Séminaire Landau

Non linear controllability of ODE and PDE with several controls

Théo Gherdaoui

under the supervision of Karine Beauchard and Frédéric Marbach

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Introduction :

Controllability : ajust a function parameter of an ODE/PDE so that the solution reaches a target at a fixed time.

ODE : affine systems (finite dimension).

PDE : bilinear Schrödinger equation with two controls.

Definition (Affine Systems)

An ODE is an affine system if it is given by :

$$x' = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \tag{1}$$

where $x(t) \in \mathbb{R}^d$ is the stat, and $u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{pmatrix} \in \mathbb{R}^m$, the control.

The function f_0 is the **drift**.

$$x' = f_0(x) + \sum_{i=1}^{m} u_i f_i(x).$$

One supposes that $f_0, \cdots, f_m$ are regular functions, and $f_0(0) = 0$ (equilibrium). One considers $x(t; u, p)$ the solution of (1) at time t, with the initial condition $x(0; u, p) = p$.

$L^\infty-$**S**mall **T**ime **L**ocally **C**ontrollable : $\forall T > 0$, $\forall \eta > 0$, $\exists \delta > 0$, so that $\forall x^* \in \mathbb{R}^d$, $\|x^*\| \leq \delta$, then, $\exists u \in L^\infty(0, T)$ such that $\|u\|_\infty \leq \eta$ and $x(T, u, 0) = x^*$.

No SNC
A few NC/SC are known $\longrightarrow$ in term of Lie Brackets.

Definition (Lie Brackets)

Let Ω be a nonempty openset of $\mathbb{R}^d$, and $X, Y : \Omega \to \mathbb{R}^d$ vector fields, $\mathcal{C}^1$. One defines the Lie Bracket of X, and Y, $[X, Y]$ as :

$$[X, Y] : x \in \Omega \mapsto DY(x)(X(x)) - DX(x)(Y(x)) \in \mathbb{R}^d.$$

$[X, Y] \in \mathcal{C}^0(\Omega, \mathbb{R}^d)$.

Definition (Iterated Lie Brackets)

Let Ω be a nonempty openset of $\mathbb{R}^d$, and $X, Y : \Omega \to \mathbb{R}^d$ vector fields, $\mathcal{C}^\infty$. One defines, for $k \in \mathbb{N}$, $ad_X^k(Y)$ by induction as :

$$ad_X^0(Y) = Y \text{ and } ad_X^{k+1}(Y) = [X, ad_X^k(Y)].$$

Theorem (Hermann and Nagano, 1963-66)

Let $f_0, \cdots, f_m$ be analytic vector fields. One supposes that (1) is STLC, then, the (following) LARC is true i.e.

$$\{g(0), g \in Lie(f_0, \cdots, f_m)\} = \mathbb{R}^d.$$

Theorem (Linear Test)

Let f_0, f_1, f_2 be analytic vector fields (and $f_0(0) = 0$). One supposes that :

$$Span(ad_{f_0}^k(f_1)(0), ad_{f_0}^k(f_2)(0) \quad k \in \mathbb{N}) = \mathbb{R}^d.$$

Then, the affine system $x' = f_0(x) + uf_1(x) + vf_2(x)$ is STLC.

Purpose : establish, with a new proof, a positive result of controllability ($S(\theta)$ condition of Sussmann) (already known) for affine systems. We want to formulate the result **in terms of Lie Brackets**.
Then, we want to adapt this proof in the framework of **PDE (infinite dimension) - bilinear Schrödinger equation**.

Strategy : approximate the solution of an ODE by a Lie Brackets series (Magnus formula) :

- solve a moment problems to give the value of the partial sum
- estimate the rest of the sum.

We construct a suitable basis $\mathcal{B}$ of $\mathcal{L}(X)$ (Viennot theorem).

$$\mathcal{B} \cap \mathcal{S}_2 = \left\{ \underbrace{(X_1 0^j, X_1 0^{j+1})0^k, (X_2 0^j, X_2 0^{j+1})0^k}_{bad}, \underbrace{(X_1 0^j, X_2)0^k}_{good}, \; j, k \geqslant 0 \right\}$$

Theorem

Let $L \in \mathbb{N}$, $\delta > 0$, $f_0, f_1, f_2 \in \mathcal{C}^\omega(B_{2\delta}, \mathbb{R}^d)$. One supposed that $f_0(0) = 0$,

$$Span\left(f_b(0),\; b \in \mathcal{B} \cap \left(\mathcal{S}_1 \cup \mathcal{S}_{2,good}\right), \quad |b| \leqslant L\right) = \mathbb{R}^d.$$

If, for all $b \in \mathcal{S}_{2,bad}$ with $|b| \leqslant L+1$,

$$f_b(0) \in \mathcal{S}_1(f)(0),$$

then, $x' = f_0(x) + uf_1(x) + vf_2(x)$ is $L^\infty - STLC$.

PDE : generalise the result of affine systems in the framework of infinite dimension.

Bilinear Schrödinger equation with two controls : for $T > 0$,

$$\left\{ \begin{array}{rcl} i\partial_t \psi(t,x) & = & -\partial_{xx}^2 \psi(t,x) - u(t)\mu_1(x)\psi - v(t)\mu_2(x)\psi(t,x), \\ & & \qquad x \in (0,1), t \in (0,T) \\ \psi(0,x) & = & \varphi_1(x) \end{array} \right. \tag{2}$$

We consider $(\varphi_j)_{j\geqslant 1}$, **the orthonormal basis** of $L^2(0,1)$, given by the **engeinvectors of the Laplacian** with Dirichlet boundary conditions, with $\lambda_j = (j\pi)^2$.

We linearise this equation (in terms of (ψ, u)) **around the trajectory** $(\psi_1, u, v \equiv 0)$ (where ψ_1 is the ground state) $\longrightarrow \Psi(T)$.

K. Beauchard and C. Laurent proved an equivalent of the linear test for (2). We want to generalise this result, with the **quadratic term** $\longrightarrow \xi(T)$.

We obtain :

$$\psi(T) \simeq \psi_1(T) + \Psi(T) + \xi(T).$$

One supposes $(\Psi(T), \varphi_K)_{L^2(0,1)} = 0$ (lost direction). Moreover,

$$(\xi(T), \varphi_K e^{-i\lambda_1 T})_{L^2(0,1)} = \int_0^T \left(\int_0^t h_1(t,s)u(s)\mathrm{d}s \right) u(t)\mathrm{d}t +$$

$$\int_0^T \left(\int_0^t h_2(t,s)v(s)\mathrm{d}s \right) u(t)\mathrm{d}t + \int_0^T \left(\int_0^t h_3(t,s)u(s)\mathrm{d}s \right) v(t)\mathrm{d}t +$$

$$\int_0^T \left(\int_0^t h_4(t,s)v(s)\mathrm{d}s \right) v(t)\mathrm{d}t,$$

where

$$h_1 = h_{\mu_1,\mu_1}, \qquad h_2 = h_{\mu_2,\mu_1}, \qquad h_3 = h_{\mu_1,\mu_2}, \qquad h_4 = h_{\mu_2,\mu_2},$$

and, for $1 \leqslant i,j \leqslant 2$,

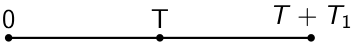
$$h_{\mu_i,\mu_j}(t,s) = -\sum_{k=1}^{+\infty} (\mu_i \varphi_1, \varphi_k)_{L^2} (\varphi_k, \mu_j \varphi_K)_{L^2} e^{i(\lambda_k(s-t)+\lambda_K(t-T)+\lambda_1(T-s))}.$$

Under good hypotheses, we obtain :

$$\left(\xi(T), \varphi_K e^{-i\lambda_1 T}\right)_{L^2(0,1)} = (-i)^n T^{n+2} \gamma_K^n \int_0^1 \bar{u}_{n+2}(t)\bar{v}(t)\mathrm{d}t + \mathcal{O}(T^{n+3}),$$

with

$$\gamma_K^n = \left(ad_\Delta^n(\mu_1), \mu_2)\varphi_1, \varphi_K\right)_{L^2(0,1)}.$$



The proof is divised in two steps :

1. In the first step, on $(0, T)$: we use moment problems to fix the value of $(\xi(T), \varphi_K)_{L^2(0,1)}$. The linear can evolve freely.
2. We use the result about small-time exact controllability in projection to force the value under :

$$\overline{\text{Span}(\varphi_j,\ j \in \mathbb{N}^* \setminus \{K\})}.$$

   We must do **carfully** in order not to destroy the first step. Then, we conclude by the Brouwer fixed-point theorem

This work is in progress...

Thank you for your attention !