Lecture 1: Probabilities

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May 16, 2016

1 Probability counting

In this class we are going to get in touch with one of the most important subjects in many fields of science: probabilities.

To use probabilities, we work with a set of outcomes. These outcomes can be grouped up by a common property that we can call an event.

Probability of an event A is given by a simple formula:
$\mathbb{P}[A] = \frac{\#\text{outcomes corresponding to } A}{\#\text{all outcomes}}$

Remark Note that $\mathbb{P}[A] \in [0, 1]$.

Example 1 (Common uniform distributions) *After tossing a coin two outcomes are possible: heads and tails.*
$\mathbb{P}[heads] = \mathbb{P}[tails] = \frac{1}{2}$
Rolling a dice produces six possible outcomes: 1,2,3,4,5,6.
Probabilities of getting any of the individual outcomes is $\frac{1}{6}$.

Problem 1 (Rolling a pair of dice) *What is the probability to get* 7 *after rolling two dice?*

Solution. The number of all possible outcomes is $6 \cdot 6 = 36$. There are 6 ways to get the sum 7: $7 = 1 + 6 = 2 + 5 = 3 + 4 = 4 + 3 = 5 + 2 = 6 + 1$.

Thus, the probability is $\mathbb{P}[\text{sum is } 7] = \frac{1}{6}$

Problem 2 *How to simulate one dice by rolling three dice?*

Remark Imagine a black box with three dice inside. The box rolls the dice for you and shows the sum. This is the only thing you can use!

Solution. The ways to get different sums are given in the table below:

$\sum$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
#	1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1

The overall number of outcomes after rolling three dice is $6^3 = 216$. Our goal is to make six groups of outcomes, so that probabilities in each of the groups are equal. The sum of outcomes in all these groups should be equal to $\frac{216}{6} = 36$.

Here are the candidates for the groups:

- $G_1 = \{\sum = 4, 5, 10\}$
- $G_2 = \{\sum = 11, 16, 17\}$
- $G_3 = \{\sum = 3, 6, 9\}$
- $G_4 = \{\sum = 12, 15, 18\}$

- $G_5 = \{\sum = 7, 8\}$
- $G_6 = \{\sum = 13, 14\}$

Problem 3 *Can you simulate a dice roll by consecutively tossing a coin?*

2 Independence

Say, a married couple has a child. The probability of him being a boy is $\frac{1}{2}$. Imagine that this couple has another child. What is the probability of him to be a boy? And if you know that the first child was a boy, what is the probability this time?

Such a phenomenon is called independence of two events.

It occurs when knowing the first event does not affect the probability of the second one AND vice versa.

The probability, which takes into account knowledge is called conditional probability and is noted by $\mathbb{P}[A_2|A_1]$ (read it as "probability of A_2 knowing A_1").

Not all events are independent, but when they are they admit a remarkable identity:

$$A_1 \text{ and } A_2 \text{ are independent} \quad \Rightarrow \quad \mathbb{P}[A_1 \text{ and } A_2] = \mathbb{P}[A_1]\mathbb{P}[A_2]$$

Problem 4 *A box contains two black balls and three white balls. What are the probabilities of all possible outcomes of taking two balls from the box?*

Are first and second takes independent?

3 Expected value

When the outcome is a number (real or integer), the whole processus can be modelled by something called a random variable. What is it value? We don't know. But we may ask ourselves a question: what is the average value it can get? To answer this, a special notion is introduced: the expected value.

Definition 1 *Let x_1, x_2,..., x_N all possible outcomes for a random variable X. The expected [or the mean] value of X is:*

$$\mathbb{E}[X] = \sum_{i=0}^{N} \mathbb{P}[X = x_1]x_1$$

In the uniform distributions of probabilities we've seen, like coin tossing or dice rolling, the expected value is no different from a regular arihmetic mean. But if we take a non-uniform distribution, this can lead to some interesting results.

Problem 5 *Coin is tossed until the outcome is heads. How many tossings is there in expected value?*

Solution.

Take on the problem from the other point of view. Say, we tossed the coin k times, that means that the coin showed tails $k-1$ times and heads one time.

The probability of such an outcome is exactly $\frac{1}{2^k}$.

So the expected value is the following infinite sum:

$$\sum_{k=1}^{\infty} \frac{k}{2^k} = 2$$

To see how to obtain the last magical transformation, you can see the next lecture on power series.

4 Classical problems

4.1 Monty Hall problem

The following problem is inspired from a famous show...

Problem 6 (Monty Hall problem (version 1)) *The player is given three doors to choose one. Behind one of the doors lies a brand new Bugatti sports car. Behind two others there are two goats. The player chooses one door. Monty Hall, the host of the show, opens another door with a goat behind and proposes the player to either stay on his choice or to change. What should he do?*

Solution. Let's count the probabilities of winning by changng the door. Two possible cases are possible.

1. If the player chose the winning door (probability $= \frac{1}{3}$), Monty Hall chooses in this case whatever of the rest of the doors. Player loses in this setting.
2. If the player chose the losing door (probability $= \frac{2}{3}$), Monty Hall has no choice but to open the other losing door. Player wins in this setting.

So the probability of winning is entirely determined by whether the player chose the right door in the beginning, hence chance of winning by changing doors is $\frac{2}{3}$.

To see the solution in a clearer way, here is another version of the same problem, this time with 100 doors.

Problem 7 (Monty Hall problem (version 2)) *The player is given 100 doors to choose one. Behind one of the doors lies a brand new Bugatti sports car. Behind 99 others there are goats. The player chooses one door. Monty Hall, the host of the show, opens 98 doors with goats behind and proposes the player to either stay on his choice or to change. What should he do?*

4.2 Bonus: Birthday paradox

Let's solve another classical problem.

Problem 8 (Birthday paradox) *Take a class consisting of 23 students. Show that the probability of "at least two people having the same birthday" is at least $\frac{1}{2}$.*

You can safely assume that February 29 doesn't exist.

Solution. Let $C = \{s_1, ..., s_{23}\}$ be the set of students in the class.

The event $A =$ "at least two people having the same birthday" is the complementary of $A' =$ "all students have different birthdays". The probability of the last is actually easier to compute because it can be decomposed into independent events of the form: $E_i =$ "s_i's birthday is different from all s_1,...,s_{i-1}".

A probability of E_i to occure is $\frac{365-i+1}{365}$, so the probability of A' is:

$\mathbb{P}[A'] = \frac{365}{365}\frac{364}{365}...\frac{(365-22)}{365} < \frac{1}{2}$.

Therefore $\mathbb{P}[A] = 1 - \mathbb{P}[A'] > \frac{1}{2}$.