

Small-Time Local Controllability of the multi-input bilinear Schrödinger equation thanks to a quadratic term

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- 1 STLC of affine systems of finite dimension
 - Definitions: STLC, Lie brackets
 - Magnus representation formula
 - Theorem and idea of proof

- 2 Small Time Local Controllability of the bilinear Schrödinger equation
 - Presentation
 - Main theorem and ideas of proof
 - Generalization
 - Conclusion and perspectives

One considers the affine system:

$$x' = \mathbf{f}_0(x) + uf_1(x) + vf_2(x), \quad (1)$$

with $f_0, f_1, f_2 \in C^\omega(\mathbb{R}^d)$. The terms \mathbf{f}_0 is called the **drift**.

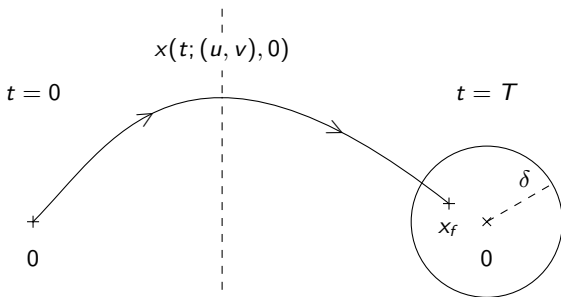
We assume that $f_0(0) = 0$, i.e. $(0, (0,0))$ is an **equilibrium** trajectory of the system (1).

We focus on small time and small controls: the solution is well-defined, and we note it $x(\cdot; (u, v), 0)$.

Definition (E-STLC)

(1) is **E – STLC** around the equilibrium if: for all $T > 0$, $\varepsilon > 0$, there exists $\delta > 0$ such that, for all target $x_f \in \mathbb{R}^d$ such that $\|x_f\| \leq \delta$, there exists $u, v \in E$ with $\|(u, v)\|_E \leq \varepsilon$ such that $x(T; (u, v), 0) = x_f$.

Historical definition: $E = L^\infty$.



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Let

$$\mathcal{F}_T : \begin{bmatrix} E^2 & \rightarrow & \mathbb{R}^d \\ (u, v) & \mapsto & x(T; (u, v), 0) \end{bmatrix}.$$

Then,

$$E - STLC \Leftrightarrow \forall T > 0, \mathcal{F}_T \text{ is locally onto at } (0, 0).$$

Definition (Lie Brackets)

For f, g , regular vectors fields on \mathbb{R}^d , we define the vector field $[f, g]$ as:

$$[f, g] : x \in \mathbb{R}^d \mapsto g'(x)f(x) - f'(x)g(x).$$

By induction, one defines: $\text{ad}_f^0 g = g$ and $\forall k \in \mathbb{N}, \text{ad}_f^{k+1}(g) = [f, \text{ad}_f^k(g)]$.

Example

One supposes $f_0(x) = \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}$ and $f_1(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then,

$$[f_1, f_0](x) = \begin{pmatrix} 0 & 2x_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ 0 \end{pmatrix}.$$

$$\text{ad}_{f_1}^2(f_0)(0) = [f_1, \text{ad}_{f_1}^1(f_0)](0) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2e_1.$$

We want to prove **sufficient conditions** of controllability in terms of the evaluation at $x = 0$ of **Lie brackets** of f_0, f_1 and f_2 .

Theorem (W.-L. Chow, 1939, P.K. Rashevski, 1938)

If $f_0 \equiv 0$ (no drift), then, the system (1) is L^∞ – STLC **iff** LARC holds, i.e.

$$\text{Lie}(f_0, f_1, f_2)(0) = \mathbb{R}^d.$$

This result is **false** in general. For example, $\begin{cases} x_1' = x_2^2 \\ x_2' = u \end{cases} \geq 0$. Then, $f_0(x) = \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}$ and $f_1(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus, $\text{Span}(f_1(0), \text{ad}_{f_1}^2(f_0)(0)) = \mathbb{R}^2$. Nevertheless, the system is not controllable.

Theorem (R. Hermann 1963, T. Nagano 1966)

If the system (1) is L^∞ – STLC, **then** LARC holds, i.e.

$$\text{Lie}(f_0, f_1, f_2)(0) = \mathbb{R}^d.$$

Theorem^[1]

The solution of (1) is given by

$$x(T; (u, v), 0) = \sum_{\substack{b \in \mathcal{B}_{[1,2]} \\ |b| \leq L}} \underbrace{\xi_b(T, (u, v))}_{\text{explicit functional in } (u,v)} \times \underbrace{f_b}_{\in \text{Lie}(f_0, f_1, f_2)}(0) + \text{remainders},$$

where $\mathcal{B}_{[1,2]}$ is a set of brackets.

The set $\mathcal{B}_{[1,2]}$ is defined as:

$$\mathcal{B}_{[1,2]} := \underbrace{\mathcal{B}_1}_{\substack{\text{linear terms: brackets} \\ \text{with } f_1 \text{ or } f_2 \text{ one time}}} \cup \underbrace{\mathcal{B}_{2,\text{good}} \cup \mathcal{B}_{2,\text{bad}}}_{\substack{\text{quadratic terms: brackets} \\ \text{with } f_1 \text{ or } f_2 \text{ two times}}}.$$

For $\tilde{b} \in \mathcal{B}_{2,\text{bad}}$,

$$\xi_{\tilde{b}}(t, (u, v)) \geq 0, \quad \text{for example } \text{ad}_{f_1}^2(f_0) \rightarrow \int_0^t \left(\int_0^s u(\sigma) d\sigma \right)^2 ds.$$

[1] Karine Beauchard, Jérémy Le Borgne, and Frédéric Marbach. "On expansions for nonlinear systems Error estimates and convergence issues". In: *Comptes Rendus. Mathématique* 361 (Jan. 2023), 97–189.

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For $\tilde{b} \in \mathcal{B}_{2,\text{good}}$,

$$\xi_{\tilde{b}}(t, (-u, v)) = -\xi_{\tilde{b}}(t, (u, v)).$$

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Theorem (Linear Test, R. Kalman 1960)

If $\{f_b(0), b \in \mathcal{B}_1\} = \mathbb{R}^d$, then system (1) is $W^{m,\infty}$ – STLC, for every $m \in \mathbb{N}$.

Idea of the proof: For all $T > 0$,

$d\mathcal{F}_T(0,0)(u, v) = X(T)$ is the solution
of the **linearized system, starting from 0**.

However,



linearized system controllable $\stackrel{\text{Kalman condition}}{\iff} \{f_b(0), b \in \mathcal{B}_1\} = \mathbb{R}^d$.



linearized system controllable $\stackrel{\text{inverse mapping theorem}}{\implies}$ STLC.

\mathcal{B}_1 is good.

Remark: For **single-control system**, $\mathcal{B}_2 = \mathcal{B}_{2,bad}$ ($\mathcal{B}_{2,good} = \emptyset$), [Beauchard, Marbach].

Theorem

Let $L > 0$. One supposes that:

$$\text{Span}(f_b(0), b \in \mathcal{B}_1 \cup \mathcal{B}_{2,good}, |b| \leq L) = \mathbb{R}^d.$$

$$\text{For all } b \in \mathcal{B}_{2,bad}, |b| \leq L \Rightarrow f_b(0) \in \mathcal{B}_1(f)(0).$$

Then, the system (1) is **smooth-STLC**, i.e. $W^{m,\infty}$ -STLC, for every $m \in \mathbb{N}$.

Example

A typical example is the following one:

$$\begin{cases} x_1' &= & u \\ x_2' &= & x_1 \\ y_1' &= & v \\ z_1' &= & x_1 y_1 - 7x_2 y_1 \\ z_2' &= & x_1^2 + x_2 \end{cases}$$

If we want to change the hypothesis as:

$$\text{For all } b \in \mathcal{B}_{2,bad}, |b| \leq L \Rightarrow f_b(0) \in \mathcal{B}_1(f)(0) + \mathcal{B}_{2,good}(f)(0).$$

we can have problems !

Example

$$\begin{cases} x_1' = u \\ y_1' = v \\ z_1' = x_1^2 + 2y_1^2 + \frac{3}{2}x_1y_1 \end{cases},$$

Indeed,

$$z_1' = \left(x_1 + \frac{3}{4}y_1\right)^2 + \frac{23}{16}y_1^2 \geq 0.$$

Work in progress..!

X Included in the H. Sussmann's $\mathcal{S}(\theta)$ condition (1987), with $\theta \rightarrow 0$.

One considers a basis of \mathbb{R}^d given by the LARC:

$$\mathbb{R}^d = \mathcal{B}_1(f)(0) \oplus \text{Spn}(f_{b_{r+1}}(0), \dots, f_{b_d}(0)),$$

with $r = \dim(\mathcal{B}_1(f)(0))$ and $b_{r+1}, \dots, b_d \in \mathcal{B}_{2, \text{good}}$. Let $m \in \mathbb{N}$.

Let $j \in \llbracket r+1, d \rrbracket$. It is sufficient to prove that we can create a motion along $f_{b_j}(0)$, i.e. there exists a continuous map $\Xi : [0, +\infty[\rightarrow \mathbb{R}^d$ with $\Xi(0) = f_{b_j}(0)$ such that for all $T > 0$, there exists $C, \rho, s_j > 0$ and a continuous map $z \in (-\rho, \rho) \mapsto (u_z, v_z) \in W^{m, \infty}(0, T)^2$ such that,

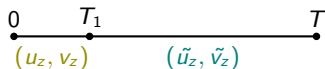
$$\forall z \in (-\rho, \rho), \quad \|x(T; (u_z, v_z), 0) - z\Xi(T)\| \leq C|z|^{1+s_j},$$

with

$$\|(u_z, v_z)\|_{W^{m, \infty}} \leq C|z|^{s_j}.$$

Then, the **Brouwer fixed-point theorem** gives the STLC result.

Idea of the proof: Let $j \in \llbracket r+1, d \rrbracket$. One considers \mathbb{P} , the linear projection on $\text{Span}(f_{b_i}(0))_{r+1 \leq i \leq d}$ parallel to $\mathcal{B}_1(f)(0)$.



The proof is divided in two steps:

1. We construct (u_z, v_z) such that:

$$\mathbb{P}(x(T_1; (u_z, v_z), 0)) = z f_{b_j}(0) + \mathcal{O}(|z|^{1+s_j}), \text{ with } s_j > 0.$$

2. STLC in $\mathcal{B}_1(f)(0)$.

Step 1: Let $\bar{u}, \bar{v} \in C_c^\infty((0, 1), \mathbb{R})$ s.t.

for every $b \in \mathcal{B}_{2,good}$ with $|b| \leq L$, $\xi_b(1, (\bar{u}, \bar{v})) = \delta_{b,b_j}$.

We need to prove the existence of such functions

Let $T_1(z) > 0$, $\varepsilon(z), \varepsilon'(z) > 0$ and $u_z, v_z : t \in (0, T_1) \mapsto \varepsilon \bar{u}\left(\frac{t}{T_1}\right), \varepsilon' \bar{v}\left(\frac{t}{T_1}\right)$.
 Then, with the Magnus formula,

$$\begin{aligned} \mathbb{P}(x(T_1; (u_z, v_z), 0)) &= \mathbb{P}\left(\sum_{b \in \mathcal{B}_1, |b| \leq L}\right) + \mathbb{P}\left(\sum_{b \in \mathcal{B}_{2,bad}, |b| \leq L}\right) \\ &\quad + \varepsilon \varepsilon' \sum_{\substack{b \in \mathcal{B}_{2,good}, \\ |b| \leq L}} T_1^{|b|} \underbrace{\xi_b(1, (\bar{u}, \bar{v}))}_{=\delta_{b,b_j}} \mathbb{P}(f_b(0)) + \text{remainders}. \end{aligned}$$

Then,

$$\mathbb{P}(x(T_1; (u_z, v_z), 0)) = \varepsilon \varepsilon' T_1^{|b_j|} f_{b_j}(0) + \mathcal{O}\left(\varepsilon \varepsilon' T_1^{|b_j|+1} + (\varepsilon + \varepsilon')^3 T_1^3\right).$$

Taking $\varepsilon = \operatorname{sgn}(z)|z|^{\sigma_1}$, $\varepsilon' = |z|^{\sigma_2}$, and $T_1 = \varepsilon = |z|^{\sigma_3}$, with $\sigma_1, \sigma_2, \sigma_3 = f^\theta(|b_j|, m)$, well chosen, one has: $\mathbb{P}(x(T_1; (u_z, v_z), 0)) = zf_{b_j}(0) + \mathcal{O}(|z|^{1+s_j})$.

Step 2: Thanks linear test, one considers $(\tilde{u}_z, \tilde{v}_z)$ s.t.

$$\mathbb{P}_{\mathcal{B}_1(f)(0)}(x(T; (0, 0), zf_{b_j}(0))) =: \mathbb{P}_{\mathcal{B}_1(f)(0)}(z\Xi(t))$$

$$\mathbb{P}_{\mathcal{B}_1(f)(0)}(x(T_1; (u_z, v_z), 0)) \xrightarrow{(\tilde{u}_z, \tilde{v}_z)} \mathbb{P}_{\mathcal{B}_1(f)(0)}(z\Xi(t))$$

Note that $\mathbb{P}_{\mathcal{B}_1(f)(0)} = I - \mathbb{P}$. Then,

$$\|x(T; (U_z, V_z), 0) - z\Xi(t)\| = \|\mathbb{P}(x(T; (U_z, V_z), 0)) - z\mathbb{P}(z\Xi(t))\|.$$

Using the explicit form of \mathcal{B}_1 , one proves that the new step doesn't destroy the first step.

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We consider the following PDE:

$$\begin{cases} i\partial_t\psi = -\partial_{xx}^2\psi - (u(t)\mu_1(x) + v(t)\mu_2(x))\psi, & (t, x) \in (0, T) \times (0, 1) \\ \psi(t, 0) = \psi(t, 1) = 0, & t \in (0, T) \\ \psi(0, x) = \psi_0(x), & x \in (0, 1) \end{cases} \quad (2)$$

$$i\partial_t\psi = f_0(\psi) + uf_1(\psi) + vf_2(\psi),$$

with

$$f_0(\psi) = -\partial_{xx}^2\psi, \quad f_i(\psi) = \mu_i \times \psi, \quad i \in \{1, 2\}.$$

Well-posedness

Let $T > 0$, $\mu_1, \mu_2 \in H^3((0, T), \mathbb{R})$, $u, v \in L^2((0, T), \mathbb{R})$, and $\psi_0 \in H_{(0)}^3(0, 1)$. There exists a unique weak solution of (2), i.e. a function $\psi \in C^0([0, T], H_{(0)}^3(0, 1))$ s.t., in $H_{(0)}^3$ for every $t \in [0, T]$:

$$\psi(t) = e^{-iAt}\psi_0 + i \int_0^t e^{-iA(t-s)} ((u(s)\mu_1 + v(s)\mu_2)\psi(s)) ds.$$

Functional analysis: $A := -\frac{d^2}{dx^2}$, $D(A) = H^2(0, 1) \cap H_0^1(0, 1)$.

- 1 eigenvalues: $\lambda_j = (j\pi)^2$, $j \geq 1$.
- 2 eigenvectors: $\varphi_j := \sqrt{2} \sin(j\pi \cdot)$, $j \geq 1$.
- 3 $(\varphi_j)_{j \geq 1}$ orthonormal basis of $L^2(0, 1)$.

Ground state: $\psi_1(t, x) := \varphi_1(x)e^{-i\lambda_1 t} = \psi(t; (0, 0), \varphi_1)$.

Definition ($L^2 - STLC$)

(2) is **$L^2 - STLC$ in $H_{(0)}^3(0, 1)$ around the ground state** if: for all $T > 0$, $\varepsilon > 0$, there exists $\delta > 0$ such that, for all target $\psi_f \in \mathcal{S} \cap H_{(0)}^3(0, 1)$ such that $\|\psi_f - \psi_1(T)\|_{H^3} \leq \delta$, there exists $u, v \in L^2(0, T)$ with $\|(u, v)\|_{L^2} \leq \varepsilon$ such that $\psi(T; (u, v), \varphi_1) = \psi_f$.

Theorem (Linear Test)^[2]

Let $\mu_1, \mu_2 \in H^3((0, 1), \mathbb{R})$ such that

$$\exists c > 0, \quad \forall j \in \mathbb{N}^*, \quad \left\| \left(\langle \mu_i \varphi_1, \varphi_j \rangle \right)_{1 \leq i \leq 2} \right\| \geq \frac{c}{j^3}.$$

Then, the bilinear Schrödinger equation (2) is L^2 -STLC in $H_{(0)}^3(0, 1)$.

Mégane Bournissou: Quadratic obstructions for the bilinear Schrödinger equation with **single-input system**^[3].

Framework of the article: $\exists K \geq 2$ such that $\langle \mu_1 \varphi_1, \varphi_K \rangle = \langle \mu_2 \varphi_1, \varphi_K \rangle = 0$.
→ use quadratic expansion of the solution to recover this direction

[2] Karine Beauchard and Camille Laurent. “Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control”. In: *Journal de Mathématiques Pures et Appliquées* 94.5 (2010), pp. 520–554.

[3] Mégane Bournissou. “Quadratic behaviors of the 1D linear Schrödinger equation with bilinear control”. In: *Journal of Differential Equations* 351 (2023), pp. 324–360.

Theorem (T.G., 2024)

One considers μ_1, μ_2 such that:

$$1 \quad \mu_1, \mu_2 \in H^3((0, 1), \mathbb{R}).$$

$$2 \quad \langle \mu_1 \varphi_1, \varphi_K \rangle = \langle \mu_2 \varphi_1, \varphi_K \rangle = 0.$$

$$3 \quad \exists c > 0, \quad \forall j \in \mathbb{N}^* \setminus \{K\}, \quad \left\| ((\mu_i \varphi_1, \varphi_j))_{1 \leq i \leq 2} \right\| \geq \frac{c}{j^3}.$$

$$4 \quad A_1^1 := \langle [\mu_1, [\mu_1, \Delta]] \varphi_1, \varphi_K \rangle = 0.$$

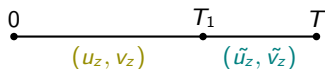
$$5 \quad A_1^2 := \langle [\mu_2, [\mu_2, \Delta]] \varphi_1, \varphi_K \rangle = 0.$$

$$6 \quad \gamma_1 := \langle [\mu_2, [\mu_1, \Delta]] \varphi_1, \varphi_K \rangle \neq 0.$$

The equation (2) is L^2 -STLC around the ground state in $H_{(0)}^3$.

- Point 1: well-posedness.
- Point 3: related to control in projection.
- Point 4 and 5: prevents the system from a drift.
- Point 6: allows us to use the bracket to recover the direction.

Idea of the proof:



The proof is divided in two steps:

1. $\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle = iz + \mathcal{O}\left(|z|^{\frac{13}{12}}\right)$.
2. **STLC in projection.** We must do it **carefully** in order not to destroy the first step (**weak norms**)

+ Brouwer fixed-point theorem

Step 1: Let $\bar{u}, \bar{v} \in L^2((0, 1), \mathbb{R})$ be such that, $\int_0^1 \bar{u}(t)dt = \int_0^1 \bar{v}(t)dt = 0$. Let $T_1(z) > 0$, $\varepsilon(z), \varepsilon'(z) > 0$ and $u_z, v_z : t \in (0, T_1) \mapsto \varepsilon \bar{u}'\left(\frac{t}{T_1}\right), \varepsilon' \bar{v}'\left(\frac{t}{T_1}\right)$. Then,

$$\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle = \mathcal{F}_{T_1}(u_z) + \mathcal{G}_{T_1}(u_z, v_z) + \mathcal{F}_{T_1}(v_z) + \mathcal{O}\left(\|(u_z, v_z)\|_{L^2}^3\right).$$

A direct computation gives:

$$\mathcal{F}_{T_1}(u_z) = -i\varepsilon^2 T_1^3 A_1^1 \int_0^1 \bar{u}(t)^2 dt + \mathcal{O}(\varepsilon^2 T_1^4) = \mathcal{O}(\varepsilon^2 T_1^4).$$

Similarly, $\mathcal{F}_{T_1}(v_z) = \mathcal{O}(\varepsilon'^2 T_1^4)$. Moreover,

$$\mathcal{G}_{T_1}(u_z, v_z) = i\varepsilon\varepsilon' T_1^3 \gamma_1 \int_0^1 \bar{u}(t)\bar{v}(t)dt + \mathcal{O}(\varepsilon\varepsilon' T_1^4).$$

Thus,

$$\begin{aligned} \langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle &= i\varepsilon\varepsilon' T_1^3 \gamma_1 \int_0^1 \bar{u}(t)\bar{v}(t)dt \\ &+ \mathcal{O}\left((\varepsilon + \varepsilon')^2 T_1^4 + (\varepsilon^3 + \varepsilon'^3) T_1^{\frac{3}{2}}\right). \end{aligned}$$

Let $\rho > 0$ and $z \in (-\rho, \rho)$. With $\varepsilon = \operatorname{sgn}(z)|z|^{\frac{3}{8}}$, $\varepsilon' = |z|^{\frac{3}{8}}$ and $T_1 = |z|^{\frac{1}{12}}$, $(\bar{u}, \bar{v}) \in C_c^\infty(0, 1)^2$ such that $\int_0^1 \bar{u}(t)\bar{v}(t)dt = \frac{1}{\gamma_1}$, one obtains:

$$\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle = iz\gamma_1 \int_0^1 \bar{u}\bar{v}' + \mathcal{O}\left(|z|^{\frac{13}{12}}\right) = iz + \mathcal{O}\left(|z|^{\frac{13}{12}}\right).$$

Theorem (T.G., 2024)

Let $n \geq 1$, $m, p \geq 0$, $K \geq 2$ such that $\lfloor \frac{n}{2} \rfloor \leq p$. Let μ_1, μ_2 such that:

- 1 $\mu_1, \mu_2 \in H^{2(p+m)+3}((0, 1), \mathbb{R})$ with $\mu^{(2k+1)}|_{\{0,1\}} = 0$, for $0 \leq k \leq p-1$.
- 2 $\langle \mu_1 \varphi_1, \varphi_K \rangle = \langle \mu_2 \varphi_1, \varphi_K \rangle = 0$.
- 3 $\exists c > 0$, $\forall j \in \mathbb{N}^* \setminus \{K\}$, $\left\| ((\mu_i \varphi_1, \varphi_j))_{1 \leq i \leq 2} \right\| \geq \frac{c}{j^{2p+3}}$.
- 4 $\forall k \in [1, \lfloor \frac{n+1}{2} \rfloor]$, $A_k^1 := \langle [ad_\Delta^{k-1}(\mu_1), ad_\Delta^k(\mu_1)] \varphi_1, \varphi_K \rangle = 0$.
- 5 $\forall k \in [1, \lfloor \frac{n+1}{2} \rfloor]$, $A_k^2 := \langle [ad_\Delta^{k-1}(\mu_2), ad_\Delta^k(\mu_2)] \varphi_1, \varphi_K \rangle = 0$.
- 6 $\gamma_n := \left\langle [ad_\Delta^{\lfloor \frac{n+1}{2} \rfloor}(\mu_1), ad_\Delta^{\lfloor \frac{n}{2} \rfloor}(\mu_2)] \varphi_1, \varphi_K \right\rangle \neq 0$.

The equation (2) is H_0^m -STLC around the ground state in $H_{(0)}^{2(p+m)+3}(0, 1)$: for all $T > 0$, $\varepsilon > 0$, there exists $\delta > 0$ such that, for all target $\psi_f \in \mathcal{S} \cap H_{(0)}^{2(p+m)+3}(0, 1)$ such that $\|\psi_f - \psi_1(T)\|_{H^{2(p+m)+3}} \leq \delta$, there exists $u, v \in H_0^m(0, T)$ with $\|(u, v)\|_{H_0^m} \leq \varepsilon$ such that $\psi(T; (u, v), \varphi_1) = \psi_f$.

Perspectives:

- 1 Several lost directions (as in finite dimension) ? An infinite number ?
- 2 **Obstruction for STLC with multi-input systems**
- 3 Other equations ? KdV ?

Théo Gherdaoui. “Small-Time Local Controllability of the multi-input bilinear Schrödinger equation thanks to a quadratic term”. In: *Preprint* (2024)

Thank you for your attention !