Lecture 5: Generating functions

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May 20, 2016

1 Modelize objects using variables

Take a set S and define a **size** to its objects. Let a_i be the number of objects in S of size i. We will represent this set as an "infinite" polynomial:

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 ... = \sum_{n=0}^{\infty} a_n x^n$$

This polynomial is also called a **generating function**, and it is an often-used tool to count objects in sets.

Several simple operations on sets are generalizable to generating functions, to name a few:

- The generating function of the disjoint union of two sets is the **sum** of their generating functions. It is actually easy to see, if you think it in this way: imagine you have two boxes with objects in them and you dump their content in one big box. It is natural that the number of objects of size i is the sum of numbers of objects of the same size from the previous boxes.
- The generating function of the cartesian product of two sets is the **product** of their generating functions. This one is a little trickier to see. Instead of taking the objects individually, you make all possible pairs out of them with one object from each set. The size of such a pair is naturally the sum of the sizes of its components. So to count all such pairs *of constant size* of size n, we just need to try to combine all possible objects from the set 1 of size k and all possible objects from the set 2 of size $n-k$.

The addition and the multiplication operations are defined in the same way as for polynomials:
If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$, then:

$$(A+B)(x) = \sum_{n=0}^{\infty} (a_n + b_n)x^n$$

$$(AB)(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$$

We have seen in the previous lectures that the expansion of $(y+x)^n$ produces binomial coefficients $\binom{n}{k}$. But why is it so? To form a monomial $x^k y^{n-k}$ we are forced in some way to choose k brackets for x's and the rest will produce y's. Therefore, the number of ways to produce the same monomial is exactly the number of ways to choose k brackets, which is exactly $\binom{n}{k}$.

We know that the product in generating functions corresponds to take couples of objects. And in the case of the product of n generating functions, it corresponds to a n-tuple. The polynomial $(x+y)$ can be seen as the generating function for a set $S = \{x, y\}$ of two objects. So the n-tuples in S^n represent all possibilities of choosing x or y in each of those sets, and for choosing exactly k x's and $(n-k)$ y's the number of choices is exactly $\binom{n}{k}$.

There are several other classical generating functions. For instance, the set $\{0, 1, 2, 3, ...\} = \mathbb{N}$ is easily represented by $1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$.

Problem 1 *You possess an infinite number of coins of value* 1¢, 2¢, 5¢, 10¢, 20¢, 50¢. *Find the generating function, where each coefficient a_n is the number of ways to compose n¢ with the aforementioned coins.*

Solution. We will construct a generating function of the sets of all compositions of coins.

Every composition can be seen as the 6-tuple $(k_1, k_2, k_5, k_{10}, k_{20}, k_{50})$, with k_i the number of coins of value i in the composition. So, what we are looking for is the cartesian product of 6 natural numbers. We are looking for the value of the sum, so naturally x would signify 1¢.

Each k_i counts for i¢, so the generating function for the set of all i¢ coins is $1+x^i+x^{2i}+x^{3i}+... = \frac{1}{1-x^i}$. So the cartesian product, that we are looking for is exactly:

$$\frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^5}\frac{1}{1-x^{10}}\frac{1}{1-x^{20}}\frac{1}{1-x^{50}}$$

2 Solve recurrencies

One of the most popular uses for the generating functions is to count recursive structures. We will see two examples.

Problem 2 *What is the number of binary trees of size n?*

Solution. A binary tree can fall in two of the following cases:

1. Either it is empty.
2. Or it has a root and two children, which are binary trees. In other words, it is a triplet $\{\text{root}, \text{leftchild}, \text{rightchild}\}$.

Let $B(x)$ be the generating function for binary trees. Then, the previous remark implies:

$$B(x) = 1 + xB(x)^2$$

If we solve this equation in terms of $B(x)$, then we find:

$$B(x) = \frac{1 \pm \sqrt{1-4x}}{2x} \tag{1}$$

The square root generating function can be obtained using the generalized binomial coefficients:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-k+1)}{k!}$$

So now we can express $(1-4x)^{1/2}$:

$$
\begin{aligned}
(1-4x)^{1/2} &= \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n \\
&= \sum_{n=0}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)...\left(\frac{1}{2}-n+1\right)}{n!}(-4)^n x^n \\
&= \sum_{n=0}^{\infty} \frac{1\cdot(-1)\cdot(-3)\cdot...(-2n+3)}{2^n n!}(-4)^n x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}\cdot 1\cdot 3\cdot...(2n-3)}{n!}(-2)^n x^n \\
&= -\sum_{n=0}^{\infty} \frac{1\cdot 3\cdot...(2n-3)}{n!}2^n x^n
\end{aligned}
$$

One can easily observe that if $n \leq 1$ then:

$$
1\cdot 3\cdot...(2n-3) = \frac{1\cdot 2\cdot 3\cdot...(2n-2)}{2\cdot 4\cdot...\cdot(2n-2)} = \frac{(2n-2)!}{2^{n-1}(n-1)!}
$$

So:

$$
\begin{aligned}
(1-4x)^{1/2} &= 1-\sum_{n=1}^{\infty} \frac{(2n-2)!}{2^{n-1}n!(n-1)!}2^n x^n \\
&= 1-2x\sum_{n=1}^{\infty} \frac{(2n-2)!}{n(n-1)!(n-1)!}x^{n-1} \\
&= 1-2x\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1}x^{n-1} \\
&= 1-2x\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}x^n
\end{aligned}
$$

The last identity gives us a hint on what sign to choose in Equation 1. If we choose + sign, then the numerator would start with 2, which can't be divided by $2x$. So we are forced to choose the $-$ sign.

Therefore:

$$
\begin{aligned}
B(x) &= \frac{1-\left(1-2x\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n\right)}{2x} \\
&= \frac{2x\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n}{2x} \\
&= \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n
\end{aligned}
$$

So the number of binary trees of size n is $\frac{1}{n+1}\binom{2n}{n}$.

The number that we just have found is called **Catalan number** and it is a number that has many interpretations in combinatorics (at least 66 `https://en.wikipedia.org/wiki/Catalan_number`).

Generating functions can be also used to solve identities that are hard to prove otherwise:

Problem 3 *Prove:*

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$$

Solution. The strategy of the proof is the following:

1. Express a generating function for the sequence (4^n) and call it f.
2. Find a generating function $g = \sum_{n=0}^{\infty} g_n x^n$, such that $\sum_{k=0}^{n} g_k g_{n-k} = 4^n$.
3. Compute g_n.

The generating function for (4^n) is of course:

$$f(x) = \sum_{n=0}^{\infty} (4x)^n = \frac{1}{1-4x}$$

Let g be a generating function such that $g(x)^2 = f(x)$. It means that $g(x) = \frac{1}{\sqrt{1-4x}} = (1-4x)^{-1/2}$. Use the generalized binomial coefficient formula:

$$\binom{-1/2}{n} = \frac{(-\frac{1}{2})(-\frac{1}{2}-1)...(-\frac{1}{2}-n+1)}{n!} = \frac{(-1)^n 1 \cdot 3 \cdot ... \cdot (2n-1)}{2^n n!} = \frac{(-1)^n (2n)!}{2^n n! 2^n n!} = \frac{(-1)^n}{4^n}\binom{2n}{n}$$

So:

$$g(x) = (1-4x)^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}\binom{2n}{n}(-4x)^n = \binom{2n}{n}x^n$$

It is exactly the generating function corresponding to the left hand side of the identity, hence we have proven it.