On bases of cyclotomic units groups

SOUANEF Rafik

Université Marie et Louis Pasteur (Besançon,France)

June 28, 2025

Introduction

Notation

Let $\mathbb{K}$ be an abelian number field.
Let $\mathrm{E}(\mathbb{K}) = \mathcal{O}_{\mathbb{K}}^{\times}$ denote the units group of $\mathbb{K}$.
Let $h(\mathbb{K})$ denote the cardinality of the class group of $\mathbb{K}$.
Let $\zeta_n = \exp(2i\pi/n)$.

Introduction - Cyclotomic units

History

Kummer had the idea to consider a certain subgroup of $\mathrm{E}(\mathbb{Q}(\zeta_p))$ to try to approximate this last group of units. He noticed that the index of this subgroup was $h(\mathbb{Q}(\zeta_p)^+)$.

Cyclotomic units

Definition

Let $\mathcal{C}_n$ denote the Galois submodule of $(\mathbb{Q}(\zeta_n)^*, \times)$ generated by the $1-\zeta_d$'s with $d \mid n$ and let $\mathrm{Was}(\mathbb{K}) = \mathrm{E}(\mathbb{K}) \cap \mathcal{C}_n$.

Definition

Let $\mathrm{Sin}(\mathbb{K})$ be the intersection of $\mathrm{E}(\mathbb{K})$ with the Galois module generated by the

$$\mathrm{N}_{\mathbb{Q}(\zeta_d)/\mathbb{K}\cap\mathbb{Q}(\zeta_d)}(1-\zeta_d)'s.$$

Cyclotomic units

Proposition

The abelian group $\mathrm{Sin}(\mathbb{K})$ is generated by:

- the roots of unity of $\mathbb{K}$
- the $N_{\mathbb{Q}(\zeta_d)/\mathbb{K}\cap\mathbb{Q}(\zeta_d)}(1-\zeta_d^a)$'s with $d \mid n$ such that d is not a prime power, $d \wedge (n/d) = 1$ and $a \wedge d = 1$
- the $N_{\mathbb{Q}(\zeta_d)/\mathbb{K}\cap\mathbb{Q}(\zeta_d)}(1-\zeta_d/1-\zeta_d^a)$'s with d being a prime power dividing n that satisfies $d \wedge (n/d) = 1$ and $a \wedge d = 1$.

Cyclotomic units

Theorem (Sinnott, 1980/1981)

Both Was($\mathbb{K}$) et Sin($\mathbb{K}$) have finite index in E($\mathbb{K}$) (that is they both are finitely generated abelian groups with rank $r_1 + r_2 - 1$). We have $[\mathrm{E}(\mathbb{K}) : \mathrm{Sin}(\mathbb{K})] \longleftrightarrow h(\mathbb{K}^+)$.

Cyclotomic units

Theorem (Sinnott, 1980/1981)

Both Was($\mathbb{K}$) et Sin($\mathbb{K}$) have finite index in E($\mathbb{K}$) (that is they both are finitely generated abelian groups with rank $r_1 + r_2 - 1$).
We have $[\mathrm{E}(\mathbb{K}) : \mathrm{Sin}(\mathbb{K})] \longleftrightarrow h(\mathbb{K}^+)$.

Warning

In the following, we quotient these groups by the group of roots of unity of $\mathbb{K}$ (and keep the same notation).

Cyclotomic units

Theorem (Sinnott, 1980/1981)

Both $\mathrm{Was}(\mathbb{K})$ et $\mathrm{Sin}(\mathbb{K})$ have finite index in $\mathrm{E}(\mathbb{K})$ (that is they both are finitely generated abelian groups with rank $r_1 + r_2 - 1$).
We have $[\mathrm{E}(\mathbb{K}) : \mathrm{Sin}(\mathbb{K})] \longleftrightarrow h(\mathbb{K}^+)$.

Warning

In the following, we quotient these groups by the group of roots of unity of $\mathbb{K}$ (and keep the same notation).

Theorem (Gold and Kim, 1989, Kučera, 1991/1992)

If $\mathbb{K} = \mathbb{Q}(\zeta_n)$, a basis of $\mathrm{Was}(\mathbb{Q}(\zeta_n)) = \mathrm{Sin}(\mathbb{Q}(\zeta_n))$ has been given explicitly.

Totally deployed fields

Definition

Suppose $\mathbb{K}$ has conductor $n = \prod_{i=1}^{r} p_i^{e_i} = \prod_{i=1}^{r} q_i$. Say $\mathbb{K}$ is totally deployed if

$$\mathbb{K} = \mathbb{K}_1 \cdots \mathbb{K}_r$$

for some $\mathbb{K}_i \subset \mathbb{Q}(\zeta_{q_i})$.

Remark

Some cases have been treated in

- Kučera, Radan "The circular units and the Stickelberger ideal of a cyclotomic field revisited", Acta Arith., 2016
- Werl, Milan "On bases of Washington's group of circular units of some real cyclic number fields", JNT, 2014

My work

Theorem (S. 2024)

If $\mathbb{K}$ is totally deployed, then $\text{Was}(\mathbb{K}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ is a direct factor of $\text{Was}(\mathbb{Q}(\zeta_n)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ and we can explicit a $\mathbb{Z}[1/2]$ basis of $\text{Was}(\mathbb{K}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$.

My work

Theorem (S. 2024)

If $\mathbb{K}$ is totally deployed, then $\text{Was}(\mathbb{K}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ is a direct factor of $\text{Was}(\mathbb{Q}(\zeta_n)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ and we can explicit a $\mathbb{Z}[1/2]$ basis of $\text{Was}(\mathbb{K}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$.

Corollary

Let $A_1, \dots, A_k$ be disjoint subsets of $[\![1, r]\!]$. Let $h_p(\mathbb{K}^+)$ denote the p-part of the class number of $\mathbb{K}^+$. For any odd prime number p, we have

$$\prod_{j=1}^{k} h_p(\mathbb{K}^+_{A_j}) \mid h_p(\mathbb{K}^+).$$