Exercise sheet 2: computing with qubits

Exercise 1. 1. Let X be a finite set. For $x \in X$, denote by $|x\rangle : X \to \mathbf{C}$ the indicator function of x. Show that the family $(|x\rangle)_{x\in X}$ forms a basis of the vector space $\mathbf{C}^X$.

2. What is the dimension of $\mathbf{C}^X$ as a $\mathbf{C}$-vector space?

3. If $\varphi \in \mathbf{C}^X$ is decomposed as $\sum_{x\in X} \alpha_x |x\rangle$, what is α_x with respect to φ?

Exercise 2. Prove that $Q_1 \times Q_1 \xrightarrow{\otimes} Q_2$ is not surjective.
Hint: one can consider the element $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in Q_2$ *as a candidate that cannot be written as* $\varphi \otimes \psi$.

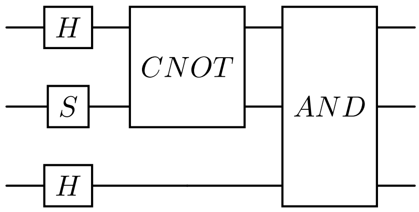
Exercise 3. Prove that there is no unitary transformation $U\colon Q_{2n} \to Q_{2n}$ that satisfies

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle$$

for all $|\psi\rangle \in Q_n$. This fact is called the *no-cloning theorem.*

Exercise 4. Let $f\colon B_1 \to B_1$ be the constant map equal to 1. Compute the matrix of the corresponding unitary transformation U_f in the basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$ of Q_2.

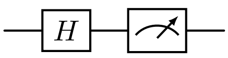
Exercise 5. We consider the same quantum circuit as in the lecture.



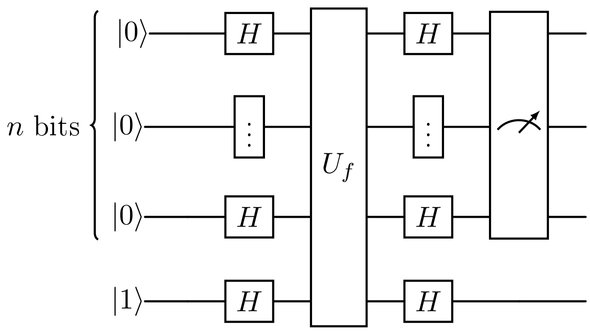
What is the image of the 3-qubit $|000\rangle$ under the unitary transformation represented by this circuit?

Exercise 6. For the 1-qubits

$$|\varphi_1\rangle = |0\rangle, \quad |\varphi_2\rangle = |1\rangle, \quad |\varphi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

determine the possible outcomes of the measurement at the end of the circuit below and their probabilities. 

Exercise 7 (General case of the Deutsch-Jozsa algorithm)**.** Let $f\colon B_n \to B_1$. We assume that we know that f is either constant or balanced (meaning the f takes as many times the value 0 as it takes the value 1). Let us consider the following circuit, that takes as an input the $(n+1)$-qubit $|0\cdots01\rangle$:



1. Show that the new state after the first column of Hadamard gates is

$$\left(\frac{1}{2^{n/2}} \sum_{x \in B_n} |x\rangle \right) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

2. Show that the state after passing the gate U_f is

$$\frac{1}{2^{n/2}} \sum_{x \in B_n} (-1)^{f(x)} |x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

3. Show that for all $x \in B_n$,

$$H^{\otimes n}(|x\rangle) = \frac{1}{2^{n/2}} \sum_{y \in B_n} (-1)^{\langle x,y \rangle} |y\rangle$$

   where $\langle x, y\rangle := \sum_{i=1}^{n} x_i y_i \pmod 2$.

4. Deduce that the state after the second column of Hadamard gates is

$$\frac{1}{2^n} \sum_{x,y \in B_n} (-1)^{f(x)+\langle x,y \rangle} |y\rangle \otimes |1\rangle$$

5. Explain why the final measurement allows us to conclude on the type of f (constant or balanced).