilon\|_{L^2}\leqslant \dfrac{C_k}{\epsilon^{N/2+k}}$.

$$\begin{aligned}
\|D^\alpha(\rho_\epsilon)\|_{L^2}^2 &= \int_{\mathbb{R}^N} D^\alpha(\rho_\epsilon)(x)^2dx\\
&= \int_{\mathbb{R}^N} \widehat{D^\alpha(\rho_\epsilon)}(\xi)^2d\xi \text{ by Fourier Plancherel}\\
&= \int_{\mathbb{R}^N} \xi^{2\alpha}\widehat{\rho}_\epsilon(\xi)^2d\xi\\
&= \int_{\mathbb{R}^N} \xi^{2\alpha}\widehat{\rho}(\epsilon\xi)^2d\xi \text{ because (8)}\\
&= \int_{\mathbb{R}^N} \frac{u^{2\alpha}}{\epsilon^{2|\alpha|}}\widehat{\rho}(u)^2\frac{1}{\epsilon^N}du \text{ because of the change of variable } \epsilon\xi=u\\
&= \frac{1}{\epsilon^{2k}\epsilon^N}\|u^\alpha\mathcal{F}(\rho)(u)\|_{L^2}^2
\end{aligned}$$

So we have

$$\begin{aligned}\|D^\alpha(\rho_\epsilon)\|_{L^2} &= \frac{1}{\epsilon^k \epsilon^{N/2}}\|\mathcal{F}(D^\alpha\rho)\|_{L^2} \\ &\leqslant \frac{1}{\epsilon^{N/2+k}}\|D^\alpha\rho\|_{L^2} \\ &\leqslant \frac{1}{\epsilon^{N/2+k}} \underbrace{\sup_{|\alpha|\leqslant k}\|D^\alpha\rho\|_{L^2}}_{C_k}\end{aligned}$$

Then, since (10) and while passing to the supremum, we have the result

$$\|\rho_\epsilon \star D^\alpha v\|_{L^\infty} \leqslant \frac{C_k}{\epsilon^{N/2+k}}\|v\|_{L^2}.$$

□

1.4.2 About traces

Exercise 9. Let $\vec{u} = (u_1, \ldots, u_N) \in \mathcal{D}(\Omega)^N$ with Ω bounded of class $\mathcal{C}^1$. We recall that $\operatorname{div} \vec{u} = \sum_i D_i u_i$. We define

$$H^p_{\operatorname{div}}(\Omega) = \{\vec{u} \in L^p(\Omega)^N, \operatorname{div} \vec{u} \in L^p(\Omega)\}$$

1) Show that $H^p_{\operatorname{div}}(\Omega)$ is a Banach space.

2) Prove that $\vec{u}.\vec{n}_{|\Gamma}$ can be defined in an appropriate space to be determined. ($\vec{u}.\vec{n}_{|\Gamma}$ = "normal trace of u")

Solution. 1) We put on $H^p_{\operatorname{div}}(\Omega)$ the norm

$$\|\vec{u}\|_{\operatorname{div}} := \sum_{i=1}^{N} \|u_i\|_{L^p(\Omega)} + \|\operatorname{div} \vec{u}\|_{L^p(\Omega)}$$

We can easily check that $(H^p_{\operatorname{div}}(\Omega), \|\cdot\|_{\operatorname{div}})$ is a normed vector space.

We want to prove now that it is complete.

Let $(\vec{u}_n)$ be a Cauchy sequence in $H^p_{\operatorname{div}}(\Omega)$.

$$\forall \epsilon > 0, \exists n_0 \geqslant 0, \forall n, p \geqslant n_0, \|\vec{u}_n - \vec{u}_p\|_{\operatorname{div}} \leqslant \epsilon$$

By definition of $\|\cdot\|_{\operatorname{div}}$ and because $(L^p, \|\cdot\|_{L^p})$ is complete, we have

$$u_{n,i} \xrightarrow[n\to\infty]{L^p} u_i \quad \forall i \in \{1, \cdots, N\}$$

and

$$\operatorname{div} \vec{u}_n \xrightarrow[n\to\infty]{L^p} v$$

Then we have by weak convergence,

$$u_{n,i} \underset{n\to\infty}{\overset{\mathcal{D}'}{\rightharpoonup}} u_i \text{ weakly} \quad \forall i \in \{1, \cdots, N\}$$

and so

$$D_i u_{n,i} \underset{n\to\infty}{\overset{\mathcal{D}'}{\rightharpoonup}} D_i u_i \text{ weakly} \quad \forall i \in \{1, \cdots, N\}$$

It follows that

$$\text{div } \vec{u}_n \underset{n\to\infty}{\overset{\mathcal{D}'}{\rightharpoonup}} \text{div } \vec{u} \text{ weakly}$$

By limit uniqueness and div $\vec{u}_n \xrightarrow[n\to\infty]{L^p} v$, we have $v = \text{div } \vec{u}$.

Hence, we have

$$u_{n,i} \xrightarrow[n\to\infty]{L^p} u_i \quad \forall i \in \{1, \cdots, N\}$$

and

$$\text{div } \vec{u}_n \xrightarrow[n\to\infty]{L^p} \text{div } \vec{u}$$

which prove that $\vec{u}_n$ goes to $\vec{u}$ in $H^p_{\text{div}}(\Omega)$ and that $\vec{u} \in H^p_{\text{div}}(\Omega)$.

2) I am sorry but I didn't find/take the time to thing about that question...

□

1.5 Sobolev compact embeddings

Exercise 10. Let Ω bounded of class $\mathcal{C}^1$, $1 \leqslant p < +\infty$.

Show that

$$N : u \mapsto \left(\int_\Omega |\nabla u|^p + \int_\Gamma |tr(u)|^p \right)^{1/p}$$

is a norm over $W^{1,p}(\Omega)$, equivalent to $\|.\|_{1,p,\Omega}$.

Solution. **Step 1 :** We want to show that N is a norm over $W^{1,p}(\Omega)$.

- absolutely homogeneous: For all $u \in W^{1,p}(\Omega)$, for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} N(\lambda u) &= \left(\int_\Omega |\nabla(\lambda u)|^p + \int_\Gamma |tr(\lambda u)|^p\right)^{1/p} \\ &= \left(\int_\Omega |\lambda|^p |\nabla(u)|^p + \int_\Gamma |\lambda|^p |tr(u)|^p\right)^{1/p} \\ &= \left(|\lambda|^p \left(\int_\Omega |\nabla(u)|^p + \int_\Gamma |tr(u)|^p\right)\right)^{1/p} \\ &= |\lambda| \left(\int_\Omega |\nabla(u)|^p + \int_\Gamma |tr(u)|^p\right)^{1/p} \\ &= |\lambda| N(u) \end{aligned}$$

- point-separating: If $u = 0$ then $N(u) = 0$. Conversely, for all $u \in W^{1,p}(\Omega)$ such that $N(u) = 0$. Then, we have

$$\|\nabla u\|^p_{L^p} = \int_\Omega |\nabla u|^p = 0 \text{ and } \int_\Gamma |tr(u)|^p = 0$$

  Therefore, we have $tr(u(x)) = 0$ for all $x \in \Gamma$, so $u \in W_0^{1,p}(\Omega)$.

  Then by Poincaré inequality, we have

$$\|u\|_{L^p} \leqslant C \|\nabla u\|_{L^p}$$

  So we have $\|u\|_{L^p} = 0$, and $u = 0$ because $\|.\|_{L^p}$ is a norm.

- triangle inequality: For all $u, v \in W^{1,p}(\Omega)$,

$$\begin{aligned} N(u+v) &= \left(\|\nabla(u+v)\|^p_{L^p(\Omega)} + \|tr(u+v)\|^p_{L^p(\Gamma)}\right)^{\frac{1}{p}} \\ &\leqslant \left(\left(\|\nabla u\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)}\right)^p + \left(\|tr(u)\|_{L^p(\Gamma)} + \|tr(v)\|_{L^p(\Gamma)}\right)^p\right)^{\frac{1}{p}} \end{aligned}$$

  We have just used the triangle inequality for $\|.\|_{L^p}$.

$$\begin{aligned} N(u+v) &\leqslant \left(\|\nabla u\|^p_{L^p(\Omega)} + \|v\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} + \left(\|tr(u)\|^p_{L^p(\Gamma)} + \|tr(v)\|^p_{L^p(\Gamma)}\right)^{\frac{1}{p}} \\ &\leqslant N(u) + N(v) \end{aligned}$$

  We used the Minkowski discrete inequality.

$$\left(\sum_i |x_i + y_i|^p\right)^{\frac{1}{p}} \leqslant \left(\sum_i |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_i |y_i|^p\right)^{\frac{1}{p}}$$

  For $x_1 = \|\nabla u\|_{L^p}, \quad x_2 = \|tr(u)\|_{L^p}$ and $y_1 = \|\nabla v\|_{L^p}, \quad y_2 = \|tr(v)\|_{L^p}$

Therefore, N is well a norm over $W^{1,p}(\Omega)$.

Step 2 : We want to show that there exists C_1 such that $N(\cdot) \leqslant C_1\|\cdot\|_{1,p,\Omega}$.

We will use the continuity of the trace,

$$i.e. \quad \forall u \in W^{1,p}(\Omega), \quad \|tr(u)\|_{L^p(\Gamma)} \leqslant C\|u\|_{1,p,\Omega}$$

Then we can prove the wanted inequality. For all $u \in W^{1,p}(\Omega)$,

$$\begin{aligned} N(u)^p &= \|\nabla u\|^p_{L^p(\Omega)} + \|tr(u)\|^p_{L^p(\Gamma)} \\ &\leqslant \|\nabla u\|^p_{L^p(\Omega)} + C^p\|u\|^p_{1,p,\Omega)} \\ &\leqslant \|u\|^p_{1,p,\Omega} + C^p\|u\|^p_{1,p,\Omega)} \\ &\leqslant C'\|u\|^p_{1,p,\Omega} \end{aligned}$$

Then, by passing to the power $\frac{1}{p}$, we have the result.

Step 3 : We want to show that there exists C_2 such that $\|\cdot\|_{1,p,\Omega} \leqslant C_2 N(\cdot)$.

We work for a contradiction. Suppose that we have for all $n \in \mathbb{N}$, $u_n \in W^{1,p}(\Omega)$ such that

$$\|u_n\|_{L^p(\Omega)} + \|\nabla u_n\|_{L^p(\Omega)} \geqslant n\left(\|\nabla u_n\|_{L^p(\Omega)} + \|tr(u_n)\|_{L^p(\Gamma)}\right) \tag{11}$$

Let consider $v_n = \dfrac{u_n}{\|u_n\|_{1,p,\Omega}}$. So we have

$$\|v_n\|_{1,p,\Omega} = 1 \tag{12}$$

The inequality (11) becomes

$$\|\nabla v_n\|_{L^p(\Omega)} + \|tr(v_n)\|_{L^p(\Gamma)} \leqslant \frac{1}{n}$$

Then we have

$$\nabla v_n \xrightarrow[n\to\infty]{L^p(\Omega)} 0 \quad \text{and} \quad tr(v_n) \xrightarrow[n\to\infty]{L^p(\Gamma)} 0 \tag{13}$$

We use the Poincaré Wirtinger inequality to have

$$\|v_n - \overline{v_n}\|_{L^p(\Omega)} \leqslant C\|\nabla v_n\|_{L^p(\Omega)} \xrightarrow[n\to\infty]{} 0$$

where $\overline{v_n} = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega v_n$. We can remark that

$$|\overline{v_n}| \leqslant \frac{1}{|\Omega|}\|v_n\|_{L^1(\Omega)} \leqslant \frac{C}{|\Omega|}\|v_n\|_{L^p(\Omega)} \leqslant \frac{C}{|\Omega|} \text{ because (12).}$$

Hence, $(\overline{v_n})$ is bounded and we can extract a subsequence, still called $(\overline{v_n})$, that goes to a constant c. By the dominated convergence theorem, we have $\overline{v_n} \xrightarrow[n\to\infty]{L^p(\Omega)} c$ because Ω is bounded then $|\overline{v_n}| \leqslant \dfrac{C}{|\Omega|}$ that is $L^1(\Omega)$.

But $v_n - \overline{v_n} \xrightarrow[n\to\infty]{L^p(\Omega)} 0$, hence

$$v_n \xrightarrow[n\to\infty]{L^p(\Omega)} c$$

By the continuity of the trace, we have

$$tr(v_n) \xrightarrow[n\to\infty]{L^p(\Gamma)} tr(c) = c\sigma(\Gamma) = 0 \text{ because of (13)}$$

where $\sigma(\Gamma)$ est the measure of Γ. It follows that $c = 0$.

We conclude because

$$v_n \xrightarrow[n\to\infty]{L^p(\Omega)} 0 \quad \text{and} \quad \nabla v_n \xrightarrow[n\to\infty]{L^p(\Omega)} 0$$

So

$$v_n \xrightarrow[n\to\infty]{W^{1,p}(\Omega)} 0$$

It is a contradiction with (12). □

Exercise 11. Find an example of $u \in W^{1,N}(\mathbb{R}^N)$ such that $u \notin L^\infty(\mathbb{R}^N)$.

Solution. Let take

$$u(x) = \ln\left(\frac{1}{|x|}\right)^\alpha$$

. We can notice that u is not bounded around zero, then $u \notin L^\infty$. We want to find a condition on α such as $u(x) \in W^{1,N}(B_{\mathbb{R}^N}(0, \frac{1}{2}))$.

Step 1 : We want to show that $u \in L^N(B_{\mathbb{R}^N}(0, \frac{1}{2}))$.

$$\begin{aligned}
\int_{B(0,1/2)} |u|^N &= \int_{B(0,1/2)} \ln\left(\frac{1}{|x|}\right)^{\alpha N} dx \\
&= K\int_0^{1/2} \ln\left(\frac{1}{\rho}\right)^{\alpha N} \rho^{N-1} d\rho \text{ as the change of variable in Exercise 5 Step 1} \\
&= K\int_{\ln(2)}^{+\infty} x^{\alpha N} \mathrm{e}^{-x(N-1)} \mathrm{e}^{-x} dx \text{ by the change of variable } \rho = \mathrm{e}^{-x} \\
&= K\int_{\ln(2)}^{+\infty} \underbrace{x^{\alpha N}\mathrm{e}^{-xN}}_{=O\left(\frac{1}{x^2}\right)} dx < +\infty
\end{aligned}$$

Then we have the result that $u \in L^N(B_{\mathbb{R}^N}(0, \frac{1}{2}))$.

Step 2 : We want to show that $\frac{\partial}{\partial x_i} u \in L^N(B_{\mathbb{R}^N}(0, \frac{1}{2}))$ for all $i \in [\![1, N]\!]$.

Let take $i \in [\![1, N]\!]$.

We start to find $\frac{\partial}{\partial x_i} u$. For the same reasons of the Exercise 4 Step 3, the weak derivative is equal to the classical derivative.

$$\begin{aligned}
\frac{\partial}{\partial x_i} \ln\left(\frac{1}{|x|}\right)^\alpha &= \alpha \ln\left(\frac{1}{|x|}\right)^{\alpha-1} \frac{-1}{2}\frac{2x_i}{|x|^3}|x| \\
&= -\alpha \ln\left(\frac{1}{|x|}\right)^{\alpha-1} \frac{x_i}{|x|^2}
\end{aligned}$$

$$
\begin{aligned}
\int_{B(0,1/2)} \left| \frac{\partial}{\partial x_i} u \right|^N &= \int_{B(0,1/2)} \left| \alpha \ln\left(\frac{1}{|x|}\right)^{\alpha-1} \frac{x_i}{|x|^2} \right|^N dx \\
&\leqslant \int_{B(0,1/2)} \alpha^N \ln\left(\frac{1}{|x|}\right)^{N(\alpha-1)} \frac{1}{|x|^N} dx \\
&\leqslant K \int_0^{1/2} \ln\left(\frac{1}{\rho}\right)^{N(\alpha-1)} \frac{1}{\rho^N} \rho^{N-1} d\rho \\
&\leqslant K \int_{\ln(2)}^{+\infty} x^{N(\alpha-1)} \frac{1}{\mathrm{e}^{-x}} \mathrm{e}^{-x} dx \\
&\leqslant K \int_{\ln(2)}^{+\infty} x^{N(\alpha-1)} dx
\end{aligned}
$$

And $x \mapsto x^{N(\alpha-1)}$ is integrable on $[\ln(2), +\infty[$ if and only if $N(\alpha-1) < -1$. In other words, we take $\boxed{\alpha < 1 - \frac{1}{N}}$.

It follows that, for $\alpha < 1 - \frac{1}{N}$, we have $u \in W^{1,N}(B(0,1/2))$.

Step 3 : We want to extend u to be in $W^{1,N}(\mathbb{R}^N)$.

Let take a function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that the support of φ is in $B(0,1/2)$ and $\varphi = 1$ on a neighboorhood of zero.

Then $\boxed{u\varphi \text{ is in } W^{1,N}(\mathbb{R}^N)}$ because

$$
u \in W^{1,N}(B(0,1/2)) \text{ and } \operatorname{supp}(\varphi) \subset B(0,1/2).
$$

And $\boxed{u\varphi \text{ is not in } L^\infty(\mathbb{R}^N)}$ since the problem of u in zero. □

2 Elliptic problems

2.1 Linear problems

Exercise 12. Prove the following inequality :

$$
\int_{\mathbb{R}^N} \frac{1}{\|x\|^2} u(x)^2 dx \leqslant 4 \int_{\mathbb{R}^N} \|\nabla u\|^2 dx
$$

for all $N \geqslant 3$ and for all $u \in H^1(\mathbb{R}^N)$

Remark :

In that exercise, the norm $\|\cdot\|$ denotes the euclidean norm on $\mathbb{R}^N$, *i.e.* $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$.

Solution. **Step 1 :** Let prove the result for $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Let take $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$.

We can write

$$u^2(x) = -\int_1^{+\infty} 2u(tx)\nabla u(tx).xdt$$

because an antiderivative of $2u(tx)\nabla u(tx).x$ is $u^2(tx)$ and when $t \to +\infty$, $u(tx) \to 0$ since $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$.

Then, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{\|x\|^2} u^2(x)dx \\
&= \int_{\mathbb{R}^N} \frac{2}{\|x\|^2} \left| \int_1^{+\infty} u(tx)\nabla u(tx).xdt \right| dx \\
&\leqslant \int_{\mathbb{R}^N} \frac{2}{\|x\|^2} \int_1^{+\infty} |u(tx)|\|\nabla u(tx)\|\|x\|dtdx \text{ by Cauchy Schwarz} \\
&\leqslant \int_{\mathbb{R}^N} 2\int_1^{+\infty} \frac{|u(tx)|}{\|x\|}\|\nabla u(tx)\|dtdx \\
&\leqslant \int_{\mathbb{R}^N} 2\int_1^{+\infty} t\frac{|u(tx)|}{\|tx\|}\|\nabla u(tx)\|dtdx \text{ because } t>0 \\
&\leqslant \int_1^{+\infty} 2t\int_{\mathbb{R}^N} \frac{|u(tx)|}{\|tx\|}\|\nabla u(tx)\|dxdt \text{ by Fubini} \\
&\leqslant \int_1^{+\infty} 2t\int_{\mathbb{R}^N} \frac{|u(y)|}{\|y\|}\|\nabla u(y)\|\frac{1}{t^N}dydt \text{ by the change of variable } y=tx \\
&\leqslant \int_1^{+\infty} \frac{2}{t^{N-1}}dt \left(\int_{\mathbb{R}^N} \frac{|u(y)|^2}{\|y\|^2}dy\right)^{1/2} \left(\int_{\mathbb{R}^N} \|\nabla u(y)\|^2 dy\right)^{1/2} \text{ by Cauchy-Schwarz}
\end{aligned}$$

It follows that

$$\left(\int_{\mathbb{R}^N} \frac{|u(y)|^2}{\|y\|^2}dy\right)^{1/2} \leqslant \left(\int_1^{+\infty} \frac{2}{t^{N-1}}dt\right) \left(\int_{\mathbb{R}^N} \|\nabla u(y)\|^2 dy\right)^{1/2}$$

And $\displaystyle\int_1^{+\infty} \frac{2}{t^{N-1}}dt = \left[\frac{2}{(2-N)t^{N-2}}\right]_1^{+\infty} = \frac{2}{N-2} \leqslant 2$ because $n \geqslant 3$.

Finally, while passing to the square, we find

$$\int_{\mathbb{R}^N} \frac{|u(y)|^2}{\|y\|^2}dy \leqslant 4\int_{\mathbb{R}^N} \|\nabla u(y)\|^2 dy$$

Step 2 : The general case.

By density of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$ for the H^1-norm, for all $u \in H^1(\mathbb{R}^N)$, there exists a sequence $(u_n)_n \in \mathcal{C}_0^\infty(\mathbb{R}^N)^{\mathbb{N}}$ such that

$$u_n \xrightarrow[n\to\infty]{H^1(\mathbb{R}^N)} u$$

In other words,

$$u_n \xrightarrow[n\to\infty]{L^2(\mathbb{R}^N)} u \quad \text{and} \quad \nabla u_n \xrightarrow[n\to\infty]{L^2(\mathbb{R}^N)} \nabla u$$

We have, for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} \frac{1}{\|x\|^2} u_n(x)^2 dx \leqslant 4 \underbrace{\int_{\mathbb{R}^N} \|\nabla u_n(x)\|^2 dx}_{=\|\nabla u_n\|^2_{L^2(\mathbb{R}^N)}} \tag{14}$$

We use Fatou's lemma (because the functions we use are non negative) to have

$$\begin{aligned}
\int_{R^N} \frac{1}{\|x\|^2} u_(x)^2 dx &= \int_{\mathbb{R}^N} \liminf \frac{1}{\|x\|^2} u_n(x)^2 dx \\
&\leqslant \liminf \int_{\mathbb{R}^N} \frac{1}{\|x\|^2} u_n(x)^2 dx \\
&\leqslant 4 \liminf \|\nabla u_n\|^2_{L^2(\mathbb{R}^N)} \text{ by (14)} \\
&\leqslant 4 \|\nabla u\|^2_{L^2(\mathbb{R}^N)} \\
&\leqslant 4 \int_{\mathbb{R}^N} \|\nabla u(x)\|^2 dx
\end{aligned}$$

So we have the result for all $u \in H^1(\mathbb{R}^N)$. □

Exercise 13. Let $n = 2$, $\frac{1}{2} < \alpha < 1$. Let

$$\Omega_\alpha = \left\{ (r, \theta), 0 < r < 1, 0 < \theta < \frac{\pi}{\alpha} \right\}$$

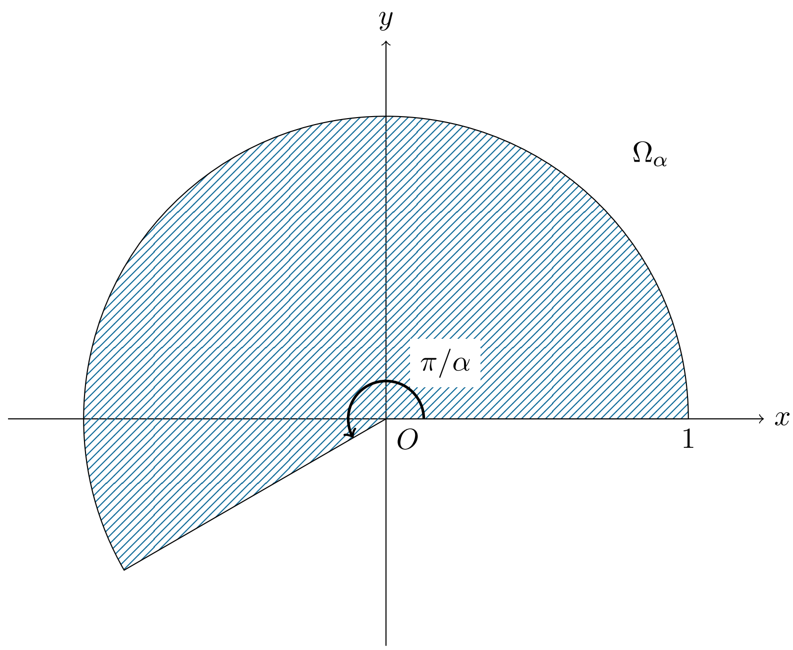
and

$$u(r, \theta) = (r^{-\alpha} - r^{\alpha}) \sin(\alpha\theta)$$

1) Show that $\exists q^* > 1$ such that $\forall q \in [1, q^*[$, $u \in W^{1,q}(\Omega_\alpha)$.

2) Calculate $-\Delta u$

3) What's up ?



Solution. 1) **Step 1 :** We want to show that $u \in L^q(\Omega_\alpha)$ for $q < \dfrac{2}{\alpha}$.

$$\int_{\Omega_\alpha} |u|^q = \underbrace{\int_0^{\frac{\pi}{\alpha}} |\sin(\alpha\theta)|^q d\theta}_{<\infty} \int_0^1 \underbrace{\left(\frac{1}{r^\alpha} - r^\alpha\right)^q r\,dr}_{\underset{r\sim 0}{\sim} \frac{1}{r^{\alpha q - 1}}}$$

The function $r \mapsto \dfrac{1}{r^{\alpha q-1}}$ is integrable on $]0,1[$ if and only if $\alpha q - 1 < 1$, *i.e.* $q < \dfrac{2}{\alpha}$.
Hence, $u \in L^q(\Omega_\alpha)$ for $q < \dfrac{2}{\alpha}$.

Step 2 : We want to show that $\dfrac{\partial}{\partial x}u, \dfrac{\partial}{\partial y}u \in L^q(\Omega_\alpha)$ for $q < \dfrac{2}{\alpha+1}$.
While deriving u, we have

$$\frac{\partial}{\partial r}u = \sin(\alpha\theta)(-\alpha r^{-\alpha-1} - \alpha r^{\alpha-1}) \tag{15}$$

$$\frac{\partial}{\partial \theta}u = \alpha\cos(\alpha\theta)(r^{-\alpha} - r^{\alpha}) \tag{16}$$

But

$$\begin{cases} \dfrac{\partial}{\partial r}u = \cos(\theta)\dfrac{\partial}{\partial x}u + \sin(\theta)\dfrac{\partial}{y}u \\ \dfrac{1}{r}\dfrac{\partial}{\partial \theta}u = -\sin(\theta)\dfrac{\partial}{\partial x}u + \cos(\theta)\dfrac{\partial}{\partial y}u \end{cases}$$

Hence, if $\dfrac{\partial}{\partial r}u$ and $\dfrac{1}{r}\dfrac{\partial}{\partial \theta}u$ are in L^q, then $\dfrac{\partial}{\partial x}u, \dfrac{\partial}{\partial y}u$ will be in L^q.

- $$\int_{\Omega_\alpha} \left|\frac{\partial}{\partial r}u\right|^q = \underbrace{\int_0^{\frac{\pi}{\alpha}} |\alpha \sin(\alpha\theta)|^q d\theta}_{<\infty} \int_0^1 \underbrace{\left|\frac{1}{r^{\alpha+1}} + r^{\alpha-1}\right|^q r\,dr}_{\underset{r\sim 0}{\sim}\frac{1}{r^{(\alpha+1)q-1}}}$$

  The function $r \mapsto \frac{1}{r^{(\alpha+1)q-1}}$ is integrable on $]0,1[$ if and only if $(\alpha+1)q - 1 < 1$, *i.e.* $q < \frac{2}{\alpha+1}$.

- $$\int_{\Omega_\alpha} \left|\frac{1}{r}\frac{\partial}{\partial \theta}u\right|^q = \underbrace{\int_0^{\frac{\pi}{\alpha}} |\alpha \cos(\alpha\theta)|^q d\theta}_{<\infty} \int_0^1 \underbrace{\frac{1}{r^q}\left|\frac{1}{r^{\alpha}} - r^{\alpha}\right|^q r\,dr}_{\underset{r\sim 0}{\sim}\frac{1}{r^{(\alpha+1)q-1}}}$$

  The function $r \mapsto \frac{1}{r^{(\alpha+1)q-1}}$ is integrable on $]0,1[$ if and only if $(\alpha+1)q - 1 < 1$, *i.e.* $q < \frac{2}{\alpha+1}$.

It follows that $\frac{\partial}{\partial x}u, \frac{\partial}{\partial y}u \in L^q(\Omega_\alpha)$ for $q < \frac{2}{\alpha+1}$.

We take $\boxed{q^* = \frac{2}{\alpha+1}}$ to have $u \in W^{1,q}(\Omega_\alpha)$ for all $q \in [1, q^*[$.

2) We have $\Delta u = \frac{\partial^2}{\partial r^2}u + \frac{1}{r}\frac{\partial}{\partial r}u + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}u$.

And we can compute

$$\begin{aligned}
\frac{\partial^2}{\partial r^2}u &= \sin(\alpha\theta\left(\alpha(\alpha+1)r^{-\alpha-2} - \alpha(\alpha-1)r^{\alpha-2}\right) \text{ while deriving (15)}\\
\frac{1}{r}\frac{\partial}{\partial r}u &= \sin(\alpha\theta)(-\alpha r^{-\alpha-2} - \alpha r^{\alpha-2}) \text{ while using (15)}\\
\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}u &= -\alpha^2\sin(\alpha\theta)(r^{-\alpha-2} - r^{\alpha-2}) \text{ while deriving (16)}
\end{aligned}$$

While computing the laplacian, we find that $\Delta u = 0$.

3) It follows that the PDE

$$\begin{cases} -\Delta v = 0 \\ v_{|\partial\Omega_\alpha} = 0 \end{cases}$$

has two solutions in $W_0^{1,q}(\Omega_\alpha)$: u and the zero function.

The problem is that Ω_α is not of class $\mathcal{C}^1$ because at the point $(0,0)$ we have an angle. Then we lose the uniqueness of the solution.

□