Include Course Administrative Data
CEA CAPA Paris Study Center
Course Code & Course Title
Instructor Name:
Date: (date of exam)

Final Examination
Time Allotted: 120 Minutes

Student CEA CAPA ID number : ____

EXAMINATION RULES

1. Desks must be cleared of all books, notes and papers. All unauthorized materials must be put away and remain out of sight throughout the examination.
2. Special needs or accommodations must be discussed with and approved by the local Academic Director or Center Director before the examination begins.
3. All telephones and electronic devices must be turned off and stored out of reach during the examination.
4. Talking to or communicating with other students during the exam is forbidden. Should assistance be required, please attract the attention of the instructor.
5. Anyone found cheating will earn a "0" on this examination. Additional sanctions may apply.
6. Anyone arriving after the start of the examination will not be allowed into the classroom to take the examination, subject to the decision of the instructor.
7. At the end of the exam, remain seated until your blue book is collected. No one may leave the room until excused.
8. Violations of these rules will result in a "0" on this examination. Additional sanctions may apply.

The exam is 2 hours long.
No calculator, use of internet or use of AI are allowed.
Score 80 or higher to get 100% .

1. (10 points) Pay attention to clarity and readability.

1 Exercises

2. (3 points) Find the total differential of $w = (x+y)/(z-3y)$.

3. (4 points) Show that $f_x(0,0)$ and $f_y(0,0)$ both exist but that f is not differentiable at $(0,0)$:

$$f(x,y) = \begin{cases} 1, & x = 0 \text{ or } y = 0, \\ 0, & \text{elsewhere.} \end{cases}$$

4. (3 points) Find the directional derivative of f in the direction of $\boldsymbol{v}$ at the point P :

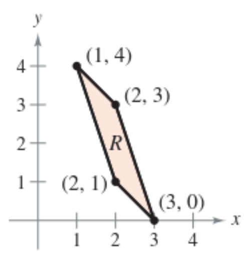
$$f(x,y) = x^3 - y^3, \quad P(4,3), \quad \boldsymbol{v} = \frac{\sqrt{2}}{2}(\boldsymbol{i}+\boldsymbol{j}).$$

5. (6 points) Let $\boldsymbol{F}(x,y) = (x^3 + e^y)\boldsymbol{i} + (xe^y - 6)\boldsymbol{j}$.
   (a) Determine whether $\boldsymbol{F}$ is conservative, and if it is give a potential function.
   (b) Let $\mathcal{C}$ be a closed loop on the triangle defined by the points $(0,0)$, $(1,0)$ and $(1,1)$. Compute the following integral :

$$\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

6. (8 points) (a) Sketch the image S in the uv-plane of the region R in the xy-plane using the given transformation :

$$\begin{cases} x = & \frac{1}{2}(v-u), \\ y = & \frac{1}{2}(3u-v). \end{cases}$$



   (b) Use the indicated change of variables to evaluate the double integral

$$\int_R \int (2y - x)\, dA.$$

7. (5 points) Find $\displaystyle\iint_S \boldsymbol{F} \cdot \boldsymbol{N} dS$ where $\boldsymbol{N}$ is the unit outward normal vector to $S : z = 1 - x - y$, first octant, and $\boldsymbol{F}(x,y,z) = 3z\boldsymbol{i} - 4\boldsymbol{j} + y\boldsymbol{k}$.

8. (5 points) Let $\boldsymbol{r}(u,v) = u\boldsymbol{i}+v\boldsymbol{j}+\sqrt{uv}\boldsymbol{k}$. Find an equation of the tangent plane to the surface represented by $\boldsymbol{r}$ at the point $(1,1,1)$.

9. (12 points) The goal is to find the solid region V that minimizes the value of

$$\iiint_V (x^2+y^2+z^2-4)\,dV.$$

(a) Give the function $f(\rho,\theta,\phi)$ defined when we use a change of coordinates from Cartesian to spherical:

$$\iiint_V (x^2+y^2+z^2-4)\,dV = \int_{\rho=a}^{b}\int_{\phi=\phi_1}^{\phi_2}\int_{\theta=\theta_1}^{\theta_2} f(\rho,\theta,\phi)\,d\theta\,d\phi\,d\rho,$$

where $0 \le \theta_1 \le \theta_2 \le 2\pi$, $0 \le \phi_1 \le \phi_2 \le \pi$, and $a \le b$.

(b) Prove

$$\iiint_V (x^2+y^2+z^2-4)\,dV = (\theta_2-\theta_1)(\cos\phi_1-\cos\phi_2)\left(\frac{b^5}{5}-\frac{4b^3}{3}-\frac{a^5}{5}+\frac{4a^3}{3}\right).$$

(c) Let $g(a,b) = \dfrac{b^5}{5}-\dfrac{4b^3}{3}-\dfrac{a^5}{5}+\dfrac{4a^3}{3}$. We find the global minimum of the function $g(a,b)$ on the closed, constrained domain defined by the physical geometry of integration boundaries:

$$D = \{(a,b)\in\mathbb{R}^2 \mid 0 \le a \le b \le 4\}.$$

i. Prove that if the interior domain is

$$\overset{\circ}{D} = \{\ (a,b)\ /\ 0 < a < b < 4\},$$

then no critical points hold in $\overset{\circ}{D}$.

ii. What theorem allows us to conclude that the minimum *must* exist on the boundary of D?

iii. By checking on the boundaries $(a,b)=(0,t)$ and $(a,b)=(t,t)$, prove that $(a,b)=(0,2)$ is a global minimum for g.

(d) Give without justification angles $0 \le \theta_1 \le \theta_2 \le 2\pi$ and $0 \le \phi_1 \le \phi_2 \le \pi$ such that the angular multiplier is *maximized*:

$$\max_{\substack{0\le\theta_1\le\theta_2\le2\pi\\0\le\phi_1\le\phi_2\le\pi}} \Big((\theta_2-\theta_1)(\cos\phi_1-\cos\phi_2)\Big).$$

(e) Give the absolute minimum value of the total integral, knowing :

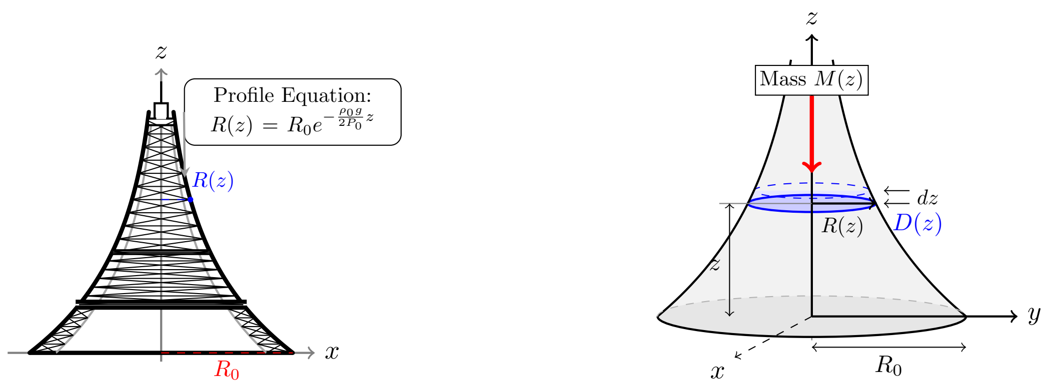
$$\max_V \iiint_V x^2+y^2+z^2-4\ dV = \left(\max_{\substack{0\le\theta_1\le\theta_2\le2\pi\\0\le\phi_1\le\phi_2\le\pi}} \Big((\theta_2-\theta_1)(\cos\phi_1-\cos\phi_2)\Big)\right) g(0,2).$$

2 Problem

> The problem is partitioned into 3 independents parts.
> If a result from another part is needed, it is restated in the question.

Consider a rigid vertical tower modeled as a 3D solid of revolution E centered around the z-axis. The solid is bounded below by the ground plane $z = 0$, and extends infinitely upward ($z \to \infty$). At any height z, the horizontal cross-section $D(z)$ is a circular disk of radius $R(z)$. The bounding surface of the tower is given in cylindrical coordinates by $r = R(z)$.

Our architectural goal is to design the profile $R(z)$ such that the internal compressive pressure $P(z)$ caused by the weight of the material above it is perfectly uniform across every horizontal cross-section: $P(z) = P_0$ (a constant). The material has a uniform mass density ρ_0 and g is the acceleration due to gravity.



(a) View in the yz-plane of the structure.

(b) Continuous 3D model with slice elements.

Figure 1: Mathematical and structural representations of the tower.

10. (22 points) **Part 1 : Pressure equilibrium to construct a tower.**

(a) For a fixed z, state the region E_z inside the solid and above the cross section $D(z)$ in the cylindrical coordinates. You may use $\tilde{z}$ as the dummy coordinate for the vertical axis.

(b) i. State the triple integral describing **total mass** $M(z)$ of the portion of the tower resting **above** a specific height z in cylindrical coordinates.

ii. Deduce from the triple integral that :

$$M(z) = \int_z^{+\infty} \rho_0 \pi [R(\tilde{z})]^2 \, d\tilde{z}.$$

(c) Given that pressure is defined as $P_0 = \frac{\text{Force}}{\text{Area}} = \frac{M(z)g}{\pi[R(z)]^2}$, show that the boundary curve $R(z)$ must satisfy the master integral equation:

$$\pi[R(z)]^2 P_0 = \rho_0 g \int_z^{+\infty} \pi[R(\tilde{z})]^2 \, d\tilde{z}.$$

(d) Deduce the first-order differential equation for $R(z)$:

$$\frac{dR}{dz} = -\frac{\rho_0 g}{2P_0} R(z).$$

(e) Prove that the explicit solution for $R(z)$ given the initial base boundary condition $R(0) = R_0$ at ground level is :

$$R(z) = R_0 e^{-\frac{\rho_0 g}{2P_0} z}.$$

(f) What famous tower does this structure remind you of ?

11. (17 points) **Part 2 : vector fields through the tower.**

(a) Consider a horizontal cross-sectional slice of the tower $D(z)$ at a fixed height z. The boundary of this disk is a circle C_z of radius $R(z)$, oriented counterclockwise when viewed from above.

i. State Green's Theorem for a 2D vector field $\mathbf{F}(x, y) = L(x, y)\mathbf{i} + M(x, y)\mathbf{j}$ over the region $D(z)$.

ii. Use Green's Theorem with the vector field $\mathbf{F}(x, y) = \langle -y, x\rangle$ to compute the line integral $\oint_{C_z} \mathbf{F} \cdot d\mathbf{r}$, and show that it directly confirms the cross-sectional area formula $\text{Area}(D(z)) = \pi[R(z)]^2$.

(b) Let E_h be the truncated 3D solid portion of the tower between the ground $z = 0$ and a finite height $z = h$. The boundary surface ∂E_h consists of three components: the base disk $D(0)$, the top disk $D(h)$, and the outer lateral surface S_L. Consider the 3D vector field $\mathbf{G}(x, y, z) = \langle x, y, 0\rangle$.

i. Calculate the divergence $\vec{\nabla} \cdot \mathbf{G}$.

ii. Set up the volume integral $\iiint_{E_h} \vec{\nabla} \cdot \mathbf{G}\, dV$ using the exponential profile $R(z) = R_0 e^{-\frac{\rho_0 g}{2P_0} z}$ and prove :

$$\iiint_{E_h} \vec{\nabla} \cdot \mathbf{G}\, dV = \frac{2\pi R_0^2 P_0}{\rho_0 g} \left(1 - e^{-\frac{\rho_0 g}{P_0} h}\right).$$

iii. Use the Divergence Theorem to compute the outward-pointing flux through the curved lateral surface S_L :

$$\iint_{S_L} \mathbf{G} \cdot \hat{\mathbf{n}}\, dS.$$

12. (15 points) **Part 3 : Tangent plane to the tower.**

(a) The outer boundary surface of the optimal tower can be described in Cartesian coordinates as the level surface of a function $F(x, y, z) = 0$, where:

$$F(x, y, z) = x^2 + y^2 - [R(z)]^2$$

i. Compute the partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, and $\frac{\partial F}{\partial z}$, and state the gradient vector $\vec{\nabla} F(x, y, z)$ in terms of x, y, and the profile derivative $\frac{dR}{dz}$.

ii. Substitute the known physical profile derivative $\frac{dR}{dz} = -\frac{\rho_0 g}{2P_0} R(z)$ into your gradient expression. Then, find the equation of the tangent plane to the tower surface at a specific point (x_0, y_0, z_0) lying on the boundary.

(b) At the top cap of the tower, the local height is given by the function function $f(x, y) = 4-x^2-y^2+xy$. Find the critical point of $f(x, y)$ and use the Second Derivative Test to classify it (as a local maximum, local minimum, or saddle point).

13. (10 points) Prove

$$\int_0^{+\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

Hint : use the polar coordinates.

Question	Points	Score
1	10	
2	3	
3	4	
4	3	
5	6	
6	8	
7	5	
8	5	
9	12	
10	22	
11	17	
12	15	
13	10	
Total:	120	