Cryptographic Primitives in Quantum Idealized Models

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PhD Defense

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Outline

Introduction

Context

Cryptography

Cryptography is the science of designing mathematical methods to protect information and ensure security properties such as confidentiality, integrity, and authenticity.

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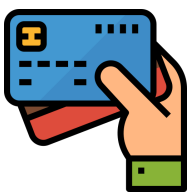

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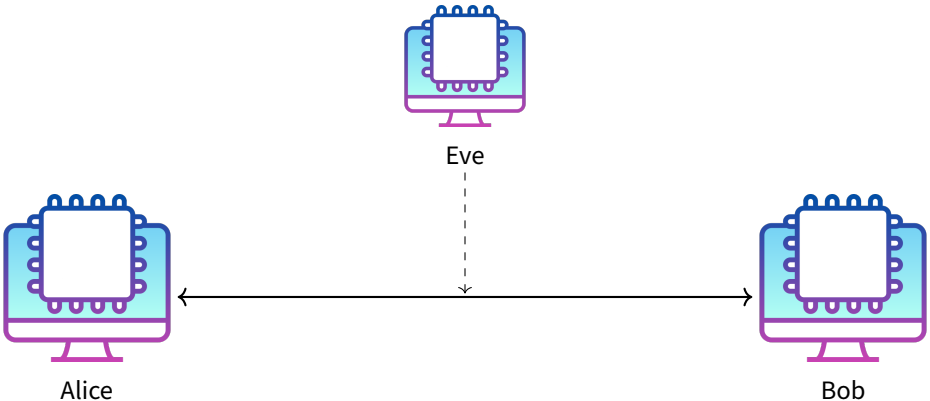
Classical Cryptography



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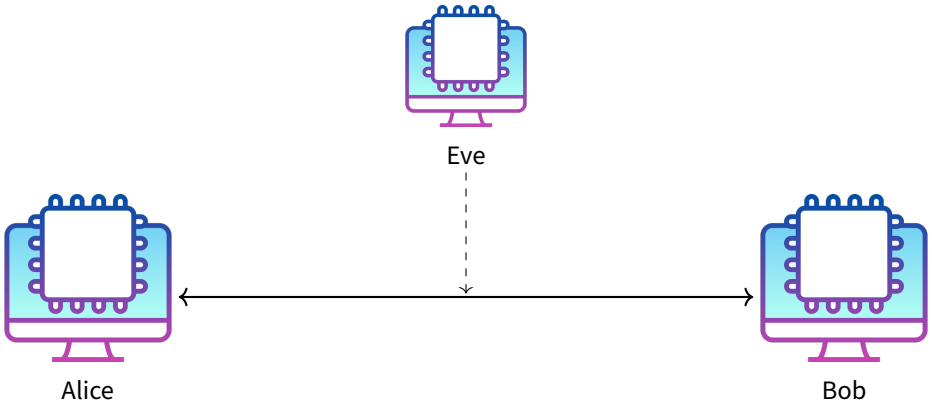
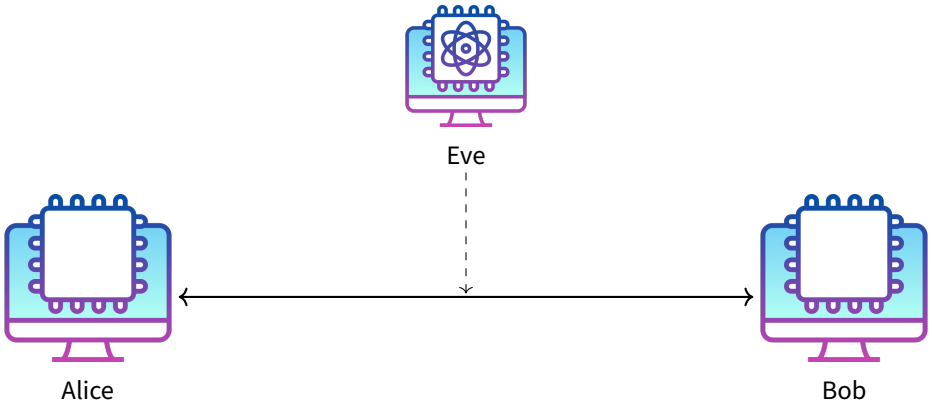
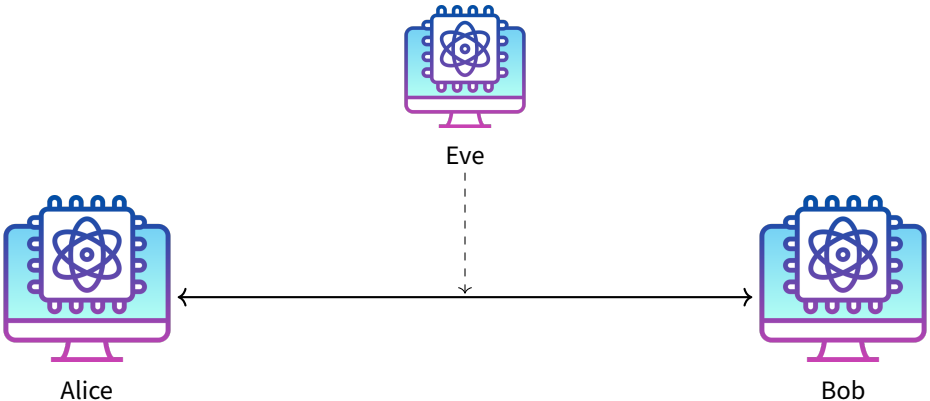


Quantum computers can be used to break classical cryptography [Sho94]!

Post-Quantum Cryptography



Quantum Cryptography



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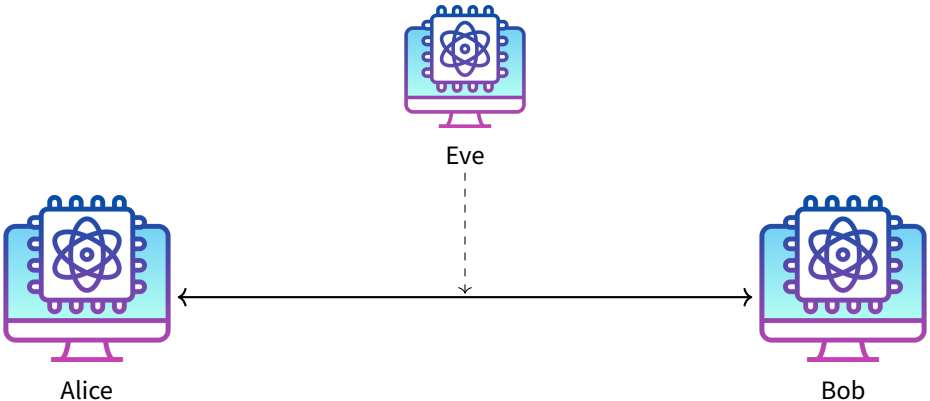
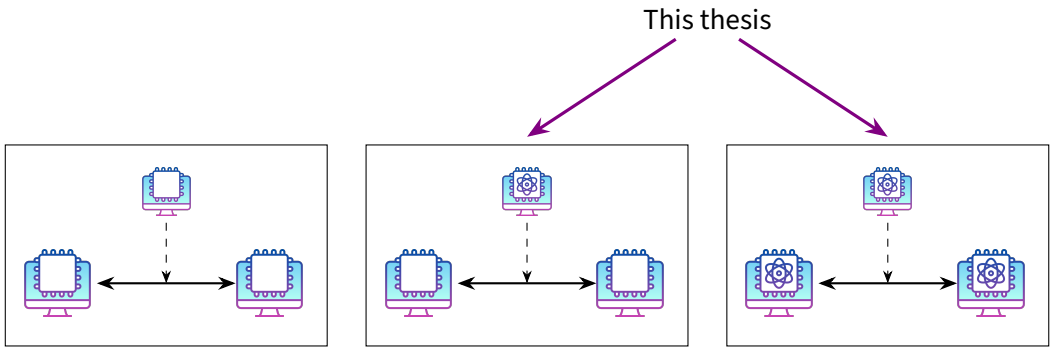


Example: Quantum Key Distribution (QKD) [BB84].

Context


This thesis
Classical Cryptography
Post-Quantum Cryptography
Quantum Cryptography

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- Foundation work
- The goal is to understand the strength of assumptions and primitives

Results

This thesis is separated in three parts:

- Quantum security of Subset Cover Problems [BGV23] (Part I)
- Key Agreements and Public Key Encryption from One-Way Functions [BGVV24]
- On Limits on the Provable Consequences of Quantum Pseudorandomness [BM24, BHMV25] (Part II)

Quantum Security of the Subset Cover Problems

Signatures

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- Alice wants to send a message to Bob
- Bob wants to make sure that the message comes from Alice
- Bob wants to verify the integrity of the message

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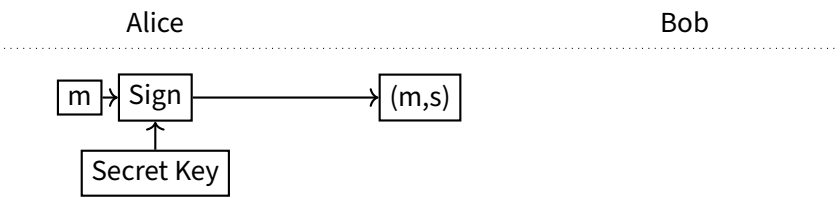

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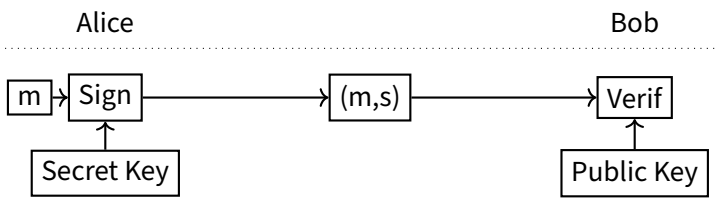

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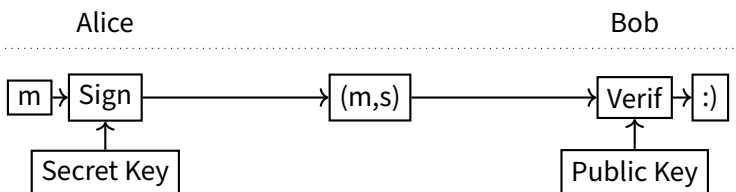

NIST Competition

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- To build SPHINCS(+), we rely on *hash functions*.

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Hash functions

- To build SPHINCS(+), we rely on *hash functions*.
- A hash function maps data of arbitrary length to fixed-sized values.
- We usually require collision resistance and pre-image resistance, but here we need subset-cover resistance.

Subset Cover

Relation between SPHINCS and the SC problem

- A lower bound for the SC problem is necessary for SPHINCS's security.

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Our model

We have quantum access to $h_1, \cdots, h_k$ such that $h_i : \mathcal{X} \to \mathcal{Y}$, and $|\mathcal{Y}| = N$.

Subset Cover

Definition

A (r, k)-subset cover ((r, k)-SC) for $h_1, \ldots, h_k$ consist of $r + 1$ elements $x_0, x_1, \ldots, x_r$ in $\mathcal{X}$ such that:

$$x_0 \notin \{x_1, \ldots, x_r\}, \text{ and}$$

$$\{h_i(x_0) | 1 \leq i \leq k\} \subseteq \bigcup_{j=1}^{r} \{h_i(x_j) | 1 \leq i \leq k\}.$$

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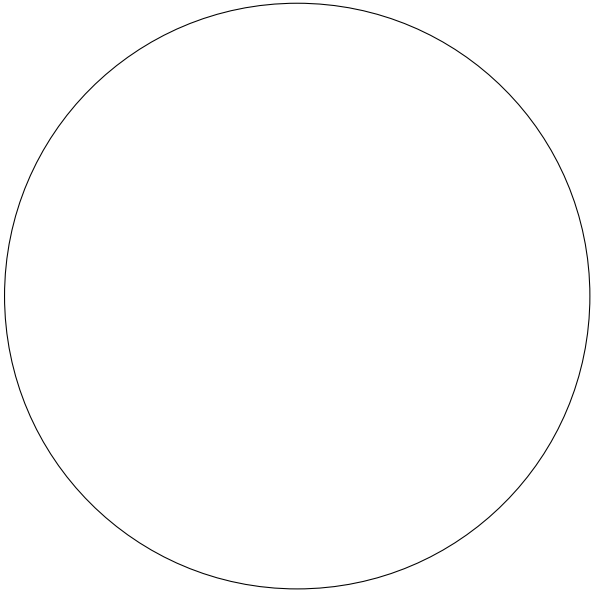

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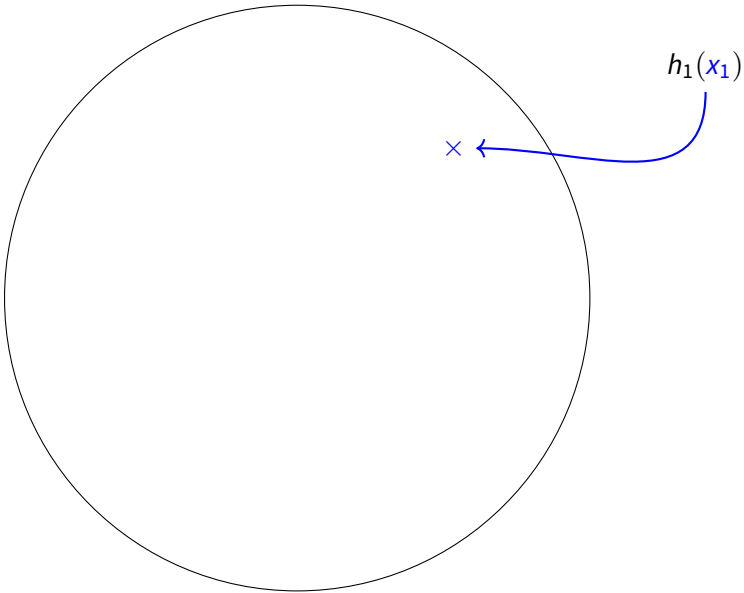

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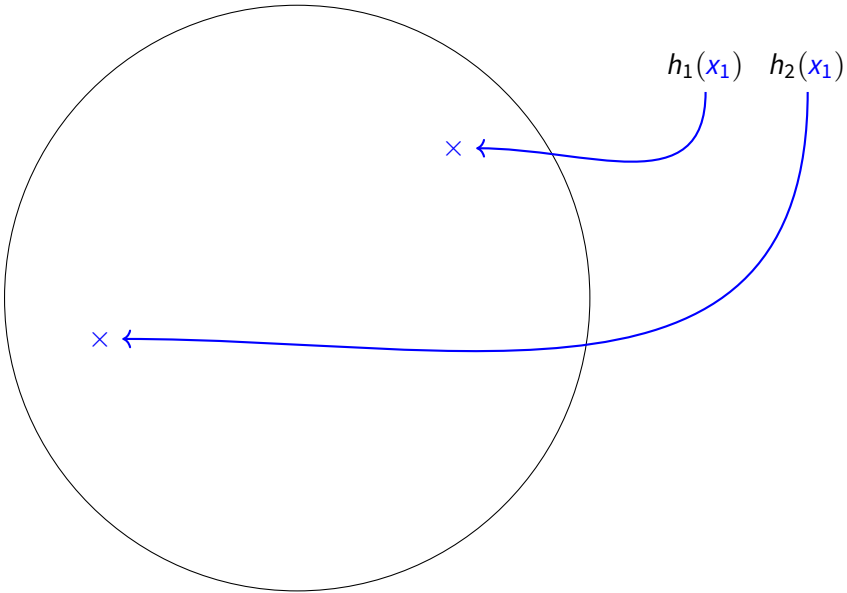

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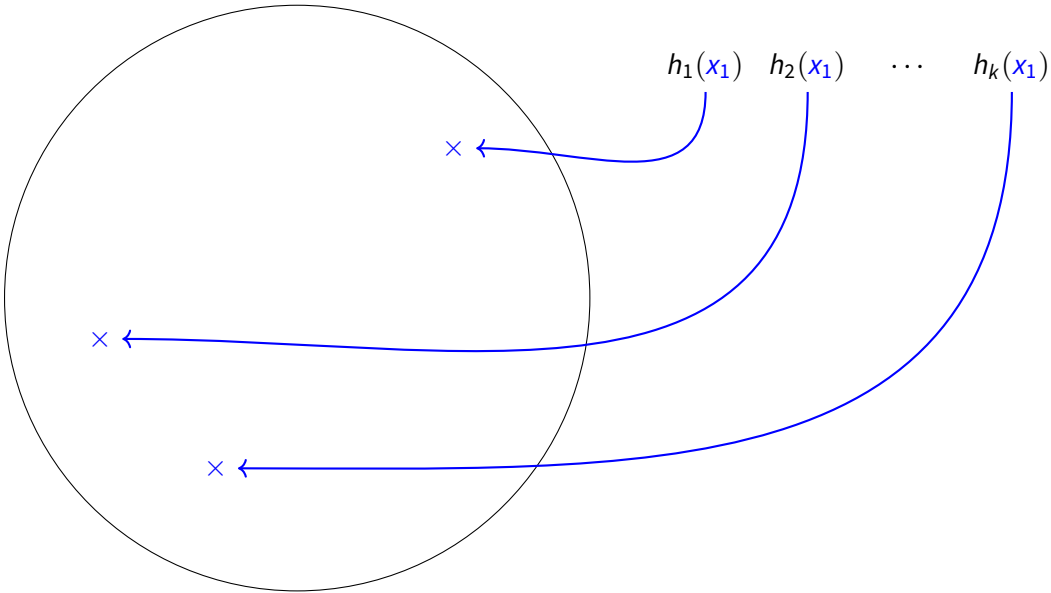

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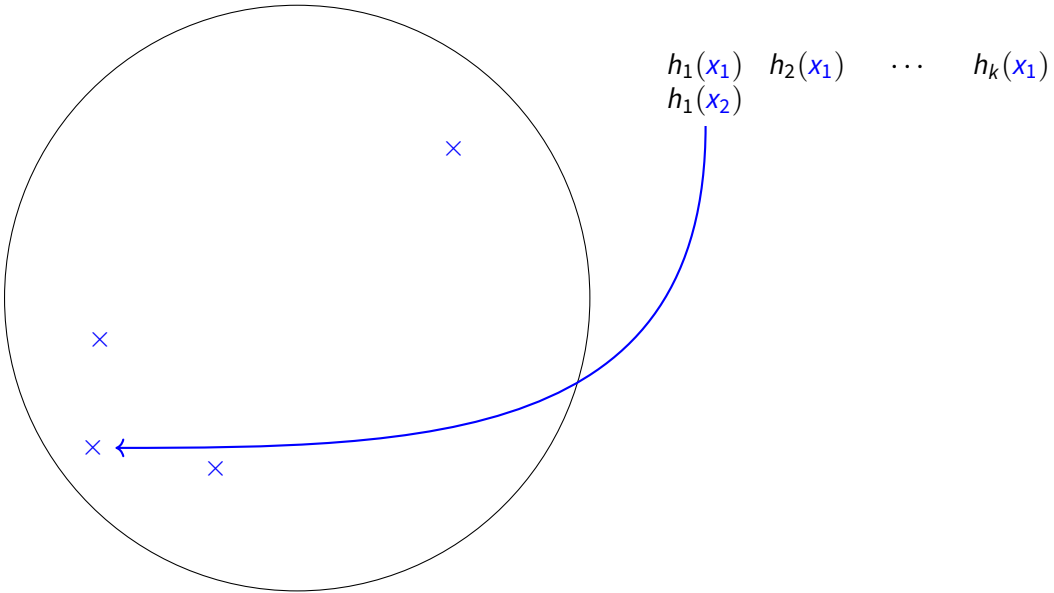

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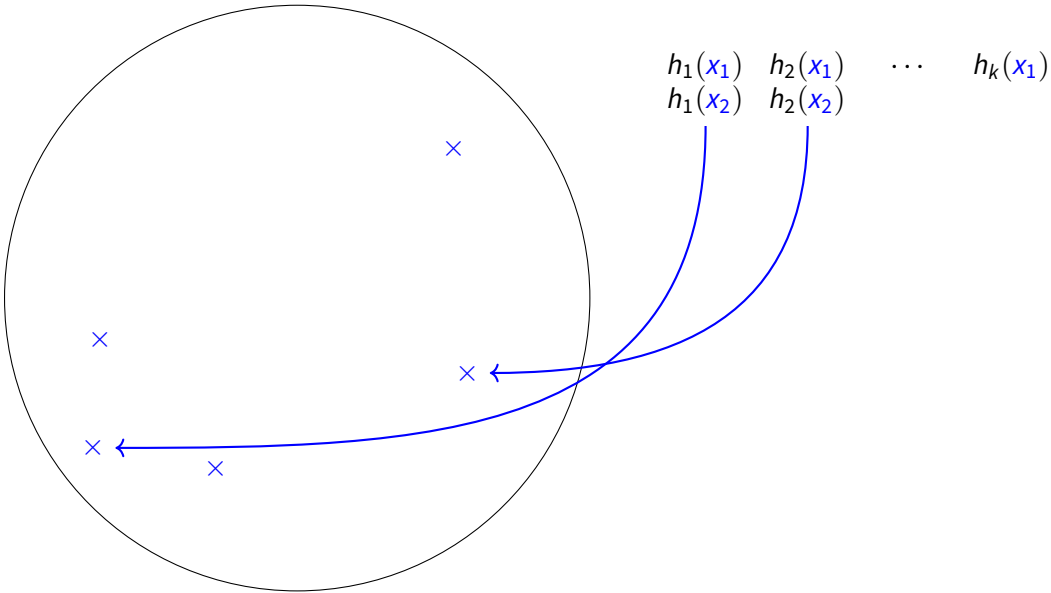

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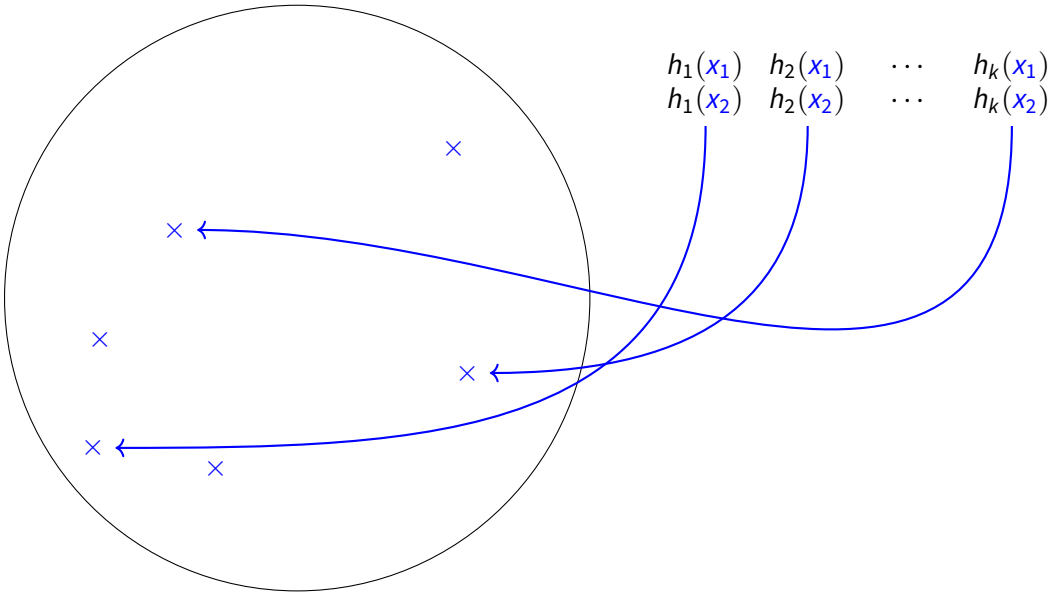

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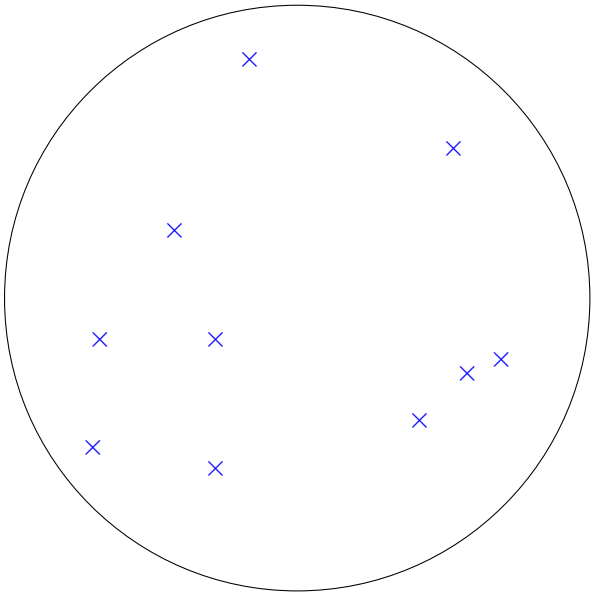

$$\begin{matrix} h_1(x_1) & h_2(x_1) & \cdots & h_k(x_1) \\ h_1(x_2) & h_2(x_2) & \cdots & h_k(x_2) \\ & \vdots & & \\ h_1(x_r) & h_2(x_r) & \cdots & h_k(x_r) \end{matrix}$$

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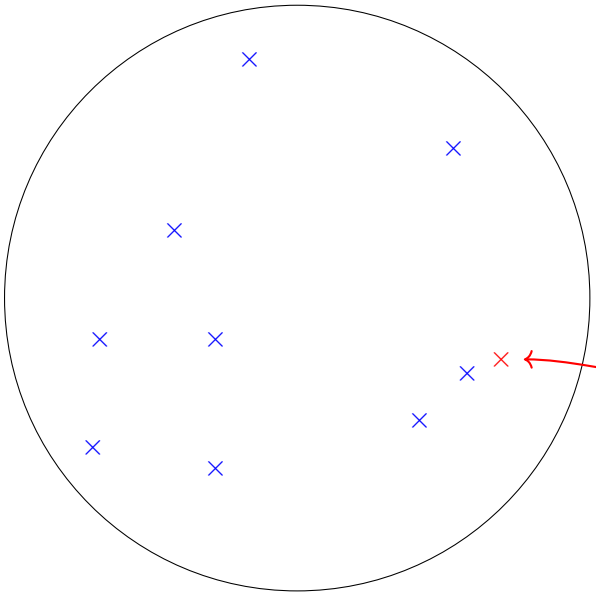

$$
\begin{array}{cccc}
h_1(x_1) & h_2(x_1) & \cdots & h_k(x_1) \\
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\end{array}
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$h_1(x_0)$

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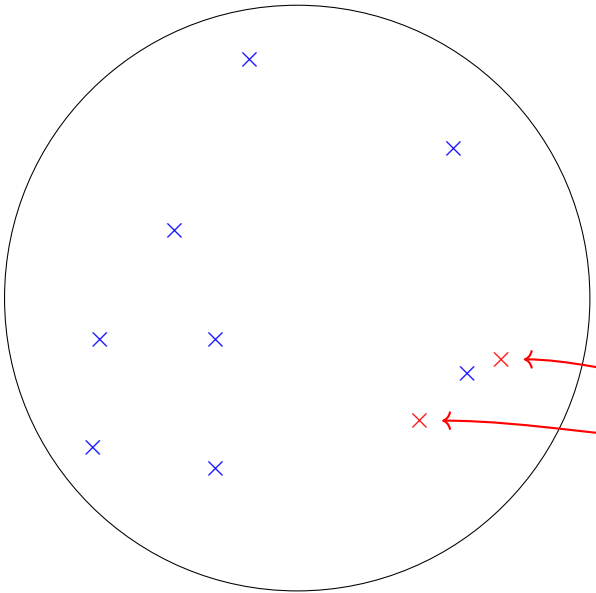

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h_1(x_0) \quad h_2(x_0)
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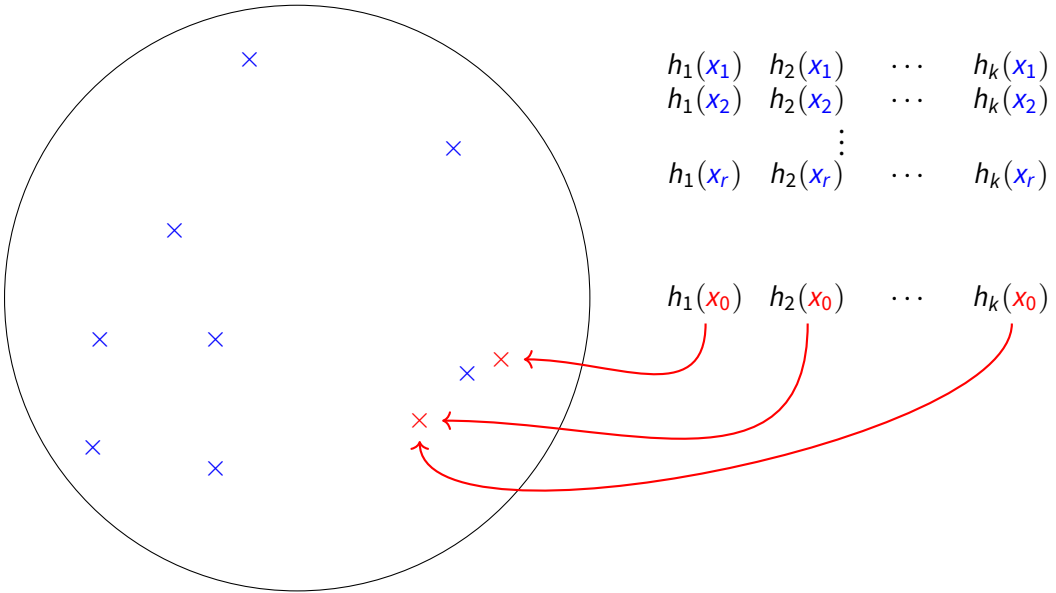
Subset Cover



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Restricted Subset Cover

Definition

A k-restricted subset cover (k–RSC) for $h_1, \ldots, h_k$ consist of $k+1$ elements $x_0, x_1, \ldots, x_k$ in $\mathcal{X}$ such that:

$$x_0 \notin \{x_1, \ldots, x_k\}, \text{ and}$$

$$\begin{aligned} h_1(x_0) &= h_1(x_1), \\ h_2(x_0) &= h_2(x_2), \\ &\vdots \\ h_k(x_0) &= h_k(x_k), \end{aligned}$$

State of the art

Theorem 1 ([YTA22])

There exists an algorithm that find a k–RSC by making $O\left(k \cdot N^{\frac{2^k-1}{2^{k+1}-1}}\right)$ *queries to* $h_1, \ldots, h_k$.

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No other work! (For quantum cryptanalysis)

Our results on k–RSC

Our goal is to start the quantum cryptanalysis of these problems.

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Lower bound on k–RSC

We prove that $\Omega\left(k^{-\frac{2^k}{2^{k+1}-1}} \cdot N^{\frac{2^k-1}{2^{k+1}-1}}\right)$ quantum queries to the idealized hash functions are needed to find a k–RSC with constant probability.

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Lower bound on 2–RSC

Given two random functions $h_1, h_2 : \mathcal{X} \to \mathcal{Y}$ where $|N| = \mathcal{Y}$, a quantum algorithm needs to make $\Omega(N^{3/7})$ queries to h_1 and h_2 to find a 2–RSC with a constant probability.

Quantum Random Oracle

Definition

- In the **Random Oracle Model** (ROM), a hash function is modelized as a *random* function $\mathcal{H} : \mathcal{X} \to \mathcal{Y}$.

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- In the **Random Oracle Model** (ROM), a hash function is modelized as a *random* function $\mathcal{H} : \mathcal{X} \to \mathcal{Y}$.
- In the **Quantum Random Oracle Model** (QROM), the random function $\mathcal{H}$ can be queried in superposition. The function $\mathcal{H}$ is fixed before the computation.

$$\sum_{x,y} \alpha_{x,y} |x\rangle |y\rangle \xmapsto{\text{StO}} \sum_{x,y} \alpha_{x,y} |x\rangle |y \oplus \mathcal{H}(x)\rangle$$

Our Model

The Compressed Oracle Model [Zha19]

- We add a new register, for the database: $|d\rangle$. Let $\mathcal{D}$ the set of all possible database.

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- When we make a query to $\mathcal{O}$:

$$\sum_{x,y,D} \alpha_{x,y,D} |x\rangle |y\rangle |D\rangle \xmapsto{\mathcal{O}} \sum_{x,y,D} \alpha_{x,y,D} |x\rangle |y\rangle |D \oplus (x,y)\rangle$$

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This model is equivalent to the QROM.

Using CROM to compute lower bounds

Database properties [CFHL21]

- A database property P is a subset of $\mathcal{D}$.
- For example, $P_{pre-image} = \{D | \exists x \in \mathcal{X}, (x, 0) \in D\}$.
- Any database property P can be seen as a projector on $\mathcal{D}$, as follows:

$$\sum_{D \in P} |D\rangle \langle D|$$

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Computing lower bounds

If we write $|\psi_i\rangle = \mathcal{O} \cdot \mathrm{U_i} \cdot \mathcal{O} \cdot \mathrm{U_{i-1}} \cdot \cdots \cdot \mathcal{O} \cdot \mathrm{U_1} |0\rangle^{\otimes n}$ the state of an algorithm after i queries to the oracle, we want to bound:

$$|P |\psi_i\rangle| \leq f(i)$$

Such that $f(i)$ does not depend of the algorithm.

Sketch of the proof

Lower bound on 2–RSC

Given two random functions $h_1, h_2 : \mathcal{X} \to \mathcal{Y}$ where $|N| = \mathcal{Y}$, a quantum algorithm needs to make $\Omega(N^{3/7})$ queries to h_1 and h_2 to find a 2–RSC with a constant probability.

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Database properties

- P_2 as the set of databases that contain a 2-RSC.

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- P'_ℓ as the set of databases that contain *at least* ℓ *distinct* collisions on h_1.

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Database properties

- P_2 as the set of databases that contain a 2-RSC.
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Writing $|\psi_i\rangle$ the state after the i^{th} query to $H = (h_1, h_2)$, our goal is to bound $|P_2 |\psi_i\rangle|$.

Sketch of the proof

Claim 1

We have that:

$$\textit{Find with } i \textit{ queries} \leq \textit{Find with } i-1 \textit{ query} + \textit{Find with the } i^{th} \textit{ query.}$$

Sketch of the proof

Claim 1

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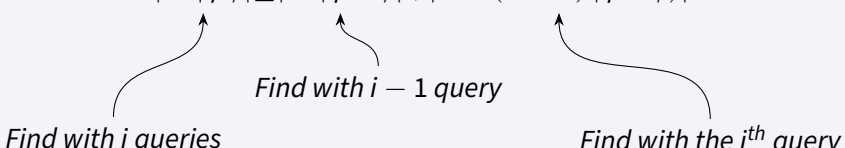
This translates to:

$$|P_2 |\psi_i\rangle| \leq |P_2 |\psi_{i-1}\rangle| + |P_2 \mathcal{O}(I - P_2) |\psi_{i-1}\rangle)|$$



Sketch of the proof

$$|P_2\mathcal{O}(I-P_2)\,|\psi_{i-1}\rangle| = \left|P_2\mathcal{O}\sum_{\substack{x,y\\ D\text{: no 2–RSC}}}\alpha_{x,y,D}\,|x,y,D\rangle\right|$$

Sketch of the proof

$$|P_2\mathcal{O}(I-P_2)\,|\psi_{i-1}\rangle| = \left|P_2\mathcal{O}\sum_{\substack{x,y\\ D\text{: no 2–RSC}}}\alpha_{x,y,D}\,|x,y,D\rangle\right|$$

$$= \left|P_2\sum_{\substack{x,y\\ D\text{: no 2–RSC}}}\alpha_{x,y,D}\,|x,y,D\oplus(x,y)\rangle\right|$$

Sketch of the proof

$$|P_2\mathcal{O}(I-P_2)\,|\psi_{i-1}\rangle| = \left|P_2\mathcal{O}\sum_{\substack{x,y\\ D\text{: no 2–RSC}}}\alpha_{x,y,D}\,|x,y,D\rangle\right|$$

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\end{aligned}
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Sketch of the proof

2-RSC

Recall that a 2–RSC consist of x_0, x_1, x_2 such that:

- $h_1(x_0) = h_1(x_1)$
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We have three possible ways to get from D that does not have a 2–RSC to $D \cup (x, y')$ that has a 2–RSC:

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If there are ℓ collisions on h_1, there are ℓ values of y' such that $D \cup (x, y')$ has a 2–RSC.

$$2 \cdot \sum_{\ell \geq 0} \frac{\ell}{N^2} \left| P'_\ell \left| \psi_{i-1} \right\rangle \right|$$

Sketch of the proof

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There are $(i-1)^2$ values of y' such that $D \cup (x, y')$ has a 2–RSC.

$$\frac{(i-1)^2}{N^2}$$

Sketch of the proof

Thus,

$$\left|P_2\mathcal{O}(I-P_2)\left|\psi_{i-1}\right\rangle\right| \leq \sqrt{2\cdot\sum_{\ell\geq 0}\frac{\ell}{N}\left|P'_\ell\left|\psi_{i-1}\right\rangle\right|^2} + \frac{i-1}{N}$$

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Giving:

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$$\begin{aligned}\left|P_2\left|\psi_i\right\rangle\right| &\leq \left|P_2\left|\psi_{i-1}\right\rangle\right| + \sqrt{2\sum_{\ell\geq 0}\frac{\ell}{N}\left|P'_\ell\left|\psi_{i-1}\right\rangle\right|^2}+\frac{i-1}{N}\\ &\leq \dots\\ &\leq \sum_{s=0}^{i-1}\left(\sqrt{2\sum_{\ell\geq 0}\frac{\ell}{N}\left|P'_\ell\left|\psi_s\right\rangle\right|^2}+\frac{s}{N}\right).\end{aligned}$$

Sketch of the proof

Lemma 2

For every $i \in \mathbb{N}$, we have that:

$$\left|P_2 \left|\psi_i\right\rangle\right| \leq 4\sqrt{e}\frac{i^{7/4}}{N^{3/4}} + 4\frac{i^2}{N} + O\left(N^{-1/48}\right).$$

Sketch of the proof

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For every $i \in \mathbb{N}$, we have that:

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Lower bound on 2–RSC

Given two random functions $h_1, h_2 : \mathcal{X} \to \mathcal{Y}$ where $|N| = \mathcal{Y}$, a quantum algorithm needs to make $\Omega(N^{3/7})$ queries to h_1 and h_2 to find a 2–RSC with a constant probability.

Results on (r, k)–SC

Lower bound on $(1, k)$–SC

We prove that $\Omega\left((k!)^{-1/5} \cdot N^{k/5}\right)$ quantum queries to the idealized hash functions are needed to find a $(1, k)$–SC with constant probability.

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Upper bound on (r, k)–SC

We design a quantum algorithm that finds a (r, k)–SC with constant probability by making $O\left(N^{k/(2+2r)}\right)$ queries to the hash functions.

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Input: $t \in \mathbb{N}, k' \in \mathbb{N}$

1. Apply the algorithm recursively to find t distinct $(r-1, k')$–SC.
2. Apply Grover's algorithm to find a (r, k)–SC.

Summary

Problem	Lower bound	Upper bound	Tight?
k-RSC	$\Omega\left(N^{\frac{2^k-1}{2^{k+1}-1}}\right)$	$O\left(N^{\frac{2^k-1}{2^{k+1}-1}}\right)$	Yes
(r,k)-SC	$\Omega\left(N^{k/(2+2r)}\right)$	$O\left(N^{k/(2+2r)}\right)$	Yes

- State of the art [YTA22]
- Our contribution
- Concurrent work [YTA23]

On Limits on the Provable Consequences of Quantum Pseudorandomness

Minimal assumption for classical cryptography

One-Way Functions

A function $f : \{0,1\}^n \to \{0,1\}^n$ is a One-Way Function (OWF) if:

1. $f(x)$ can be computed efficiently.
2. Given $y = f(x)$, it is hard to compute x.

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One-Way Functions are minimal for cryptography

- Most advanced cryptographic schemes require one-way functions.
- For example, a hash function has to be a one-way function.
- There is nothing (interesting) weaker.

Pseudorandomness

Pseudorandom Number Generator

A function $g : \{0,1\}^n \to \{0,1\}^\ell$ is a Pseudorandom Number Generator (PRNG) if:

1. $g(x)$ can be computed efficiently.
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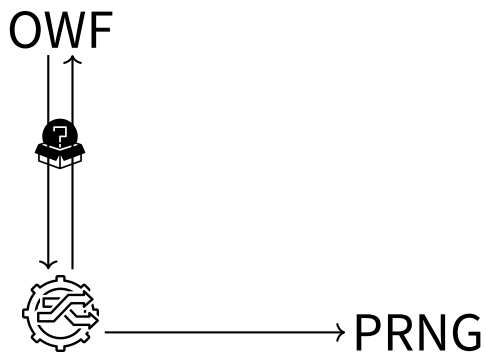
Theorem 3 ([HILL99])

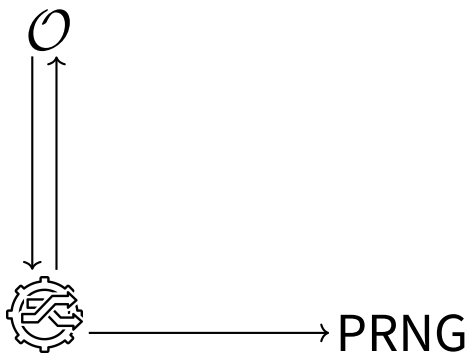
PRNG and OWF are equivalent in a black-box way.

Black-box constructions



Black-box constructions

$\mathcal{O}$ such that OWF exist



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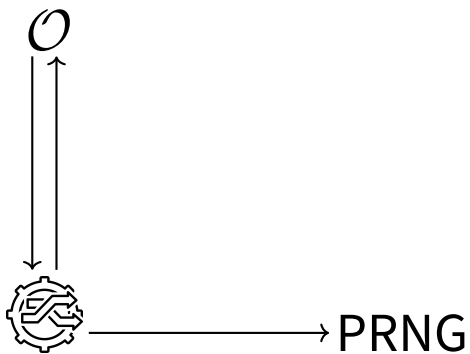

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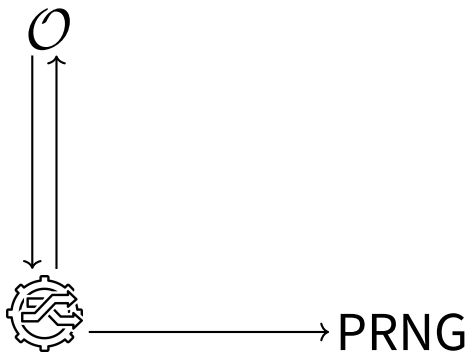


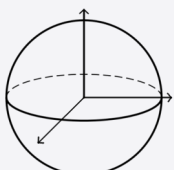
A black-box impossibility construction of $\mathcal{A}$ from $\mathcal{B}$ is an oracle $\mathcal{O}$ such that, relative to $\mathcal{O}$, $\mathcal{B}$ exists but not $\mathcal{A}$.

Quantum Pseudorandomness

Quantum Randomness

We can also consider quantum randomness.

The equivalent to the uniform distribution is the *Haar measure* μ_{2^n}.



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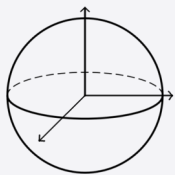

Pseudorandom Quantum States Generators [JLS18]

A function $F : \{0,1\}^\lambda \to \left(\mathbb{C}^2\right)^{\otimes n}$ is a Pseudorandom Quantum State generator (PRS) if:

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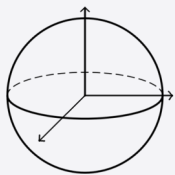

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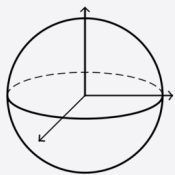

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Claim 2

We can build Quantum Cryptography from PRSs.

oblivious transfer, multi party computation, public key encryption with quantum keys, quantum one-time digital signatures, pseudo one-time pad encryption schemes, statistically binding and computationally hiding commitments and quantum computational zero knowledge proofs, bit commitments...

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There can be quantum pseudorandomness even if

“P = NP”

Different type of Quantum Pseudorandomness

PRU, PRFS, PRS, 1PRS, EFI pairs, OWSG...

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In the quantum setting however...

The Landscape of Quantum Assumptions

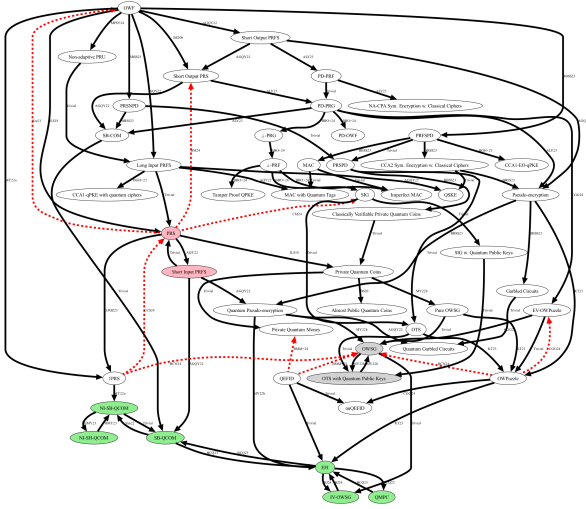

Figure: https://sattath.github.io/microcrypt-zoo/

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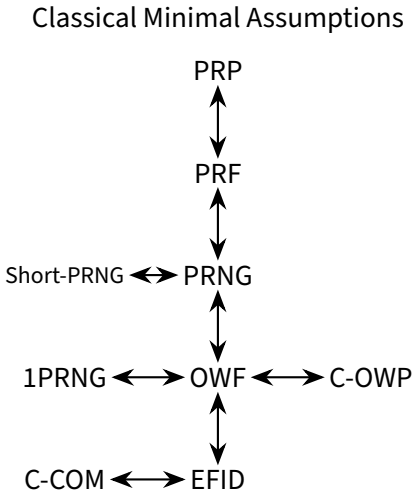

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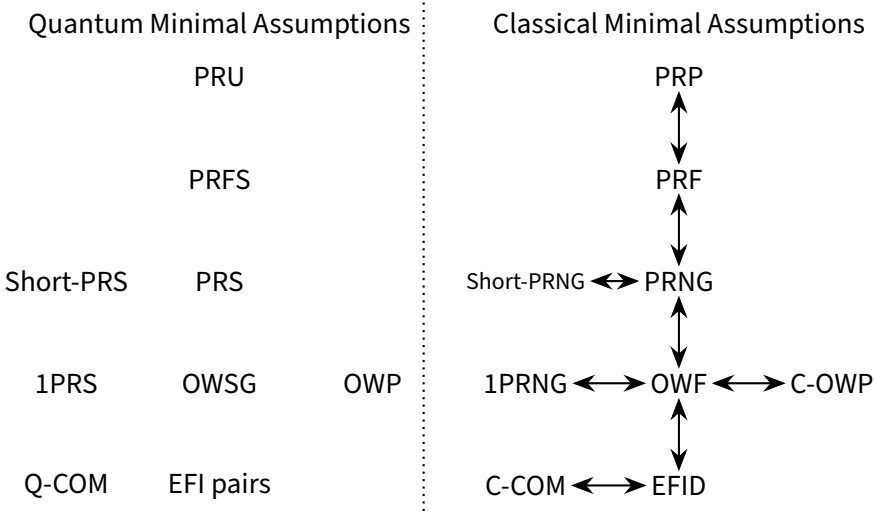

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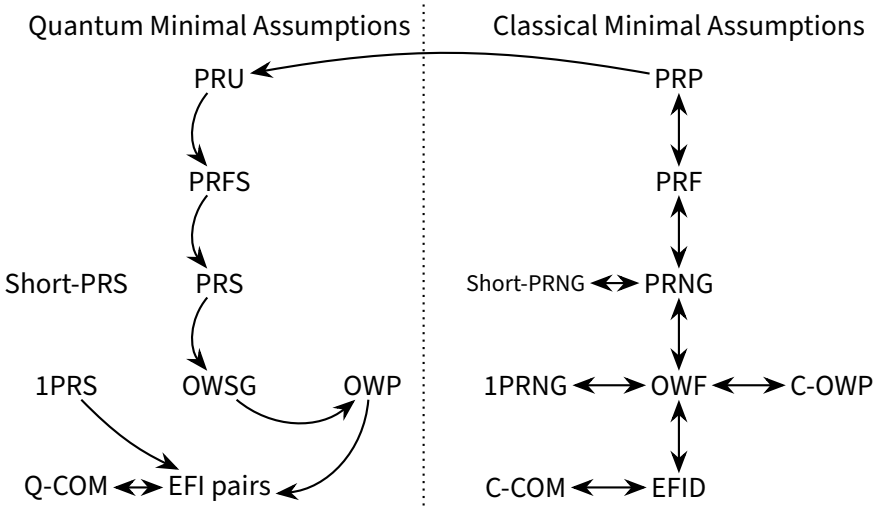

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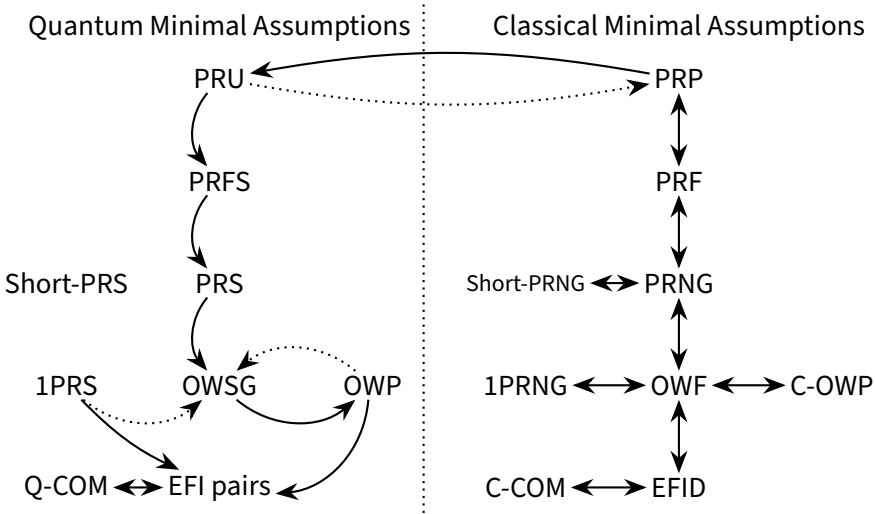

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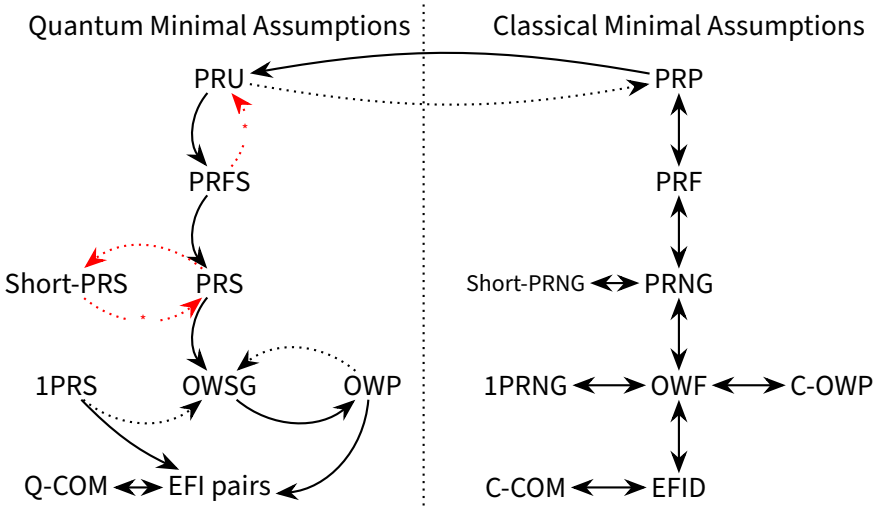

Formal definition of PRSs

Definition 6 (Pseudorandom quantum states [JLS18])

A keyed family of n-qubit quantum states $\{|\varphi_k\rangle\}_{k\in\{0,1\}^\lambda}$ is *pseudorandom* if the following two conditions hold:

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Main result

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Assuming Conjecture 1, there exists an oracle $\mathcal{O}$ relative to which short-PRSs exist, but long-PRSs (with reversible generation algorithm) do not.

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Remember: A black-box impossibility construction of $\mathcal{A}$ from $\mathcal{B}$ is an oracle $\mathcal{O}$ such that, relative to $\mathcal{O}$, $\mathcal{B}$ exists but not $\mathcal{A}$.

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Now let's rule out long-PRSs!

Breaking PRSs

By contradiction, consider the long-PRS algorithm $\{G_k\}_{k\in\{0,1\}^*}$:

$$G_k : |0\rangle^n \mapsto |\psi_k\rangle = U_T^{(k)} \; S_{x_T^{(k)}} \; \ldots \; U_1^{(k)} \; S_{x_1^{(k)}} \; U_0^{(k)} \; |0\rangle^n$$

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We assume there is no trace out.

Attack idea

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- However, we want $\mathcal{O}_2$ not to break short-PRS! To do so, we make it independent of $\mathcal{O}_1$.
- We will approximate G_k without querying $\mathcal{O}_1$, and $\mathcal{O}_2$ will be independent of $\mathcal{O}_1$.

Uncomputing the Generation

Claim 5

Because the output is pure, we can write G_k as:

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- We can exploit the *structure* of the input state, by uncomputing and applying a product state test.
- Problem: how to learn the x_i?

The Concentration Inequality

Concentration inequality (informal)

Let $\mathcal{X}$ be the product space of pure quantum states with the corresponding product Haar measure σ_N. Suppose that $f : \mathcal{X} \to \mathbb{R}$ is L-Lipschitz in the Frobenius norm. For every $t > 0$, it holds that

$$\Pr_{U \leftarrow \sigma_N} \left[f(U) \geq \mathbb{E}_{V \leftarrow \sigma_N} [f(V)] + t \right] \leq \exp\left(-\frac{(N-2)t^2}{24L^2} \right).$$

Consequence

We cannot extract information from long-PRS.

Our Conjecture

Conjecture 1 (informal)

Let X be the product space of pure quantum states with the corresponding product Haar measure σ. If S_0, S_1 are two measurable subsets of X such that $\sigma(S_0), \sigma(S_1) \geq A$, and if $d(S_0, S_1) \geq B$ for some distance d on X, then $\sigma(X \setminus (S_0 \cup S_1)) \geq \text{poly}(A, B)$.

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Prepares $\Phi = (|\rho\rangle \otimes |\rho\rangle)^{\otimes M}$ for some large M and do for all k:

P_k: Apply $(G_k^\dagger \otimes G_k^\dagger)^{\otimes M}$, apply M product-tests on each copy; if they pass, return 1. Otherwise, return 0.

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Inefficient, but we can chose $\mathcal{O}_2 =$ QPSPACE! (Quantum OR Lemma)
Relative to $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)$ =(CHS,QPSPACE), we have short-PRSs but not long-PRSs!

Summary

Theorem 8 ([BHMV25])

Assuming Conjecture 1,

$$\exists \textit{short-PRS} \not\Rightarrow \exists\ \textit{long-PRS}$$

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Theorem 10

$$\exists \textit{PRU} \not\Rightarrow \exists \textit{ (some) qPKE}$$

Conclusion

On subset cover

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- We study problems in the Quantum Random Oracle Model that are linked to post-quantum schemes, namely SPHINCS and SPHINCS+.

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Perspectives

- However, it remains to prove the security of these schemes against quantum adversaries.

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Thank you for your attention.

Bibliography


Charles Bennett and Gilles Brassard.
Quantum cryptography: Public key distribution and coin tossing.
volume 560, pages 175–179, 01 1984.


Samuel Bouaziz-Ermann, Alex B. Grilo, and Damien Vergnaud.
Quantum security of subset cover problems.
In Kai-Min Chung, editor, *ITC 2023*, volume 267 of *LIPIcs*, pages 9:1–9:17. Schloss Dagstuhl, June 2023.


Samuel Bouaziz-Ermann, Alex B. Grilo, Damien Vergnaud, and Quoc-Huy Vu.
Towards the impossibility of quantum public key encryption with classical keys from one-way functions.
CiC, 1(1):32, 2024.


Samuel Bouaziz–Ermann, Minki Hhan, Garazi Muguruza, and Quoc-Huy Vu.
On limits on the provable consequences of quantum pseudorandomness.
2025.


Samuel Bouaziz–Ermann and Garazi Muguruza.
Quantum pseudorandomness cannot be shrunk in a black-box way.
arXiv preprint arXiv:2402.13324, 2024.


Kai-Min Chung, Serge Fehr, Yu-Hsuan Huang, and Tai-Ning Liao.

The tools

Lemma 11 (Quantum OR lemma)

Let $\{\Pi_i\}_{i\in[N]}$ be POVMs. Let $0 < \varepsilon < 1/2$ and $\delta > 0$. Let Ψ be a quantum state such that either

1. *there exists $i \in [N]$ such that $\mathrm{Tr}[\Pi_i\Psi] \geq 1-\varepsilon$, or*
2. *for all $i \in [N]$, $\mathrm{Tr}[\Pi_i\Psi] \leq \delta$.*

Then, there is a quantum circuit C, such that in case i)

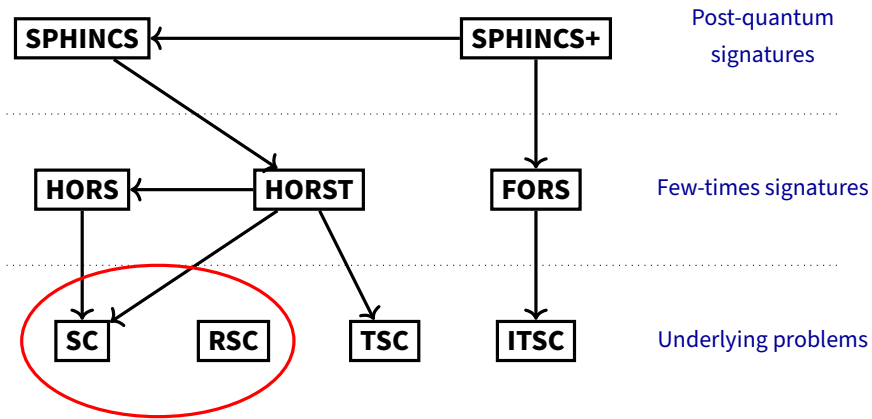
$$\Pr(1 \leftarrow C(\Psi)) \geq \frac{(1-\varepsilon)^2}{7},$$

and in case ii),

$$\Pr(1 \leftarrow C(\Psi)) \leq 4N\delta.$$

The circuit C can be implemented in QPSPACE.

Motivation


SPHINCS
SPHINCS+
Post-quantum signatures
HORS
HORST
FORS
Few-times signatures
SC
RSC
TSC
ITSC
Underlying problems

Our algorithm

Algorithm for finding a (r, k)–SC

Input: $t \in \mathbb{N}, k' \in \mathbb{N}$.

1. Execute the $(r-1, k')$–SC algorithm t times to find t distinct $(r-1, k')$–SC in $h_1, \ldots, h_{k'}$.
   Let T be the set of these $(r-1, k')$–SC.
2. Define $F : \mathcal{X} \to \{0, 1\}$ as follows:

$$F(x) = \begin{cases} 1, & \text{if there exists } (x_0, x_1, \ldots, x_{r-1}) \in T \text{ such that} \\ & \forall 1 \leq m \leq k - k', h_m(x) = h_{k'+m}(x_0), \\ 0, & \text{otherwise.} \end{cases}$$

3. Execute Grover's algorithm to find an x such that $F(x) = 1$
4. Find $(x_0, x_1, \ldots, x_{r-1})$ in T and output $(x_0, x_1, \ldots, x_{r-1}, x)$.