1. (10 points) Pay attention to clarity and readability.

Section 10

2. (5 points) Classify the following conics as *circle*, *ellipse*, *parabola* or *hyperbola* (explain your reasoning).

A. $y^2 + 6y + 8x + 25 = 0$.

**Solution:**

$$y^2 + 6y + 8x + 25 = 0 \iff (y - (-3))^2 = (4)(-2)(x - (-2)).$$

Horizontal parabola.

B. $9x^2 + 4y^2 + 36x - 24y = -36$.

**Solution:**

$$9x^2 + 4y^2 + 36x - 24y = -36 \iff \frac{(x-(-2))^2}{4} - \frac{(y-3)^2}{9} = 1.$$

Since $2 < 9$, its an ellipse of vertical major axis.

C. $x^2 - 2x - 4y - 7 = 0$.

**Solution:**

$$x^2 - 2x - 4y - 7 = 0 \iff (x-1)^2 = (4)(1)(y+2).$$

Vertical parabola.

D. $y^2 = 16x^2 - 64x + 208$.

**Solution:**

$$y^2 = 16x^2 - 64x + 208 \iff \frac{y^2}{9 \times 16} - \frac{(x-2)^2}{9} = 1.$$

Hyperbola of vertical transverse axis.

E. $y^2 - 4y = x + 5$.

**Solution:**

$$y^2 - 4y = x + 5 \iff (y-2)^2 = (4)\frac{1}{4}(x+9).$$

Horizontal parabola.

F. $25x^2 - 10x - 200y - 119 = 0$.

**Solution:**

$$25x^2 - 10x - 200y - 119 = 0 \iff (x - 1/5)^2 = (4)(2)(y + 3/5).$$

Horizontal parabola.

3. (10 points) ($\star$)

(a) Find the vertex, focus, and directrix of the following parabola, and sketch its graph :

$$y^2 + 6y + 8x + 25 = 0.$$

**Solution:**

$$y^2 + 6y + 8x + 25 = 0 \iff (y+3)^2 = (4)(-2)(x+2).$$

Vertex $(h, k) = (-2, -3)$.
Focus $(h + p, k) = (-4, -3)$.
Directrix $x = h - p = 0$.

(b) Find the center, foci, vertices, and eccentricity of the following ellipse, and sketch its graph :

$$(x+4)^2 + 4(y+6)^2 = 1.$$

**Solution:** Center $(h, k) = (-4, -6)$.
Major and minor axis of length $a = 1, b = 1/2$.
Foci distance $c = \sqrt{a^2 - b^2} = \sqrt{3}/2$. Foci at $(h \pm c, k) = (-4 \pm \sqrt{3}/2, -6)$. Vertices at $(h \pm a, k) = (-5, -6), (-3, -6)$.
Eccentricity $e = \frac{c}{a} = \sqrt{3}/2$.

(c) Find the center, foci, vertices, and eccentricity of the following hyperbola, and sketch its graph using asymptotes as an aid :

$$9x^2 - y^2 - 36x - 6y + 18 = 0.$$

**Solution:**

$$9x^2 - y^2 - 36x - 6y + 18 = 0 \iff \frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1.$$

Center $(h, k) = (2, -3)$.
Major and minor axis length : $(a, b) = (1, 3)$.
Distance to the foci $c = \sqrt{a^2 + b^2} = \sqrt{10}$.
Foci $(h \pm c, k) = (2 \pm \sqrt{10}, -3)$.
Vertices $(h \pm a, k) = (1, -3), (3, -3)$.
Eccentricity $e = \frac{c}{a} = \sqrt{10}$.

4. (4 points) ($\star$) Probably the most famous of all comets, Halley's comet, has an elliptical orbit with the sun at one focus. Its maximum distance from the sun is approximately 35.29 AU (1 astronomical unit is approximately 92.956 $\times$ 106 miles), and its minimum distance is approximately 0.59 AU.
Find the eccentricity of the orbit.

**Solution:** Minimum distance from focus (sun) to the ellipse = distance from focus to closest vertice $= a - c = 0.59$.
Maximum distance from focus (sun) to the ellipse = distance from focus to farthest vertice $= a + c = 35.29$.

$$\begin{cases} a - c = 0.59 \\ a + c = 35.29 \end{cases} \iff \begin{cases} 2a = 35.29 + 0.59, \\ 2c = 35.29 - 0.59. \end{cases}$$

Eccentricity $e = \frac{c}{a} = \frac{2c}{2a} = \frac{35.29-0.59}{35.29+0.59} = \frac{34.70}{35.88} (= 0.9671)$.

5. (3 points) True or False ? If it is false, explain why or give an example.
   (a) The graph of the parametrics equations $x = t^2$ and $y = t^2$ is the line $y = x$.

   **Solution:** False : $x = y = -1$ is not on the curve of $x = t^2, y = t^2$.

   (b) If $y$ is a function of $t$ and $x$ is a function of $t$ then $y$ is a function of $x$.

   **Solution:** False : $x = 0, y = t$, $y$ cannot be a function of $x$ otherwise $y(0)$ must have several values.

   (c) The curve represented by the parametrics equations $x = t$ and $y = \cos t$ can be written as an equation of the form $y = f(x)$.

   **Solution:** True.

6. (10 points) ($\star$) LORAN (long distance radio navigation) for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci. Assume that two stations, 300 miles apart, are positioned on a rectangular coordinate system at (-150, 0) and (150, 0) and that a ship is traveling on a path with coordinates (x, 75) (see figure).
Find the x-coordinate of the position of the ship when the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).

**Solution:** $c = 150$.
Let $d_1$ is the distance between $(x, 75)$ and the focus 1, $d_2$ is the distance between $(x, 75)$ and the focus 2.
We know $v = \dfrac{d_1}{t_1} \iff t_1 \cdot v = d_1$. Also, $d_2 = t_2 \cdot v$.

$$2a = |d_2 - d_1| = v\,|t_2 - t_1| = v\Delta t = 0.001(186,000) \Rightarrow a = 93.$$

Also

$$b = \sqrt{c^2 - a^2} = \sqrt{150^2 - 93^2}.$$

Thus the equation of the ellipse is :

$$\frac{x^2}{93^2} - \frac{y^2}{150^2 - 93^2} = 1.$$

For y = 75 :

$$x^2 = 93^2(1 + \frac{75^2}{150^2 - 93^2}).$$

Taking the square root gives the value.

7. (4 points) Match each set of parametrics equations to a graph. Explain your reasoning:
   A. Lissajous curve : $x = 4\cos\theta$, $y = 2\sin 2\theta$.
   B. Evolute of ellipse : $x = \cos^3\theta, y = 2\sin^3\theta$.
   C. Involute of circle: $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$.
   D. Serpentine curve: $x = \cot\,\theta$, $y = 4\sin\theta\cos\theta$.

**Solution:** A $\leftrightarrow$ (d) : $x(\theta = 0) = 4$, $y(\theta = 0) = 0$ and only (d) admits the point $(4, 0)$.
B $\leftrightarrow$ (a) : $\theta = \pi/2$, $(x, y) = (0, 2)$ thus (a).
C $\leftrightarrow$ (b) : (b) is the only one where $y > 2$ is possible.
D $\leftrightarrow$ (c) : $\text{cotan}(0) = \infty \Rightarrow x = \infty$ is on the graph which can only be (c).

8. (7 points) ($\star$) The curve represented by the equation $r = ae^{b\theta}$, where $a$ and $b$ are constants, is called a logarithmic spiral. The figure shows the graph of $r = e^{\theta/6}$, $-2\pi \le \theta \le 2\pi$. Find the area of the shaded region.

**Solution:** The area of the whole figure (blue + interior in white) is given by

$$\int_0^{2\pi} r^2(\theta)d\theta.$$

The interior white area is given by

$$\int_{-2\pi}^{0} r^2(\theta)d\theta.$$

Thus the blue area A is given by

$$A = \frac{1}{2}\int_0^{2\pi} r^2(\theta)d\theta - \frac{1}{2}\int_{-2\pi}^{0} r^2(\theta)d\theta = \frac{3}{2}\left[e^{2\pi/3} + e^{-2\pi/3} - 2\right].$$

9. (15 points) ($\star$) Use the parametric equations $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, $a > 0$, to answer the following.
   (a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

**Solution:**

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta}{1-\cos\theta}$$

and

$$\frac{d^2y}{dx^2} = \frac{\cos\theta(1-\cos\theta) - \sin\theta^2}{a(1-\cos\theta)(1-\cos\theta)^2} = \frac{-1}{a(\cos\theta - 1)^2}.$$

(b) Find the equation of the tangent line at the point where $\theta = \pi/6$.

**Solution:** At $\theta = \frac{\pi}{6}$, $\dfrac{dy}{dx}(\theta) = \frac{1/2}{1-\sqrt{3}/2} = \frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}$.
$(x(\pi/6), y(\pi/6)) = (a(\pi/6 - 1/2), a(1-\sqrt{3}/2))$.

$$y_{\text{tangent}}(x) = (2+\sqrt{3})(x - a(\pi/6 - 1/2)) + a(1-\sqrt{3}/2).$$

(c) Find all points of horizontal tangency.

**Solution:** $\dfrac{dy}{dx} = 0 \iff \sin\theta = 0 \iff \theta = 0, 1-\cos\theta \neq 0.$
Thus there exists $k$ such that $\theta = k\pi$ and $\theta \neq 2k\pi$ i.e. $\theta = (2k+1)\pi$.
The points are then

$$(x, y) = (a(2k+1)\pi, 2a).$$

(d) Determine where the curve is concave upward or concave downward.

**Solution:** $\frac{d^2y}{dx^2} < 0$ for all $\theta \neq 2k\pi$ and undefined for the others so the curve is concave downward for all the open intervals $]2k\pi; 2(k+1)\pi[$.

(e) Find the length of one arc of the curve.

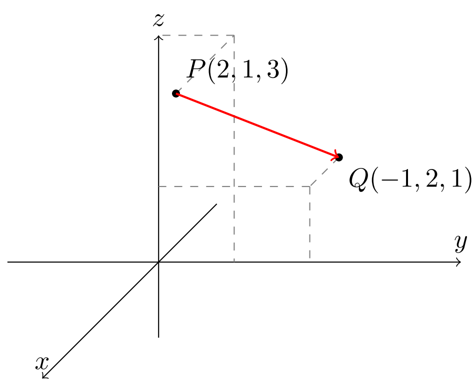
**Solution:**

$$\begin{aligned}
s &= \int_0^{2\pi} \sqrt{\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2} \\
&= \int_0^{2\pi} \sqrt{a^2\sin^2\theta + a^2(1-\cos\theta)^2} \\
&= a\int_0^{2\pi} \sqrt{2-2\cos\theta} \\
&= a\int_0^{2\pi} \sqrt{4\sin^2(\frac{\theta}{2})} \\
&= 2a\int_0^{2\pi} \sin(\frac{\theta}{2}) \\
&= 4a\left[\cos(\theta/2)\right]_0^{2\pi} \\
&= 8a.
\end{aligned}$$

Section 11

10. (3 points) The initial and terminal points of a vector $\boldsymbol{v}$ are given: P(2,1,3) and Q(-1,2,1).

(a) Sketch the given directed line segment.




**Solution:**

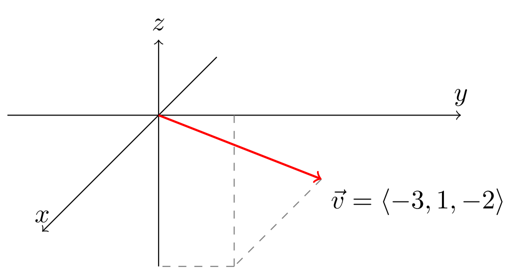
(b) Write the vector in a component form.

**Solution:** $\vec{PQ} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$.

(c) Write the vector as the linear combination of standard unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$.

**Solution:** $\vec{PQ} = -3\vec{i} + \vec{j} - 2\vec{k}$.

(d) Sketch the vector with its initial point at the origin.




**Solution:**

(e) Calculate the magnitude (norm) of the vector.

**Solution:** $\left\|\vec{PQ}\right\| = \sqrt{9+1+4} = \sqrt{14}$.

(f) Indicate the unit vector associated with $\boldsymbol{v}$.

**Solution:** $\vec{PQ} / \left\|\vec{PQ}\right\|$.

11. (6 points) ($\star$) To carry a 100-pound cylindrical weight, two workers lift on the ends of short ropes tied to an eyelet on the top center of the cylinder. One rope makes a 20° angle away from the vertical and the other makes a 30° angle (see figure).

   (a) Find each rope's tension when the resultant force is vertical.
   *Hint : To lift the weight vertically, the sum of the vertical components of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *must be 100 and the sum of the horizontal components must be 0.*

   **Solution:** If the tension of $\vec{v}$ is $v = \|\vec{v}\|$ and the tension of $\vec{u}$ is $u = \|\vec{u}\|$ then

   $$\vec{v} = \begin{bmatrix} -v\sin 20 \\ v\cos 20 \end{bmatrix}, \vec{u} = \begin{bmatrix} u\sin 30 \\ u\cos 30 \end{bmatrix}.$$

   The resultant force is :

   $$\vec{F} = \vec{v} + \vec{u} = \begin{bmatrix} -v\sin 20 + u\sin 30 \\ v\cos 20 + u\cos 30 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \end{bmatrix}.$$

   Thus :

   $$\begin{cases} -v\sin 20 + u\sin 30 = 0 \\ v\cos 20 + u\cos 30 = 100 \end{cases} \iff \begin{cases} -v\sin 20 + u/2 = 0 \\ v\cos 20 + u\frac{\sqrt{3}}{2} = 100 \end{cases} \iff \begin{cases} -\sqrt{3}v\sin 20 + \frac{\sqrt{3}}{2}u = 0 \\ v(\cos 20 + \sqrt{3}\sin 20) = 100 \end{cases}$$

   From this, $v = \frac{100}{\cos 20 + \sqrt{3}\sin 20}$ and then $u = 2v\sin 20$.

   (b) Find the vertical component of each worker's force.
   $\sin 20 = 0.34$ *and* $\cos 20 = 0.94$

   **Solution:** For $\vec{u}$ : $u_y = u\cos 30 = \frac{\sqrt{3}}{2}\frac{200\sin 20}{\cos 20 + \sqrt{3}\sin 20}$.
   For $\vec{v}$ : $v_y = v\cos 20 = \frac{100\cos 20}{\cos 20 + \sqrt{3}\sin 20}$.

12. (3 points) Let $\boldsymbol{u} = \boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{v} = \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{w} = a\boldsymbol{u} + b\boldsymbol{v}$.

   (a) Sketch $\boldsymbol{u}$ and $\boldsymbol{v}$.

   (b) If $\boldsymbol{w} = 0$, show that a and b must be both 0.

   **Solution:**

   $$\vec{w} = \vec{0} \iff \begin{bmatrix} a \\ a+b \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} a &= 0, \\ a+b &= 0, \\ b &= 0. \end{cases}$$

   (c) Find a and b such that $\boldsymbol{w} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$.

   **Solution:**

   $$\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \iff \begin{bmatrix} a \\ a+b \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \iff \begin{cases} a &= 1, \\ a+b &= 2, \\ b &= 1. \end{cases}$$

   Thus $a = b = 1$.

13. (5 points) (a) ($\star$) Prove the triangular inequality : $\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$.
*Hint : use* $\|\boldsymbol{u}\|^2 = \boldsymbol{u}\cdot\boldsymbol{u}$.

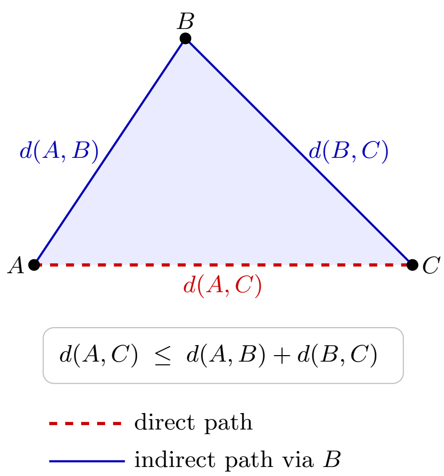
**Solution:**

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\
&= \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \\
&= \|\vec{u}\|^2 + 2\,\|\vec{u}\|\,\|\vec{v}\|\underbrace{\cos\theta}_{\leq 1} + \|\vec{v}\|^2 \\
&\leq \|\vec{u}\|^2 + 2\,\|\vec{u}\|\,\|\vec{v}\| + \|\vec{v}\|^2 \\
&= (\|\vec{u}\| + \|\vec{v}\|)^2.
\end{aligned}$$

Thus by taking the square root :

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|\,.$$

(b) Make a drawing highlighting this property.




**Solution:**

14. (7 points) ($\star$) Both the magnitude and the direction of the force on a crankshaft change as the crankshaft rotates. Find the torque on the crankshaft using the position and data shown in the figure.

**Solution:**

$$\vec{PQ} = \begin{bmatrix} 0 \\ 0 \\ 0.16 \end{bmatrix}, \vec{F} = \begin{bmatrix} 0 \\ -2000\sin 60 \\ -2000\cos 60 \end{bmatrix}.$$

Torque :

$$\vec{PQ}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 0.16 \\ 0 & -1000\sqrt{3} & -1000 \end{vmatrix} = 160\sqrt{3}\,\vec{i}.$$

15. (3 points) (a) Find the set of parametric equations of the line that passes through the point (-2,0,3) and is parallel to the vector $\boldsymbol{v} = 2\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$.

> **Solution:**
> $$x + 2 = 2t, \quad y = 4t, \quad z - 3 = -2t.$$

(b) Find the equation of the plane that passes through the point (3,2,2) and is perpendicular to the vector $\boldsymbol{n} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$.

> **Solution:**
> $$2(x - 3) + 3(y - 2) - (z - 2) = 0.$$

16. (2 points) Convert the point (2,-2,-4) from rectangular coordinates to cylindrical coordinates, and to spherical coordinates.

> **Solution:**
> $$(x, y, z) = (2, -2, -4)$$
> For the cylindrical coordinates : $\theta = \arctan(y/x) = \arctan(-1) = -\pi/4$.
> $r = \sqrt{x^2 + y^2} = \sqrt{8}$ and $z = -4$.
>
> For the spherical coordinates :
> $\rho = \sqrt{4 + 4 + 16} = 2\sqrt{6}$. $\theta = -\pi/4$ and $\Phi = \arccos(z/\rho) = \arccos(-2/\sqrt{6})$.

17. (4 points) Match the correspond equation (written in cylindrical coordinates or spherical coordinates) to its graph. A. $r = 5$. B. $\theta = \pi/4$. C. $\rho = 5$. D. $\Phi = \pi/4$. E. $r^2 = z$. F. $\rho = 4/\cos\Phi$.

> **Solution:** A $\leftrightarrow$ (d) : cylindrical coordinates with constant $r$ is a cylinder.
> B $\leftrightarrow$ (e) : it's the only one with $\theta = \pi/4$ constant since (e) is for $\Phi$.
> C $\leftrightarrow$ (c) : spherical coordinates with constant $\rho$ is a sphere.
> D $\leftrightarrow$ (a) : constant $\phi$ and any $\rho, \theta$ create a circle of growing radius $R = \rho\sin(\Phi)$.
> E $\leftrightarrow$ (f) : equation of a parabola in cylindrical coordinates.
> F $\leftrightarrow$ (b) : $\rho\cos\Phi = 4 = z$ is constant.

Section 12

18. (2 points) (a) Evaluate the vector-valued function $\boldsymbol{r}(t) = \begin{pmatrix} \cos t \\ 2\sin t \end{pmatrix}$ at each given value of $t$.

A. $t = 1$. B. $t = 0$. C. $\boldsymbol{r}(1+s)$. D. $\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)$.

**Solution:** A : $(\cos(1), 2\sin(1))$.
B : $(1, 0)$.
C : $(\cos(1+s), 2\sin(1+s))$.
D : $(\cos(t+\Delta t) - \cos(t), \sin(t+\Delta t) - \sin(t))$.

(b) Compute $\boldsymbol{r}'(t)$.

**Solution:**

$$\boldsymbol{r}'(t) = \begin{bmatrix} -\sin t \\ 2\cos t \end{bmatrix}.$$

19. (4 points) True or False ? If false, explain why or give an example that shows it is false.

(a) If a particle moves along a sphere centered at the origin, then its derivative vector is always tangent to the sphere.

**Solution:** True since a derivative vector is always tangent to the curve.

(b) The definite integral of a vector-valued function is a real number.

**Solution:** False : $\int_0^1 \vec{i} + \vec{j} = \vec{i} + \vec{j}$.

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t}[\|\boldsymbol{r}(t)\|] = \|\boldsymbol{r}'(t)\|$.

**Solution:** False : see Problem solving 4 of Section 11.

(d) If $\boldsymbol{r}$ and $\boldsymbol{u}$ are differentiable vector-valued functions of $t$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{r}(t) \cdot \boldsymbol{u}(t)] = \boldsymbol{r}'(t) \cdot \boldsymbol{u}'(t).$$

**Solution:** False :

$$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{r}(t) \cdot \boldsymbol{u}(t)] = \boldsymbol{r}'(t) \cdot \boldsymbol{u}(t) + \boldsymbol{r}(t) \cdot \boldsymbol{u}'(t).$$

20. (7 points) ($\star$) A psychrometer (an instrument used to measure humidity) weighing 4 ounces is whirled horizontally using a six-inch string (see figure). The string will break under a force of 2 pounds. Find the maximum speed in feet/s the instrument can attain without breaking the string. (Use $\boldsymbol{F} = m\boldsymbol{a}$, where $m = 1/128$.).
*Hint : for a rotating motion,* $\|\boldsymbol{a}(t)\| = b\omega^2$ *where* $b$ *is the radius of the circle described by* $\boldsymbol{r}(t)$.
*Hint : 12 inches is 1 foot and 1 pound is 16 ounces.*

**Solution:**

$$\vec{F} = m\vec{a} \Rightarrow \left\|\vec{F}\right\| = 2 = \|m\vec{a}\| = b\omega^2 \iff 2 = \frac{1}{128}\frac{1}{2}\omega^2.$$

Thus $\omega^2 = 512 \Rightarrow \omega = 16\sqrt{2}$.

$$v = b\omega = 8\sqrt{2} feet.s^{-1}.$$

21. (4 points) An object moves along the path given by $\boldsymbol{r}(t) = 3t\boldsymbol{i} + 4t\boldsymbol{j}$.

(a) Find $\boldsymbol{v}(t)$, $\boldsymbol{a}(t)$, $\boldsymbol{T}(t)$, and $\boldsymbol{N}(t)$ (if it exists).

**Solution:**

$$\boldsymbol{v}(t) = 3\boldsymbol{i} + 4\boldsymbol{j}.$$

$$\boldsymbol{a}(t) = \vec{0}.$$

$$\|\boldsymbol{v}(t)\| = 5.$$

$$\boldsymbol{T}(t) = 3/5\boldsymbol{i} + 4/5\boldsymbol{j}$$

$\boldsymbol{T}' = 0 \Rightarrow \boldsymbol{N}$ does not exists.

(b) What is the form of the path?

**Solution:** $x(t) = 3t \iff t = x/3, y(t) = 4t = 4x/3$.
It's the equation of a line

(c) Is the speed of the object constant or changing?

**Solution:** $v = 5$ is constant.

22. (3 points) Find $\boldsymbol{r}(t)$ that satisfies the initial condition

$$\boldsymbol{r}'(t) = 2t\boldsymbol{i} + e^t\boldsymbol{j} + e^{-t}\boldsymbol{k}, \qquad \boldsymbol{r}(0) = \boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k}.$$

**Solution:**

$$\boldsymbol{r}(t) = t^2\boldsymbol{i} + e^t\boldsymbol{j} - e^{-t}\boldsymbol{k} + \boldsymbol{C}.$$

Then we evaluate for $t = 0$ :

$$\begin{aligned} \boldsymbol{r}(0) &= \boldsymbol{j} - \boldsymbol{k} + \boldsymbol{C} \\ &= \boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k} \\ \Rightarrow \boldsymbol{C} &= \boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}. \end{aligned}$$

Thus :

$$\boldsymbol{r}(t) = (t^2 + 1)\boldsymbol{i} + (e^t + 2)\boldsymbol{j} - (e^{-t} + 4)\boldsymbol{k}.$$

23. (13 points) ($\star$) The quarterback of a football team releases a pass at a height of 7 feet above the playing field, and the football is caught by a receiver 30 yards directly downfield at a height of 4 feet. The pass is released at an angle of 35° with the horizontal.

(a) Find the speed of the football when it is released.
*Hint : 30 yards = 90 feet. Don't compute* $\cos 35$ *or* $\sin 35$.

**Solution:** Let $v_0$ be the initial speed. We know :

$$\boldsymbol{r}(t) = \begin{bmatrix} (v_0 \cos 35)t \\ h + (v_0 \sin 35)t - \frac{g}{2}t^2 \end{bmatrix}.$$

The ball is catched at $x(t_f) = 90$ feet.

$$x(t_f) = 90 \iff t_f = \frac{90}{v_0 \cos 35}.$$

The ball is catched at a height of 4 feet :

$$y(t_f) = 4 \iff h + (v_0 \sin 35)t_f - \frac{g}{2}t_f^2 = 4.$$

Solving for $v_0$ gives :

$$v_0^2 = \frac{16 \cdot 90^2}{\cos^2 35(90 \tan 35 + 3)} = 54.1\text{feet.s}^{-1}.$$

(b) Find the maximum height of the football.

**Solution:** Maximum height = maximum of $y(t) = h + (v_0 \sin 35)t - \frac{g}{2}t^2$ which is maximum when $t = \frac{v_0 \sin 35}{g} = \frac{v_0 \sin 35}{32} = 0.969$ s.
$y(0.969) = 22$ feet.

(c) Find the time the receiver has to reach the proper position after the quarterback releases the football.

**Solution:** $x(t) = 90$ feet $\Rightarrow t = t_f = \frac{90}{v_0 \cos 35} = 2.0$ s.

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 5 | |
| 3 | 10 | |
| 4 | 4 | |
| 5 | 3 | |
| 6 | 10 | |
| 7 | 4 | |
| 8 | 7 | |
| 9 | 15 | |
| 10 | 3 | |
| 11 | 6 | |
| 12 | 3 | |
| 13 | 5 | |
| 14 | 7 | |
| 15 | 3 | |
| 16 | 2 | |
| 17 | 4 | |
| 18 | 2 | |
| 19 | 4 | |
| 20 | 7 | |
| 21 | 4 | |
| 22 | 3 | |
| 23 | 13 | |
| Total: | 134 | |