École Normale Supérieure de Rennes

Internship report

GROUP REPRESENTATIONS, SYMMETRIC GROUP, & MODULAR CHARACTERS

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Abstract

In this report, we study the concept of group representations and character theory. We will apply the results of this theory to the symmetric group and find all irreducible representations of the symmetric group in characteristic 0. We then present a few rules that allow to compute irreducible characters of the symmetric group[1]. Finally, we will travel to the confines of space and character theory[2] by describing BRAUER's theory of modular characters.

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[1] The term "symmetric group" hides a lot behind the singular phrasing: there are a *lot* of different symmetric groups.
[2] I may be lying a bit.
[3] and the tea she brought.

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Introduction

Notations

We will use the following notations.

— $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}^* = \{1, 2, 3, \dots\}$.

— Elements of a vector space will be denoted in boldface, like for example $\mathbf{v}$.

— The symmetric (*resp.* alternating) group of order n will be denoted by $\mathcal{S}_n$ (*resp.* $\mathcal{A}_n$).

— If G is a group, we write its identity element 1_G.

— If V and W are vector spaces, $\mathscr{L}(V, W)$ is the vector space of linear transforms from V to W.

— If V is a vector space, its general linear group will be denoted by $\mathrm{GL}(V)$.

- If $d \in \mathbb{N}^*$ and $\mathbb{K}$ is a field, we write $\mathcal{M}_d(\mathbb{K})$ for the set of square matrices with entries in $\mathbb{K}$ of dimensions $d \times d$. The group of invertible matrices in $\mathcal{M}_d(\mathbb{K})$ will be denoted by $\mathrm{GL}_d(\mathbb{K})$.
- If $n \in \mathbb{N}^*$, we write $\lambda \vdash n$ if λ is partition of n, that is to say $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a non-increasing sequence of integers such that $n = \lambda_1 + \cdots + \lambda_k$.
- If $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, we let $\lambda! := \lambda_1! \cdots \lambda_k!$.

1 $\mathbb{K}[G]$-modules

1.1 Linear actions and matrices

Let $\mathbb{K}$ be a field (usually $\mathbb{C}$, or an algebraically closed field of characteristic zero), and let V be a vector space over $\mathbb{K}$ (generally of finite dimension d). Let G be a group (usually finite).

Definition 1.1.1. *V is said to be a G-*module *if there exists a group homomorphism*

$$\rho : G \longrightarrow \mathrm{GL}(V).$$

It is equivalent to say that G acts linearly on V. The integer $\dim V$ *is called the degree of ρ (denotated by* $\deg \rho$*).*

Remark 1.1.2. (V, ρ) is called a *group representation* of G. We usually only write V instead of (V, ρ). In this case if $\mathbf{v} \in V$, we write

$$g \cdot \mathbf{v} = \rho(g)(\mathbf{v}),$$

or simply $g\mathbf{v}$.

Such an homomorphism endows V with a $\mathbb{K}[G]$-*module* structure. Conversely, if V is a $\mathbb{K}[G]$-module, we can define a group representation on V. This $\mathbb{K}[G]$-module structure is often shortened to "G-module" like in the definition, or even "module" when the context is clear.

We also define matrix representations in the same way.

Definition 1.1.3. *A matrix representation of the group G is a group homomorphism*

$$X : G \longrightarrow \mathrm{GL}_d(\mathbb{K}).$$

The integer d is called the degree of X. Similarly, $\dim V$ *is called the degree of ρ.*

Remark 1.1.4. A matrix representation of degree d is nothing than a G-module on $V = \mathbb{K}^d$. Conversely, a G-module V defines a matrix representation: choose a basis $\mathcal{B}$ of V and let $X(g) = \mathrm{Mat}_{\mathcal{B}}(\rho(g))$. We will go back and forth between matrix representations and G-modules as they are the same thing.

1.2 Examples

Let us now consider a few examples. Some are trivial while the others will be used extensively here.

1.2.1 Trivial representation

Given any group G and $\mathbb{K}$-vector space V, we can define

$$\rho : \left| \begin{array}{ccc} G & \longrightarrow & \mathrm{GL}(V) \\ g & \longmapsto & \mathrm{id}_V \end{array} \right. ,$$

which is clearly a group homomorphism.

1.2.2 Representations of the cyclic groups

Let $n \geq 1$ and $\mathbb{K} = \mathbb{C}$. Let us find all one-dimensional representations of $\mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \ldots, \overline{n-1}\}$. Let $X : (\mathbb{Z}/n\mathbb{Z}, +) \longrightarrow (\mathbb{C}^*, \cdot\,)$ be such a (matrix) representation (note that we identify $\mathrm{GL}_1(\mathbb{C})$ with $\mathbb{C}^*$). We have

$$1 = \rho(\bar{0}) = \rho(\bar{n}) = \rho(\bar{1})^n.$$

This means that $\rho(\bar{1})$ is an n-th root of unity in $\mathbb{C}$. As $\mathbb{Z}/n\mathbb{Z}$ is cyclic and generated by $\bar{1}$, this determines the whole representation. Set $\omega := \rho(\bar{1})$ (an n-th root of unity), then

$$\rho(\bar{k}) = \omega^k$$

for all $\bar{k} \in \mathbb{Z}/n\mathbb{Z}$. Conversely, a n-th root of unity gives a group representation in the same way, and so we have found all one-dimensional group representations of the cyclic groups.

1.2.3 Matrix representations of the symmetric group

Let $n \geq 1$. We define two matrix representation of the symmetric group.

— *Sign representation.* Let $\mathrm{sgn} : \mathcal{S}_n \longrightarrow \{-1, 1\}$ be the sign function. Then, sgn (seen as a function taking its values in $\mathbb{K}^*$) is a one-dimensional matrix representation.

— *Defining representation.* We define the following degree n representation: if $\pi \in \mathcal{S}_n$, set

$$X(\pi) = (\delta_{i,\pi(j)}).$$

$X(\pi)$ is often known as the permutation matrix of π.

1.2.4 Permutation representation

Let $S = \{s_1, \dots, s_n\}$ be a set upon which acts G. We can transform this action into a G-module: let $\mathbb{K}\mathbf{S} = \mathbb{K}\{\mathbf{s}_1, \dots, \mathbf{s}_n\}$ be the vector space generated by S over $\mathbb{K}$, which consists of the formal sums

$$\sum_{i=1}^{n} \lambda_i \mathbf{s}_i = \lambda_1 \mathbf{s}_1 + \cdots + \lambda_n \mathbf{s}_n.$$

If $g \in G$, we can define the action by

$$g\left(\sum_{i=1}^{n} \lambda_i \mathbf{s}_i\right) = \sum_{i=1}^{n} \lambda_i (g \cdot \mathbf{s}_i).$$

In this way, we found a G-module structure on $\mathbb{K}\mathbf{S}$ by linearly extending the action of G. This representation is called the *permutation representation.*

1.2.5 Left regular representation

Let $G = \{g_1, \dots, g_n\}$ be a finite group. G acts naturally on itself by setting $g \cdot h = gh$ for all $g, h \in G$. We consider the algebra $\mathbb{K}[\mathbf{G}]$ consisting of the formal sums

$$\sum_{i=1}^{n} \lambda_i \mathbf{g}_i = \lambda_1 \mathbf{g}_1 + \cdots + \lambda_n \mathbf{g}_n.$$

Its G-action is defined by

$$g\left(\sum_{i=1}^{n} \lambda_i \mathbf{g}_i\right) = \sum_{i=1}^{n} \lambda_i g \mathbf{g}_i.$$

Obviously, this is a special case of the permutation representation with $S = G$ but this structure is richer as $\mathbb{K}[\mathbf{G}]$ is an algebra. For this reason, it is called the group algebra of G over $\mathbb{K}$, denoted $\mathbb{K}[\mathbf{G}]$. The square brackets are used to indicate that it is an algebra and not only a vector space (denoted by $\mathbb{K}\mathbf{S}$). This representation is called the *left regular representation.*

1.2.6 Left coset representation

Let G be a finite group and let H be a subgroup of G. Let $\mathcal{H} := G/H$ and let $g_1, \dots, g_k$ be a transversal for H (or a set of distinguished representatives of H in G). This means we have $\mathcal{H} = \{g_1 H, \dots, g_k H\}$. The group G acts on $\mathcal{H} = G/H$ by setting $g' \cdot gH := (g'g)H$ for all $g, g' \in G$. As G acts on $\mathcal{H}$, we can consider the G-module $\mathbb{K}\boldsymbol{\mathcal{H}}$. For $\lambda_1 \mathbf{g}_1 \mathbf{H} + \cdots + \lambda_n \mathbf{g}_n \mathbf{H} \in \mathbb{C}\boldsymbol{\mathcal{H}}$ and $g \in G$, we have

$$g(\lambda_1 \mathbf{g}_1 \mathbf{H} + \cdots + \lambda_n \mathbf{g}_n \mathbf{H}) = \lambda_1 \mathbf{g}\mathbf{g}_1 \mathbf{H} + \cdots + \lambda_n \mathbf{g}\mathbf{g}_n \mathbf{H}.$$

If $H = G$, this representation is the trivial representation. If $H = \{1_G\}$, $\mathcal{H} = G$ and this is the regular representation.

1.2.7 Tensor product

We now give a way to construct a tensor product of G-modules: if (V_1, ρ_1) and (V_2, ρ_2) are representations of G, we define $\rho_1 \otimes \rho_2$ to be the representation given by

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2)$$

for all $g \in G$, and $v_1 \in V_1, v_2 \in V_2$. It is a well defined representation of G on $V_1 \otimes V_2$.

1.3 Submodules

In all fields of algebra after defining a structure, one quickly wants to study the "substructures" associated to an object (group–subgroup, vector space–subspace, ...). Obviously in our case, we will want submodules to be vector spaces. Not only that, a submodule will have to be closed under the action of G.

The introduction of submodules will help us decompose representations into "smaller" already known representations. In matrix terms, if $\rho : G \longrightarrow \mathrm{GL}(V)$ is a matrix representation, we want to find a basis $\mathcal{B}$ of V such that for all $g \in G$,

$$X(g) := \mathrm{Mat}_{\mathcal{B}}(\rho(g)) = \left(\begin{array}{c|c} A(g) & B(g) \\ \hline (0) & C(g) \end{array}\right)$$

where the sizes of the matrices $A(g)$, $B(g)$, and $C(g)$ are independant of g. Ultimately, we will want to find a basis such that $\mathrm{Mat}_{\mathcal{B}}(\rho(g))$ are all block diagonal matrices of dimensions independant of g. This motivates the idea of submodules and the necessity of decomposing modules into submodules until they cannot be decomposed furthermore.

Definition 1.3.1. *Let V be a G-module and W a subspace of V. W is said to be a submodule of V if*

$$g\mathbf{w} \in W$$

for all $g \in G$ and $\mathbf{w} \in W$.

Example 1.3.2. $\{0_V\}$ and V are always submodules of V (and are said to be its *trivial* submodules).

Example 1.3.3. Let $n \geq 2$ and $G = \mathcal{S}_n$. Consider the G-module $V = \mathbb{K}[\mathbf{1}, \dots, \mathbf{n}]$, and $W = \mathbb{K}[\mathbf{1} + \mathbf{2} + \cdots + \mathbf{n}]$ (the subspace spanned by $\mathbf{1} + \mathbf{2} + \cdots + \mathbf{n}$). If $\pi \in G$, then $g(\mathbf{1} + \mathbf{2} + \cdots + \mathbf{n}) = \mathbf{1} + \mathbf{2} + \cdots + \mathbf{n}$ because π is a permutation of $[\![1, n]\!]$. This means[4] W is a submodule of V.

[4]Note that we have only checked that W is closed under the action of G on a basis of W, which is enough.

More generaly, consider a finite group G and the module $V = \mathbb{K}[\mathbf{G}]$. Then the subspace $W = \mathbb{K}[\sum_{g \in G} \mathbf{g}]$ is a submodule of W: if $h \in G$ then

$$h\left(\sum_{g \in G} \mathbf{g}\right) = \sum_{g \in G} \mathbf{hg} = \sum_{g \in G} \mathbf{g}.$$

Example 1.3.4. If $V = \mathbb{K}[\boldsymbol{\mathcal{S}}_n]$, then the submodule $\mathbb{K}[\sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi)\ \boldsymbol{\pi}]$ has the sign representation.

Definition 1.3.5. *A G-module is irreducible (or simple) if it contains no non-trivial submodule.*

Example 1.3.6. Any 1-dimensional module is irreducible. Representations of examples 1.3.3 and 1.3.4 are not irreducible when $n \geq 2$ and $|G| \geq 2$ as we described some of their nontrivial submodules.

1.4 Maschke's theorem

It is extremely convenient in linear algebra to describe a vector space in terms of direct sums of vector spaces. The first necessity given a vector space V and a subspace W is to be able to find another subspace W' such that $V = W \oplus W'$. Now if V is a G-module and W is a submodule, nothing can ensure that W' is also a submodule. However, W' is not unique as a vector space complement of V, so there is a chance to find one that is also a submodule. To achieve this, we will use inner products.

Notation. If $n, m \geq 1$, $A \in \mathcal{M}_n(\mathbb{K})$, and $B \in \mathcal{M}_m(\mathbb{K})$, let

$$A \oplus B = \left(\begin{array}{c|c} A & (0) \\ \hline (0) & B \end{array}\right) \in \mathcal{M}_{n+m}(\mathbb{K}).$$

If X is a matrix representation and $X(g) = A(g) \oplus B(g)$ for all $g \in G$ (such that the sizes of $A(g)$ and $B(g)$ are independant of g), we write $X = A \oplus B$.

1.4.1 $\mathbb{K} = \mathbb{C}$

First, we study reduction of submodules when $\mathbb{K} = \mathbb{C}$.

Proposition 1.4.1. *Suppose $\mathbb{K} = \mathbb{C}$. Let V be a G-module and let W be a submodule of V. Let $\langle \cdot, \cdot \rangle$ be an inner product on V invariant under the action of G,* i.e

$$\langle g\mathbf{v}_1, g\mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $g \in G$. Then, $W^\perp$ is also a submodule.

Proof. Let $w \in W$, $u \in W^\perp$ and $g \in G$. We can write

$$\begin{aligned} \langle g\mathbf{u}, \mathbf{w} \rangle &= \left\langle g^{-1}g\mathbf{u}, g^{-1}\mathbf{w} \right\rangle \\ &= \left\langle \mathbf{u}, g^{-1}\mathbf{w} \right\rangle \\ &= 0 \end{aligned}$$

(because $g^{-1}\mathbf{w} \in W$). This means that $\langle g\mathbf{u}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$, and so $g\mathbf{u} \in W^\perp$ for all $g \in G$.

□

Example 1.4.2. Consider $V = \mathbb{C}\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ upon which $\mathcal{S}_3$ acts, and the submodule $W = \mathbb{C}\{\mathbf{1} + \mathbf{2} + \mathbf{3}\}$ (example 1.3.3). As vector spaces, we have

$$\mathbb{C}\{\mathbf{1}, \mathbf{2}, \mathbf{3}\} = \mathbb{C}\{\mathbf{1} + \mathbf{2} + \mathbf{3}\} \oplus \mathbb{C}\{\mathbf{2}, \mathbf{3}\}$$

but $\mathbb{C}\{\mathbf{2}, \mathbf{3}\}$ is not a submodule as $(1\,2)(\mathbf{2}) = \mathbf{1} \notin \mathbb{C}\{\mathbf{2}, \mathbf{3}\}$. To find a complement to W that is also a submodule, we can consider the inner product defined by $\langle \mathbf{i}, \mathbf{j} \rangle = \delta_{i,j}$ for all $i, j \in \{1, 2, 3\}$ and then extended linearly on the left and antilinearly on the right. It is invariant under the action of $\mathcal{S}_3$ as for every $\pi \in \mathcal{S}_3$ and $i, j \in \{1, 2, 3\}$,

$$\langle \pi\mathbf{i}, \pi\mathbf{j} \rangle = \delta_{\pi(i),\pi(j)} = \delta_{i,j} = \langle \mathbf{i}, \mathbf{j} \rangle$$

for π is injective (note we only checked the invariance on a basis of $\mathbb{C}\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$). We can now consider

$$\begin{aligned} W^\perp &= \{v \in V \mid \forall w \in W, \langle v, w \rangle = 0\} \\ &= \{\lambda_1\mathbf{1} + \lambda_2\mathbf{2} + \lambda_3\mathbf{3} \in \mathbb{C}\{\mathbf{1}, \mathbf{2}, \mathbf{3}\} \mid \langle \lambda_1\mathbf{1} + \lambda_2\mathbf{2} + \lambda_3\mathbf{3}, \mathbf{1} + \mathbf{2} + \mathbf{3} \rangle = 0\} \\ &= \{\lambda_1\mathbf{1} + \lambda_2\mathbf{2} + \lambda_3\mathbf{3} \in \mathbb{C}\{\mathbf{1}, \mathbf{2}, \mathbf{3}\} \mid \lambda_1 + \lambda_2 + \lambda_3 = 0\}. \end{aligned}$$

Theorem 1.4.3 (Maschke, $\mathbb{K} = \mathbb{C}$)**.** *Suppose G is a finite group, $\mathbb{K} = \mathbb{C}$ and let V be a G-module. If V is nonzero, there exists irreducible submodules $(W^{(i)})_{1 \le i \le k}$ such that*

$$V = \bigoplus_{i=1} W^{(i)}.$$

Proof. The proof is a done by induction on $d = \dim V$: if V is irreducible, we are done. If not, there exists a submodule $W \subset V$. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_k, \mathbf{e}_{k+1}, \ldots, \mathbf{e}_d)$ be a base of V such that $(\mathbf{e}_1, \ldots, \mathbf{e}_k)$ is a basis of W. Then, let $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{i,j}$ for all $i, j \in [\![1, d]\!]$ and extend it to an inner product on V. Although $(\cdot\,,\,\cdot)$ may not be G-invariant, the inner product defined for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ by

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \sum_{g \in G} (g\mathbf{v}_1, g\mathbf{v}_2)$$

is G-invariant. We can then consider the submodule $W^\perp$ (where "$\perp$" is the orthogonal for $\langle \cdot, \cdot \rangle$) and induct on W and $W^\perp$. □

Remark 1.4.4. As we will see, MASCHKE's theorem also holds for all fields $\mathbb{K}$ such that its characteristic does not divide the order of $|G|$. However, the theorem becomes furiously wrong when G is infinite. Take for example the action of $\mathbb{Z}$ on $\mathbb{C}^2$ defined by

$$n \cdot (z_1, z_2) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1 + nz_2, z_2)$$

for all $n \in \mathbb{Z}$ and $(z_1, z_2) \in \mathbb{C}^2$. $V = \mathbb{C}^2$ contains the submodule $W = \mathbb{C}(1, 0)$. This means V is reducible and so the matrices

$$A_n := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

for $n \in \mathbb{Z}$ are all diagonalizable. However, A_n has the double eigenvalue 1, which means $A_n = I_2$ for all $n \in \mathbb{Z}$ which is absurd.

We have presented a proof when $\mathbb{K} = \mathbb{C}$ which is extremely convenient and uses well known tools of bilinear algebra. Its conveniences also lie in our ability to construct a complement for a submodule: constructing the inner product (given a basis) is extremely simple, and calculations of orthogonal spaces too.

We now present a different proof which works with a more general statement of the theorem.

1.4.2 MASCHKE's theorem in other fields

Proposition 1.4.5. *Let G be a finite group, $\mathbb{K}$ a field whose characteristic does not divide $|G|$, and (V, ρ) a G-module. If $W \subset V$ is a submodule, then there exists a submodule $W^\circ \subset V$ such that*

$$V = W \oplus W^\circ.$$

Proof. Let p be the projector on W. As the characteristic of $\mathbb{K}$ does not divide $|G|$, $|G| = |G| \cdot 1_{\mathbb{K}}$ is invertible in $\mathbb{K}$ and so we can define

$$p^\circ = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ p \circ \rho(g)^{-1}.$$

Let's show p° is a projector on W.

— $p^\circ(V) \subset W$. If $g \in G$ then $(p \circ \rho(g)^{-1})(V) \subset W$ and so $(\rho(g) \circ p \circ \rho(g)^{-1})(V) \subset W$ as W is closed under the action of G. In the end, $\rho^\circ(V) \subset W$.

— $p^\circ \circ p^\circ = p^\circ$. Let $\mathbf{v} \in V$, we see that

$$p^\circ(p^\circ(\mathbf{v})) = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ p \circ \rho(g)^{-1}(p^\circ(\mathbf{v})),$$

but $p^\circ(\mathbf{v}) \in W$ and so $\rho(g)^{-1}(p^\circ(\mathbf{v})) \in W$. This means that $p(\rho(g)^{-1}(p^\circ(\mathbf{v}))) = \rho(g)^{-1}(p^\circ(\mathbf{v}))$ as p is a projector on W. Then,

$$\begin{aligned} p^\circ(p^\circ(\mathbf{v})) &= \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \rho(g)^{-1}(p^\circ(\mathbf{v})) \\ &= \frac{1}{|G|} \sum_{g \in G} p^\circ(\mathbf{v}) \\ &= p^\circ(\mathbf{v}). \end{aligned}$$

This shows that p° is a projector W. Let $W^\circ = \ker(p^\circ)$. We have

$$V = W \oplus W^\circ.$$

Let's check W° is a submodule. First, notice that

$$\begin{aligned} \rho(g) \circ p^\circ &= \frac{1}{|G|} \sum_{h \in G} \rho(h) \circ p \circ \rho(h)^{-1} \\ &= \frac{1}{|G|} \sum_{h \in G} \rho(g) \circ \rho(h) \circ p \circ \rho(h)^{-1} \\ &= \frac{1}{|G|} \sum_{h \in G} \rho(gh) \circ p \circ \rho(gh)^{-1} \circ \rho(g) \\ &= p^\circ \circ \rho(g). \end{aligned}$$

And so for all $\mathbf{w}^\circ \in W^\circ$,

$$p^\circ(g \cdot \mathbf{w}^\circ) = g \cdot p^\circ(\mathbf{w}^\circ) = 0,$$

which means $g \cdot \mathbf{w}^\circ \in W^\circ$ for all $g \in G$ and $\mathbf{w}^\circ \in W^\circ$, *i.e* W° is a submodule of V.

□

Theorem 1.4.6 (Maschke)**.** *Let G be a finite group, $\mathbb{K}$ a field whose characteristic does not divide $|G|$, and let V be a G-module. If V is nonzero, there exists irreducible submodules $(W^{(i)})_{1\leq i\leq k}$ such that*

$$V = \bigoplus_{i=1} W^{(i)}.$$

Proof. By induction on $\dim V$ like in theorem 1.4.3 but using proposition 1.4.5. □

1.5 Schur's lemma

Let $\mathbb{K}$ be a field and G a group.

Definition 1.5.1 (G-homomorphism)**.** *Let V and W be G-modules. A linear transformation $\theta : V \longrightarrow W$ is said to be a G-*homomorphism *if for all $\mathbf{v} \in V$ and $g \in G$,*

$$\theta(g \cdot \mathbf{v}) = g \cdot \theta(\mathbf{v}).$$

In other words, $\theta \circ \rho_V(g) = \rho_W(g) \circ \theta$ for all $g \in G$. The set of G-homomorphisms is denoted by $\mathrm{Hom}_G(V, W)$.

Remark 1.5.2. This means that θ preserves the action of G on V and transports it to W. In matrix terms, if $\mathcal{B}_V$ and $\mathcal{B}_W$ are respective bases of V and W, and X and Y are the respective matrix representations of V and W, then

$$TX(g) = Y(g)T$$

for all $g \in G$, where $T = \mathrm{Mat}_{\mathcal{B}_V,\mathcal{B}_W}(\theta)$. If T is invertible, this means $X(g)$ and $Y(g)$ are similar.

Example 1.5.3. Let $G = \mathcal{S}_n$. Consider $V = \mathbb{C}$ with the trivial representation and $W = \mathbb{C}\{\mathbf{1}, \mathbf{2}, \dots, \mathbf{n}\}$ with the defining action. Define $\theta : V \longmapsto W$ by letting $\theta(1) = \mathbf{1} + \mathbf{2} + \cdots + \mathbf{n}$. If $\lambda \in \mathbb{C}$ and $\pi \in \mathcal{S}_n$, then

$$\theta(\pi \cdot \lambda) = \theta(\lambda) = \lambda(\mathbf{1} + \mathbf{2} + \cdots + \mathbf{n}) = \pi \cdot \theta(\lambda).$$

As in all fields of algebra, we define a G-isomorphism to be a bijective G-homorphism. In this case, we say V and W are G-isomorphic (or G-equivalent), denoted by $V \cong W$. In the case of matrices, two matrix representations are isomorphic if they differ by a change of basis.

Proposition 1.5.4. *If $\theta : V \longrightarrow W$ is a G-homomorphism, $\ker\theta$ is a submodule of V and $\operatorname{Im}\theta$ is a submodule of W.*

Remark 1.5.5. The proof is simple, and we have already come across its reasoning in the proof of 1.4.5. In fact, p° is a G-homorphism and we basically showed that $W^\circ = \ker p^\circ$ was a submodule of V.

Theorem 1.5.6 (SCHUR's lemma)**.** *Let $\theta : V \longrightarrow W$ be a G-homomorphism between two irreducible G-modules V and W. Either $\theta = 0_{\mathscr{L}(V,W)}$, or θ is an isomorphism.*

Proof. $\operatorname{Im}\theta$ and $\ker\theta$ are necessarily trivial submodules because V and W are irreducible.

□

Corollary 1.5.7. *Suppose $\mathbb{K} = \mathbb{C}$ and let V and W be irreducible G-modules. Then, either $\operatorname{Hom}_G(V, W) = \{0\}$ or $\operatorname{Hom}_G(V, W) \cong \mathbb{C}$.*

Proof. Let $\theta \in \operatorname{Hom}_G(V, W)$ and let $\lambda \in \mathbb{C}$ be an eigenvalue of θ. If θ is non-zero, we can suppose $V = W$. Then, $\theta - \lambda\,\mathrm{id}_V$ is also an homomorphism with a nontrivial kernel. By the same arguments as in the last theorem, $\theta = \lambda\,\mathrm{id}_V$. It is also clear that every homothety is an homomorphism, so $\operatorname{Hom}_G(V, V) = \mathbb{C}\,\mathrm{id}_V$.

□

Remark 1.5.8. This statement is true over all fields of characteristic 0.

Corollary 1.5.9. *Let V and W be irreducible modules and suppose V is irreducible. Then, $\operatorname{Hom}_G(V, W)$ is the multiplicity of V in W.*

1.6 Characters

1.6.1 Induced representation

We now consider what happens when we consider a representation of a group G and its restriction on a subgroup H and *vice-versa*. Here, G will be a finite group.

Definition 1.6.1. *Let (V, ρ) be a group representation of G and let H be a subgroup of H. Then,*

$$\rho{\downarrow}_H^G = \rho|_H$$

is a representation of H.

Notations. If χ is the character[5] of (V, ρ), we write $\chi{\downarrow}_H^G$ for the character of $\rho{\downarrow}_H$ (which is also the restriction of χ on H). We also write $\rho{\downarrow}_H$ when the context is clear.

Example 1.6.2. Consider the sign representation ρ on $\mathbb{K}[\sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi)\, \boldsymbol{\pi}]$. Then, $\rho{\downarrow}_{\mathcal{A}_n}$ is the trivial representation.

We now consider the opposite: from a representation of H, construct an "appropriate" representation of G.

Definition 1.6.3. *If (V, ρ) is an H-module, then we call* induced representation *from H to G the representation associated to*

$$\mathbb{K}[G] \otimes_{\mathbb{K}[H]} V.$$

It is denotated by $\rho{\uparrow}_H^G$.

Remark 1.6.4. We will not give further details about the formal construction of the induced representation. However, it is far more interesting to consider what happens with matrices. Consider $\{t_1, \dots, t_\ell\}$ a system of left coset representatives for H, *i.e*

$$G = \bigsqcup_{i=1}^{\ell} t_i H.$$

Consider a matrix representation Y of H and set $Y(g) = 0$ if $g \notin H$. For every $g \in G$, $Y{\uparrow}_H^G(g)$ is the block matrix

$$Y{\uparrow}_H^G(g) = (Y(t_i^{-1} g t_j))_{1 \leq i,j \leq j}.$$

More explicitely,

$$Y{\uparrow}_H^G(g) = \begin{pmatrix} Y(t_1^{-1} g t_1) & Y(t_1^{-1} g t_2) & \cdots & Y(t_1^{-1} g t_\ell) \\ Y(t_2^{-1} g t_1) & Y(t_2^{-1} g t_2) & \cdots & Y(t_2^{-1} g t_\ell) \\ \vdots & \vdots & \ddots & \vdots \\ Y(t_\ell^{-1} g t_1) & Y(t_\ell^{-1} g t_2) & \cdots & Y(t_\ell^{-1} g t_\ell) \end{pmatrix}.$$

[5] We will soon talk about characters.

1.6.2 Definition and properties of characters

We have studied modules and how they can be decomposed into irreducible submodules. However, group representation can contain a lot of information and so, it would be extremely convenient if there existed a scalar valued function that could determine irreducible modules. This is exactly what characters are for when studying representations of finite groups (with $G = \{g_1, \dots, g_n\}$).

G will denote a *finite* group.

Definition 1.6.5. *Let V be a $\mathbb{K}$-vector space, and $\rho : G \longrightarrow \mathrm{GL}(V)$ a group representation. The character of (V, ρ) is the function*

$$\chi : \left| \begin{array}{ccc} G & \longrightarrow & \mathbb{K} \\ g & \longmapsto & \mathrm{Tr}(\rho(g)) \end{array} \right. .$$

Remark 1.6.6. Obviously, we define characters for matrix representations the same way. Furthermore, χ is said to be *irreducible* if V is.

Example 1.6.7. Let's compute the characters of the regular representation of $\mathbb{K}[\mathbf{G}]$, and let χ denote its character. First, $\chi(1_G) = \mathrm{id}_{\mathbb{K}[\mathbf{G}]}$, and so $\chi(1_G) = |G|$. If $g \neq 1_G$, set $\mathcal{B} = (\mathbf{g}_1, \dots, \mathbf{g}_n)$ (a natural basis of $\mathbb{K}[\mathbf{G}]$), and let's study the matrix representation associated to $\mathcal{B}$. Notice that $X(g)$ is a permutation matrix, and so $\chi(g) = \mathrm{Tr}(X(g))$ is the number of fixed points of $\mathcal{B}$. If that number was nonzero, there would exists a $\mathbf{g}_i$ such that $g\mathbf{g}_i = \mathbf{g}_i$ and so $g = 1_G$, which is not possible. In conclusion,

$$\chi(g) = \begin{cases} |G| & \text{if } g = 1_G, \\ 0 & \text{otherwise.} \end{cases}$$

The first property of characters is one that will interest us throughout this subsection: characters are class functions.

Definition 1.6.8. *A function $f : G \longrightarrow \mathbb{K}$ is said to be a class function if it is constant on conjugacy classes, that is to say*

$$f(ghg^{-1}) = f(h)$$

for all $g, h \in G$.

Now, we give (without proof) a few properties of characters.

Proposition 1.6.9. *Characters verify the following properties.*

(i) *If V and W are G-modules of respective characters φ and ψ, and $V \cong W$ then $\varphi_V = \psi_W$.*

(ii) $\chi(1_G) = \dim V$.

(iii) *If $V = V_1 \oplus V_2$ and χ_1 and χ_2 are the respective characters of V_1 and V_2, then $\chi_V = \chi_1 + \chi_2$.*

(iv) *If $\dim V = 1$, $\chi = \rho$.*

(v) *If $\mathbb{K} = \mathbb{C}$, $\chi(g^{-1}) = \overline{\chi(g)}$.*

Characters table. At first, characters seem to be extremely restrictive functions. However, we will see that a character determines its associated representation. Furthermore, characters are class function. This means that if χ is a character and K is a conjugacy class in G, we can define $\chi_K = \chi(g)$ where $g \in K$. This remark makes the study of characters easier and so we can study *characters table*. A character table of a group is an array where its rows are indexed by its characters χ, and its columns by the conjugacy classes K of G.

$$\begin{array}{c|ccc} & \cdots & K & \cdots \\ \hline \vdots & & \vdots & \\ \chi & \cdots & \chi_K & \\ \vdots & & & \end{array}$$

By convention, the first row is the trivial character and the first column is $K = \{1_G\}$.

Example 1.6.10. In appendix A are presented a few characters table.

1.6.3 Characters in characteristic 0

We will now see the importance of introducing characters, and why they determine representation. As suggested by the title of this subsection $\mathbb{K}$ will be a characteristic 0 field. As usual, G will be a finite group ($G = \{g_1, \ldots, g_n\}$).

Definition 1.6.11. *Let $\varphi, \psi : G \longrightarrow \mathbb{K}$ be two functions. The* inner product *of φ and ψ is defined by*

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\psi(g^{-1}).$$

Remark 1.6.12. If $\mathbb{K} = \mathbb{C}$ and χ and ψ are characters of G, then

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

Proposition 1.6.13. *If χ and ψ are irreducible characters of G, then*

$$\langle \chi, \psi \rangle = \delta_{\chi, \psi}.$$

Basically, characters form an orthonormal family with respect to this bilinear form.

Proof. Let (V, ρ) and (W, σ) be irreducible representations of G of respective characters χ and ψ. If $f : W \longrightarrow V$ is a linear map, define

$$f^0 = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ f \circ \sigma(g^{-1}).$$

If $h \in G$, we have

$$\begin{aligned} \rho(h) \circ f^0 \circ \sigma(h)^{-1} &= \frac{1}{|G|} \sum_{g \in G} \rho(h) \circ \rho(g) \circ f \circ \sigma(g^{-1}) \circ \sigma(h^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(hg) \circ f \circ \sigma((hg)^{-1}) \\ &= f^0 \end{aligned}$$

and so $\rho(g) \circ f^0 = f^0 \circ \sigma(g)$ for all $g \in G$. This means $f^0 : W \longrightarrow V$ is a G-homomorphism. According to SCHUR's lemma 1.5.6 and corollary 1.5.7, $f^0 = 0$ if V and W are non-isomorphic. If $V \cong W$, we can assume $V = W$ and so there must exists some $\lambda \in \mathbb{K}$ such that $f^0 = \lambda \operatorname{id}_V$.

— *Case $\chi \neq \psi$.* In this case, V and W must non-isomorphic and so $f^0 = 0$. Let $\mathcal{B}_V$ and $\mathcal{B}_W$ be respective bases of V and W and let $n = \dim V$ and $m = \dim W$. Consider $R = (r_{i,j})_{i,j}$ the matrix representation associated to ρ given by $\mathcal{B}_V$. In the same way, let us consider $S = (s_{i,j})_{i,j}$ for σ and $X = (X_{i,j})_{i,j}$ the matrix of f in bases $\mathcal{B}_W$ and $\mathcal{B}_V$. As $f^0 = 0$,

$$\frac{1}{|G|} \sum_{g \in G} R(g) X(g) S(g^{-1}) = 0.$$

For every i, j, this means

$$\frac{1}{|G|} \sum_{k,\ell} \sum_{g \in G} r_{i,k}(g) x_{k,\ell} s_{\ell,j}(g^{-1}) = 0$$

but as $f : W \longrightarrow V$ can be any map, it is necessary that

$$\frac{1}{|G|}\sum_{g\in G} r_{i,k}(g)s_{\ell,j}(g^{-1}) = 0$$

for all i, j, k, ℓ. In other words,

$$\langle r_{i,k}, s_{\ell,j}\rangle = 0$$

for all i, j, k, ℓ. Furthermore,

$$\begin{aligned}
\langle \chi, \psi\rangle &= \frac{1}{|G|}\sum_{g\in G} \chi(g)\psi(g^{-1}) \\
&= \frac{1}{|G|}\sum_{g\in G} \operatorname{Tr}(\rho(g))\operatorname{Tr}(\sigma(g^{-1})) \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{i,j} r_{i,i}(g)s_{j,j}(g^{-1}) \\
&= \sum_{i,j}\frac{1}{|G|}\sum_{g\in G} r_{i,i}(g)s_{j,j}(g^{-1}) \\
&= 0.
\end{aligned}$$

— *Case* $\chi = \psi$. We can suppose $V = W$. Since we only care about traces, we can also assume $R = S$. We already know there exists some $\lambda \in \mathbb{C}$ such that $f^0 = \lambda\,\mathrm{id}_V$. This means

$$\frac{1}{|G|}\sum_{k,\ell}\sum_{g\in G} r_{i,k}(g)x_{k,\ell}s_{\ell,j}(g^{-1}) = \lambda\delta_{i,j}$$

and so $\langle a_{i,k}, b_{\ell,j}\rangle = 0$ when $i \neq j$. If $i = j$, we use the equation

$$\frac{1}{|G|}\sum_{g\in G} R(g)XR(g^{-1}) = \lambda I_n$$

and let us consider the traces:

$$\lambda n = \frac{1}{|G|}\sum_{g\in G}\operatorname{Tr}(X) = \operatorname{Tr}(X),$$

and so $\lambda = \frac{1}{n}\operatorname{Tr}(X)$. Going back to our previous equations,

$$\frac{1}{|G|}\sum_{k,\ell}\sum_{g\in G} r_{i,k}(g)x_{k,\ell}r_{\ell,i}(g^{-1}) = \frac{1}{n}\operatorname{Tr} X = \frac{1}{n}(x_{1,1} + \cdots + x_{n,n}).$$

When comparing coefficients of monomials, we get

$$\langle r_{i,k}, r_{\ell,i} \rangle = \frac{1}{|G|} r_{i,k}(g) r_{\ell,i}(g^{-1}) = \frac{1}{n}\delta_{k,\ell}.$$

Finally,

$$\langle \chi, \chi \rangle = \sum_{i,j} \langle r_{i,i}, r_{j,j} \rangle = \sum_{i=1}^{n} \langle r_{i,i}, r_{i,i} \rangle = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

□

Corollary 1.6.14. *Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a group representation of G and let χ be its character. Suppose*

$$V = \bigoplus_{i=1}^{k} m_i V_i$$

is the decomposition of V into irreducible G-modules where each V_i has character χ_i. We have the following results.

(i) *Its character verifies $\chi = \sum_{i=1}^{k} m_i \chi_i$.*

(ii) *For all $j \in [\![1, k]\!]$, $\langle \chi, \chi_j \rangle = m_j$.*

(iii) *We have $\langle \chi, \chi \rangle = \sum_{i=1}^{k} m_i^2$.*

(iv) *V is irreducible* if and only if $\langle \chi, \chi \rangle = 1$.

(v) *If (W, σ) is another representation of G with character ψ, then V and W are isomorphic G-modules* if and only if $\chi = \psi$.

1.6.4 Frobenius reciprocity

We know study a link between induction and restriction on characters. The next theorem will basically that induction and restriction are in some way adjoint operators.

Theorem 1.6.15 (Frobenius reciprocity)**.** *Let H be a subgroup of G and let χ and ψ be characters respectively of H and G. Then,*

$$\left\langle \psi\uparrow^{G}, \chi \right\rangle = \langle \psi, \chi\downarrow_{H} \rangle .$$

Remark 1.6.16. Note that the left inner product is in taken in the space of class functions in G, while the right inner product is in the space of class functions in H.

Lemma 1.6.17. *For every $g \in G$,*

$$\psi{\uparrow}^G(g) = \frac{1}{|H|} \sum_{x \in G} \psi(x^{-1}gx)$$

where we have set $\psi(y) = 0$ if $y \notin H$.

Proof. Take the trace of the matrix of 1.6.4. □

*Proof (*FROBENIUS *reciprocity).*

$$\begin{aligned}
\left\langle \psi{\uparrow}^G, \chi \right\rangle &= \frac{1}{|G|} \sum_{g \in G} \psi{\uparrow}^G(g)\chi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \frac{1}{|H|} \sum_{x \in G} \psi(x^{-1}gx)\chi(g^{-1}).
\end{aligned}$$

Now set $g' = x^{-1}gx$. This gives

$$\begin{aligned}
\left\langle \psi{\uparrow}^G, \chi \right\rangle &= \frac{1}{|G||H|} \sum_{x \in G} \sum_{g' \in G} \psi(g')\chi(xg'^{-1}x^{-1}) \\
&= \frac{1}{|G||H|} \sum_{x \in G} \sum_{g' \in G} \psi(g')\chi(g'^{-1}) \\
&= \frac{1}{|H|} \sum_{g \in G} \psi(g')\chi(g'^{-1}) \\
&= \frac{1}{|H|} \sum_{h \in H} \psi(h)\chi(h^{-1}) \\
&= \langle \psi, \chi{\downarrow}_H \rangle .
\end{aligned}$$

□

2 Application to the symmetric group

2.1 Tableaux and YOUNG subgroups

Now that we know more about representations and how they can be decomposed, we wish to find all irreducible representations of the symmetric group. In this section n will denote a positive integer. If $\lambda = (n_1, \ldots, n_k)$ is a partition of n, its FERRERS *diagram* is a left-justified array of n dots with k rows, the i-th row containing n_i points. For example if $n = 9$ and $\lambda = (4, 2, 2, 1)$, the FERRERS diagram of λ is

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & \bullet & & \\ \bullet & & & \end{array}$$

We say the shape of a FERRERS diagram is the partition λ associated. They will be particularly important as partitions determine conjugacy classes of $\mathcal{S}_n$: two permutations σ and $\tau \in \mathcal{S}_n$ are conjugate *if and only if* they have the same number of cycles of the same length. Furthermore, we know that the number of irreducible representations of a group G is exactly its number of conjugacy classes. This is why studying FERRERS diagrams will help us greatly when constructing those irreducible representations. Before that, we need to find a subgroup of $\mathcal{S}_n$ corresponding to $\lambda \vdash n$.

Definition 2.1.1 (YOUNG subgroup)**.** *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ be a partition of n. Its* YOUNG *subgroup is the group*

$$\mathcal{S}_\lambda = \mathcal{S}_{[\![1,\lambda_1]\!]} \times \mathcal{S}_{[\![\lambda_1+1,\lambda_1+\lambda_2]\!]} \times \cdots \mathcal{S}_{[\![n-\lambda_k+1,\lambda_k]\!]}.$$

Example 2.1.2. Take $n = 9$ and $\lambda = (4, 2, 2, 1)$. Then,

$$\mathcal{S}_\lambda = \mathcal{S}_{\{1,2,3,4\}} \times \mathcal{S}_{\{5,6\}} \times \mathcal{S}_{\{7,8\}} \times \mathcal{S}_{\{9\}}.$$

Remark 2.1.3. If $\lambda \vdash n$, $\mathcal{S}_\lambda$ is isomorphic to

$$\mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \cdots \times \mathcal{S}_{\lambda_k}.$$

In the last example, $\mathcal{S}_{(4,2,2,1)} \cong \mathcal{S}_4 \times \mathcal{S}_2 \times \mathcal{S}_2 \times \mathcal{S}_1$.

Furthermore, we can always consider $\mathcal{S}_\lambda$ as a subgroup of $\mathcal{S}_n$.

Definition 2.1.4 (YOUNG tableau)**.** *Let $\lambda \vdash n$. A* YOUNG *tableau of shape λ (or λ-tableau) is a* FERRERS *diagram of shape λ where the dots have been bijectively replace by $1, \ldots, n$.*

Example 2.1.5. Still with $n = 9$ and $\lambda = (4, 2, 2, 1)$,

$$\begin{array}{llll} 1 & 2 & 3 & 4 \\ 5 & 6 & & \\ 7 & 8 & & \\ 9 & & & \end{array}, \quad \begin{array}{llll} 2 & 5 & 9 & 1 \\ 3 & 4 & & \\ 6 & 8 & & \\ 7 & & & \end{array}, \quad \text{or} \quad \begin{array}{llll} 9 & 2 & 3 & 4 \\ 7 & 6 & & \\ 8 & 5 & & \\ 1 & & & \end{array}$$

are YOUNG tableau of shape λ.

If t is a tableau of shape λ, we write $\operatorname{sh} t = \lambda$. We also can write t^λ instead of t. Note that like matrices, we write $t_{i,j}$ for its entry in row i and column j (when it exists).

Remark 2.1.6. As there are as much YOUNG tableaux as there are elements of $\mathcal{S}_n$, there are exactly $n!$ YOUNG tableaux of shape λ.

We now define an equivalence relation of λ-tableaux.

Definition 2.1.7. *Let s and t be* YOUNG *tableaux of shape λ. The tableaux s and t are said to be* row equivalent *if corresponding rows of s and t contain the same elements. The equivalence class of a λ-tableau t is called a λ-*tabloid*, or* tabloid of shape λ*, denoted by $\{t\}$.*

Example 2.1.8. Consider $n = 3$, $\lambda = (2, 1)$, and

$$t = \begin{array}{cc} 1 & 2 \\ 3 & \end{array}.$$

Then,

$$\{t\} = \left\{ \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 1 \\ 3 & \end{array} \right\}.$$

If t and its entries are explicitly given, we use lines between to indicate when we are working with tabloids. With the last example, we have

$$\{t\} = \begin{array}{cc} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$$

Now let us define an action of $\mathcal{S}_n$ on tabloids. If $\lambda \vdash n$, t is a λ-tableau, and $\sigma \in \mathcal{S}_n$, we can set $\pi t := (\sigma(t_{i,j}))_{i,j}$. This action induces another action on tabloids. We can let

$$\sigma\{t\} := \{\sigma t\}.$$

This is a well defined action (independant of the choice of t). Like in 1.2.4, we can consider an $\mathcal{S}_n$-module associated to this action.

Definition 2.1.9. *if $n \geq 1$ and $\lambda \vdash n$, the module*

$$M^\lambda = \mathbb{K}\{\boldsymbol{\{t_1\}}, \dots, \boldsymbol{\{t_m\}}\}$$

(where $\{t_1\}, \dots, \{t_m\}$ is a complete of λ-tabloids) is called the permutation module *associated to λ.*

We now consider basic examples of permutation modules.

Example 2.1.10. If $\lambda = (n)$, $M^{(n)} = \mathbb{K}\left\{\overline{\underline{\mathbf{1} \quad \cdots \quad \mathbf{n}}}\right\}$. It has the trivial action.

Example 2.1.11. Let $\lambda = (1^n) = (1, \dots, 1) \vdash n$. If t is a tableau of shape λ, it consists of a single column and n rows. This means there are as many tabloids of shape $\lambda = (1^n)$ as there tableaux of shape $\lambda = (1^n)$ (*i.e* $n!$). This means we have

$$M^{(1^n)} \cong \mathbb{K}\mathcal{S}_n,$$

which has the regular representation.

Example 2.1.12. Let $\lambda = (n-1, 1) \vdash n$. Each λ-tabloid is determined by the (unique) number (between 1 and n) in its second row. This means that

$$M^{(n-1,1)} \cong \mathbb{K}\{\mathbf{1}, \mathbf{2}, \dots, \mathbf{n}\},$$

which has the defining representation.

Proposition 2.1.13. *Permutations modules are cyclic,* i.e *there exists* $\mathbf{v} \in M^\lambda$ *such that*

$$M^\lambda = \mathbb{K}\mathcal{S}_n\mathbf{v}.$$

Proof. Take for $\mathbf{v}$ any λ-tabloid.

□

We now show the link between Young subgroups and tabloids.

Theorem 2.1.14. *Let* $\lambda \vdash n$ *and*

$$\{t^\lambda\} = \begin{array}{llll} \hline 1 & 2 & \cdots & \lambda_1 \\ \hline \lambda_1 + 1 & \lambda_1 + 2 & \cdots & \lambda_1 + \lambda_2 \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline n - \lambda_k + 1 & \cdots & \cdots & n \\ \hline \end{array}.$$

Then, $V^\lambda = \mathbb{K}\mathcal{S}_n\boldsymbol{\mathcal{S}_\lambda}$ *and* M^λ *isomorphic (as* $\mathcal{S}_n$*-modules).*

For typesetting reasons, $\{t^\lambda\}$ is presented here as a square array. It is obviously not always the case.

2.2 Orders on partitions

We now study two partial orders.

Definition 2.2.1 (dominance order)**.** *Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \dots, \mu_m)$ be two partitions of $n \geq 1$. We say λ dominates μ (written $\lambda \trianglerighteq \mu$) if for every $i \geq 1$,*

$$\sum_{j=1}^{i} \lambda_i \geq \sum_{j=1}^{i} \mu_i$$

(with the convention that $\lambda_i = 0$ if $i > \ell$ and $\mu_i = 0$ if $i > m$).

Basically, λ dominating μ means that λ is fat and short, while μ is thin and tall.

Remark 2.2.2. This order is *not total.* For example with $n = 6$, (3^2) does not dominate $(4, 1^2)$ and *vice versa.*

Lemma 2.2.3 (dominance lemma)**.** *Let $\lambda, \mu \vdash n$ be two partitions of n. If there exists t^λ and s^μ (two tableaux of respective shape λ and μ) such that for each index of row i, the elements of the i-th row of s^μ are all in different columns in t^λ, then $\lambda \trianglerighteq \mu$.*

Proof. By hypothesis, we can permute the elements in each columns of t^λ in such a way that for every index of row i, the elements of rows 1 to i of s^μ all appear in rows 1 to i of t^μ. This way, $\lambda_1 + \cdots + \lambda_i$ is the number of elements in the first i rows of t^λ, which is no less than the number of elements in the first i rows of $s^m u$ which is exactly $\mu_1 + \cdots + \mu_i$.

□

We now consider the lexicographic order (denoted by $\leq$). For example when $n = 4$, $(2, 2) > (2, 1, 1)$.

Proposition 2.2.4. *The lexicographic order is a refinement of the dominance order, that is to say if $\lambda, \mu \vdash n$ are two partitions of a positive integer n such that $\lambda \trianglerighteq \mu$, then $\lambda \geq \mu$.*

Proof. Let i be the smallest index such that $\lambda_i \neq \mu_i$. Then, $\sum_{j=1}^{i-1} \lambda_j = \sum_{j=1}^{i-1} \mu_j$. By hypothesis, we necessarily have $\sum_{j=1}^{i} \lambda_i > \sum_{j=1}^{i} \mu_i$ (as $\lambda \trianglerighteq \mu$) and so $\lambda_i > \mu_i$.

□

2.3 Specht modules

We now construct all irreducible $\mathcal{S}_n$-modules.

Definition 2.3.1. *Let t be a tableau of shape $\lambda \vdash n$. Let $R_1, \ldots, R_\ell$ be its rows and $C_1, \ldots, C_k$ its columns. We define*

$$R_t = \mathcal{S}_{R_1} \times \cdots \times \mathcal{S}_{R_\ell} \quad \text{and} \quad C_t = \mathcal{S}_{C_1} \times \cdots \times \mathcal{S}_{C_k}.$$

R_t is called the row-stabilizer *of t, while C_t is the* column stabilizer *of t.*

Remark 2.3.2. A tabloid $\{t\}$ (as an equivalence class) is basically $\{t\} = R_t t$.

Definition 2.3.3. *Let H be a subgroup of $\mathcal{S}_n$. We define its* group algebra sums *by*

$$H^+ = \sum_{\pi \in H} \pi \quad \text{and} \quad H^- = \sum_{\pi \in H} \operatorname{sgn}(\pi)\pi.$$

Furthermore, if t is a tableau of shape $\lambda \vdash n$, we let

$$\kappa_t = C_t^-.$$

The element $\boldsymbol{e}_t = \kappa_t \boldsymbol{\{t\}}$ is called a polytabloid *(associated to t).*

Notation. If $H = \{\pi\}$, $\pi^- := H^-$.

Lemma 2.3.4. *Let t be a tableau and $\pi \in \mathcal{S}_n$. We have the following equalities.*

(i) $R_{\pi t} = \pi R_t \pi^{-1}$, *and* $C_{\pi t} = \pi C_t \pi^{-1}$.

(ii) $\kappa_{\pi t} = \pi \kappa_t \pi^{-1}$.

(iii) $\boldsymbol{e}_{\pi t} = \pi \boldsymbol{e}_t$.

Proof. For (i), we work with equivalences: let σ be a permutation,

$$\begin{aligned}
\sigma \in R_{\pi t} &\iff \sigma\{\pi t\} = \{\pi t\} \\
&\iff \pi^{-1}\sigma\pi\{t\} = \{t\} \\
&\iff \pi^{-1}\sigma\pi \in R_t \\
&\iff \sigma \in \pi R_t \pi^{-1}.
\end{aligned}$$

The proof is the same for $C_{\pi t}$ and (ii). For (iv):

$$\boldsymbol{e}_{\pi t} = \kappa_{\pi t}\boldsymbol{\{\pi t\}} = \pi\kappa_t\pi^{-1}\pi\boldsymbol{\{t\}} = \pi\boldsymbol{e}_t.$$

□

Definition 2.3.5. *If λ is a partition n, its corresponding* Specht *module S^λ is the submodule of M^λ spanned by the polytabloids of shape λ.*

Remark 2.3.6. S^λ is indeed a submodule of M^λ: thanks to lemma 2.3.4, we know that the action of $\mathcal{S}_n$ sends polytabloids to polytabloids (as $\pi \boldsymbol{e}_t = \boldsymbol{e}_{\pi t}$). This also means that S^λ is a *cyclic $\mathcal{S}_n$-module.*

2.4 Submodule theorem

In this subsection, $\mathbb{K}$ will be a field of characteristic 0. We will show that the S^λ form a set of irreducible pairwise non isomorphic $\mathcal{S}_n$-modules. This statement fails when $\mathbb{K}$ has positive characteristic.

For this subsection, we will need the unique bilinear form defined by

$$\langle \{\boldsymbol{t}\}, \{\boldsymbol{s}\} \rangle = \delta_{\{t\},\{s\}}.$$

Lemma 2.4.1. *Let H be a subgroup of $\mathcal{S}_n$.*

(i) If $\pi \in H$ then $\pi H^- = H^- \pi = \operatorname{sgn}(\pi) H^-$, and $\pi^- H^- = H^-$.

(ii) If $\mathbf{u}, \mathbf{v} \in M^\lambda$,

$$\left\langle H^- \mathbf{u}, \mathbf{v} \right\rangle = \left\langle \mathbf{u}, H^- \mathbf{v} \right\rangle.$$

In other words, H^- is a self-adjoint operator $M^\lambda \longrightarrow \mathbb{K}$ with respect to $\langle \cdot, \cdot \rangle$.

(iii) If H contains a transposition $(b\ \ c)$, then there exists $k \in \mathbb{K}[\boldsymbol{\mathcal{S}}_n]$ such that

$$H^- = k(\mathrm{id}_{[\![1,n]\!]} - (b\ \ c)).$$

(iv) If t is tableau such that there exists b and c in the same row of t such that $(b\ \ c) \in H$, then

$$H^- \{\boldsymbol{t}\} = \mathbf{0}.$$

Proof. (i) We have

$$\begin{aligned}\pi H^- &= \pi \sum_{\sigma \in H} \operatorname{sgn}(\sigma)\sigma \\ &= \sum_{\sigma \in H} \operatorname{sgn}(\sigma)\pi\sigma \\ &= \operatorname{sgn}(\pi) H^- \\ &= H^-\pi,\end{aligned}$$

where the third and fourth lines have been obtained by the changes of variable $\sigma \mapsto \pi\sigma$ and $\sigma \mapsto \sigma\pi$ (as H is a group).

(ii) Notice that $\langle \cdot\,, \cdot \rangle$ is $\mathcal{S}_n$-invariant. We have

$$\begin{aligned}\left\langle H^-\mathbf{u}, \mathbf{v}\right\rangle &= \sum_{\pi \in H} \langle \operatorname{sgn}(\pi)\pi\mathbf{u}, \mathbf{v}\rangle \\ &= \sum_{\pi \in H} \operatorname{sgn}(\pi) \left\langle \pi^{-1}\pi\mathbf{u}, \pi^{-1}\mathbf{v}\right\rangle \\ &= \sum_{\pi \in H} \left\langle \mathbf{u}, \operatorname{sgn}(\pi^{-1})\pi^{-1}\mathbf{v}\right\rangle \\ &= \left\langle \mathbf{u}, H^-\mathbf{v}\right\rangle\end{aligned}$$

(with the change of variable $\pi \mapsto \pi^{-1}$).

(iii) Let $K \subset H$ be the subgroup $K = \{\mathrm{id}_{[\![1,n]\!]}, (b\ \ c)\}$. Let $(k_1, \dots, k_q)$ be a collection of distinguished representatives of K in H. Then, $H = \bigsqcup_{j=1}^q k_j K$ and so

$$H^- = \left(\sum_{j=1}^q k_j\right) K^- = k(\mathrm{id}_{[\![1,n]\!]} - (b\ \ c))$$

where we set $k = \sum_{j=1}^q k_j$.

(iv) As b and c are in the same row of t, we have $(b\ \ c)\{\boldsymbol{t}\} = \{\boldsymbol{t}\}$ and so

$$H^-\{\boldsymbol{t}\} = k(\mathrm{id}_{[\![1,n]\!]} - (b\ \ c))\{\boldsymbol{t}\} = k(\{\boldsymbol{t}\} - (b\ \ c)\{\boldsymbol{t}\}) = 0.$$

□

Corollary 2.4.2. *Let* $\lambda, \mu \vdash n$. *If* $\kappa_t\{\boldsymbol{s}\} \neq \mathbf{0}$, *then* $\lambda \trianglerighteq \mu$. *In case* $\lambda = \mu$, $\kappa_t\{\boldsymbol{s}\} = \pm \boldsymbol{e_t}$.

Proof. Let b and c be two elements in the same row of s. Notice they cannot be in the same row of t (if it were the case, we would have $\kappa_t\{\boldsymbol{s}\} = \mathbf{0}$). Thanks to dominance lemma (2.2.3), we have $\lambda \geq \mu$. If $\lambda = \mu$, the argument that proved the dominance lemma establishes that there exists some $\pi \in C_t$ such that $\{s\} = \pi\{t\}$. We then have

$$\kappa_t\{\boldsymbol{s}\} = \kappa_t\pi\{\boldsymbol{t}\} = \operatorname{sgn}(\pi)\kappa_t\{\boldsymbol{t}\} = \pm\boldsymbol{e}_t.$$

□

Corollary 2.4.3. *Let $\mathbf{u} \in M^\mu$ and let t be a tableau of shape μ. Then, $\kappa_t\mathbf{u}$ is a multiple of $\boldsymbol{e}_t$.*

Proof. Write $\mathbf{u} = \sum_{j=1}^q c_j\{\boldsymbol{s_j}\}$ and use previous corollary with linearity.

□

Theorem 2.4.4 (submodule theorem)**.** *Let U be a submodule of M^μ. Then, either*

$$S^\mu \subset U \quad \text{or} \quad U \subset S^{\mu\perp}.$$

In the case where $\mathbb{K}$ is of characteristic 0, $\mathcal{S}^\mu$ is irreducible.

Remark 2.4.5. This second part of the theorem does *not* hold when $\mathbb{K}$ has positive characteristic. However, it is always true that $S^\lambda/(S^\lambda \cap S^{\lambda\perp})$ is irreducible.

Proof. Let $\mathbf{u} \in U$ and let t be a tableau of shape μ. According to the previous corollary, $\kappa_t\mathbf{u} = f\boldsymbol{e_t}$ for some scalar $f \in \mathbb{K}$ (f depends on $\mathbf{u}$ and t).

Let us suppose we can find $\mathbf{u}$ and t such that their associated f is nonzero. Then $\boldsymbol{e_t} = f^{-1}\kappa_t\mathbf{u}$ is an element of U ($\kappa_t\mathbf{u} \in U$ because U is a submodule). As S^μ is cyclic and contains a polytabloid (which spans S^μ in a cyclic sense), this means $S^\mu \subset U$.

Now suppose $f = 0$ for all $\mathbf{u} \in U$ and tableaux t of shape μ. Let $\mathbf{u} \in U$ and let $\boldsymbol{e_t}$ be a polytabloid of shape μ (whose set spans S^μ). We have

$$\langle \mathbf{u}, \boldsymbol{e}_t \rangle = \langle \mathbf{u}, \kappa_t\{\boldsymbol{t}\} \rangle = \langle \kappa_t\mathbf{u}, \{\boldsymbol{t}\} \rangle = 0$$

as $\kappa_t\mathbf{u} = \mathbf{0}$.

□

Proposition 2.4.6. *If $\mathbb{K}$ has characteristic 0 and $\theta \in \operatorname{Hom}(S^\lambda, M^\mu)$ is a nonzero homomorphism, then $\lambda \trianglerighteq \mu$. If $\lambda = \mu$, θ is an homothety.*

Proof. As θ is nonzero, there exists a polytabloid $\boldsymbol{e_t}$ such that $\theta(\boldsymbol{e}_t) \neq \mathbf{0}$. Since $\mathbb{K}$ is of chracteristic 0 and $\langle \cdot, \cdot \rangle$ is an inner product, we have $M^\lambda = S^\lambda \oplus {S^\lambda}^\perp$. We can extend θ to M^λ by setting $\theta(\mathbf{v})$ for every $\mathbf{v} \in {S^\lambda}^\perp$. We now have $\theta \in \mathrm{Hom}(M^\lambda, M^\mu)$ and it satisfies

$$0 \neq \theta(\boldsymbol{e}_t) = \theta(\kappa_t \{\boldsymbol{t}\}) = \kappa_t \theta(\{\boldsymbol{t}\}).$$

We can write $\theta(\{\boldsymbol{t}\}) = \sum_{j=1}^{q} c_j \{\boldsymbol{s_j}\}$ where s_j have shape μ. We first conclude with 2.4.2 as there must be some $j \in [\![1, q]\!]$ such that $\kappa_t \{\boldsymbol{s_j}\} \neq 0$ and so $\lambda \trianglerighteq$. If $\lambda = \mu$

If $\lambda = \mu$, corollary 2.4.3 gives that there exists some $c \in \mathbb{K}$ such that $\theta(\boldsymbol{e}_t) = c\boldsymbol{e}_t$. If $\pi \in \mathcal{S}_n$,

$$\theta(\boldsymbol{e}_{\pi t}) = \theta(\pi \boldsymbol{e}_t) = \pi\theta(\boldsymbol{e}_t) = \pi c \boldsymbol{e}_t = c\boldsymbol{e}_{\pi t},$$

and so θ is an homothety.

□

Theorem 2.4.7. *If $\mathbb{K}$ has characteristic 0, $(S^\lambda)_{\lambda \vdash n}$ is a complete list of irreducible $\mathcal{S}_n$-modules pairwise non-isomorphic.*

Proof. Those modules are irreducible and $S^\lambda \cap {S^\lambda}^\perp = \{\mathbf{0}\}$ for every $\lambda \vdash n$. We have the right number of irreducible modules as they are indexed by the partitions of n. We then have to show those modules are pairwise inequivalent: let $\lambda, \mu \vdash n$ and suppose $S^\lambda \cong S^\mu$. There must exist some $\theta \in \mathrm{Hom}(S^\lambda, M^\mu)$ and so $\lambda \trianglerighteq \mu$ according to the previous proposition. We conclude by symmetry of λ and μ.

□

Corollary 2.4.8. *If $\mu \vdash n$ and $\mathbb{K}$ has characteristic 0, then*

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda,\mu} S^\lambda.$$

Furthermore, $m_{\mu,\mu} = 1$.

Proof. If S^λ appears in a direct sum decomposition of M^μ, then we know according to proposition 2.4.6 that $\lambda \trianglerighteq \mu$. Also according to this proposition, if $\lambda = \mu$ then $\mathrm{Hom}(S^\mu, M^\mu) \cong \mathbb{K}$ and

$$m_{\mu,\mu} = \dim \mathrm{Hom}(S^\mu, M^\mu) = 1.$$

□

2.5 Branching rule

We now want to know what happens to the $\mathcal{S}_n$-modules we have studied after restriction to $\mathcal{S}_{n-1}$ or induction to $\mathcal{S}_{n+1}$. If λ is a partition of an integer n, recall its Ferrers' diagram is a Young tableau of shape λ, but represented with dots instead of integers. For example with $\lambda = (4, 4, 2, 1) \vdash 11$,

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & & & \end{array}$$

is its Ferrers diagram. For every $\lambda \vdash n$, a inner corner in the Ferrers diagram of shape λ is a dot such that after its removal, the remaining dots form a Ferrers diagram of a partition $\lambda^- \vdash n-1$. An outer dot is a dot such that it can be added to the Ferrers diagram to make a Ferrers diagram of $\lambda^+ \vdash n+1$. For example with $\lambda = (4, 4, 2, 1) \vdash 11$, we represent inner corners with larger dots and outer corners with circles.

$$\begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \large\bullet & \\ \bullet & \large\bullet & \circ & & \\ \large\bullet & \circ & & & \end{array}$$

With this example, λ^+ can be

$$(5,4,2,1) : \begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & & & \\ \bullet & & & & \end{array}, \quad (4,4,3,1) : \begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \end{array}, \quad (4,4,2,2) : \begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & \bullet & & \end{array},$$

and λ^- can be

$$(4,3,2,1) : \begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \\ \bullet & & & \end{array}, \quad (4,4,1,1) : \begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & & & \\ \bullet & & & \end{array}, \quad (4,4,2) : \begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \end{array}.$$

Those methods of "diminishing" or "augmenting" $\lambda \vdash n$ correspond exactly to restriction and induction of representations. In fact,

$$S^{(4,4,2,1)}\uparrow^{\mathcal{S}_{12}} \cong S^{(5,4,2,1)} \oplus S^{(4,4,3,1)} \oplus S^{(4,4,2,2)}$$

and

$$S^{(4,4,2,1)}\downarrow_{\mathcal{S}_{10}} \cong S^{(4,3,2,1)} \oplus S^{(4,4,1,1)} \oplus S^{(4,4,2)}.$$

This decomposition is called the *branching rule*.

To study and give a proof to this rule, we will describe (without proof) a basis for the SPECHT module S^λ. Let λ be a partition of n. A tableau t of shape λ is said to be *standard* if its columns and rows are both increasing sequences. The tabloid $\{t\}$ and polytabloid $\boldsymbol{e}_t$ are also said to be standard if t is.

Theorem 2.5.1. *The set of standard polytabloids of shape $\lambda \vdash n$ is a basis of S^λ.*

Proof. See [Sag01], theorem 2.6.5, page 74. □

Notation. We denote by f^λ the number of polytabloid of shape $\lambda \vdash n$. Thanks to the theorem above, $\dim S^\lambda = f^\lambda$.

Lemma 2.5.2. *Let $n \in \mathbb{N}^*$ and $\lambda \vdash n$. Then,*

$$f^\lambda = \sum_{\lambda^-} f^{\lambda^-}.$$

Proof. A standard tableau of shape $\lambda \vdash n$ is exactly a standard tableau of a shape of size $n-1$ where an outer corner has been added. □

Lemma 2.5.3. *Let G be a group and W be a submodule of a G-module V. Then,*

$$V \cong W \oplus (V/W).$$

Theorem 2.5.4 (branching rule)**.** *Let $\lambda \vdash n$. Then,*

$$S^\lambda\downarrow_{\mathcal{S}_{n-1}} = \bigoplus_{\lambda^-} S^{\lambda^-} \quad \text{and} \quad S^\lambda\uparrow^{\mathcal{S}_{n+1}} = \bigoplus_{\lambda^+} S^{\lambda^+}$$

Proof. We detail the proofs one after the other.

— *Restriction.* Let $r_1 < r_2 < \cdots < r_k$ be the rows of the inner dots of λ. For each $i \in [\![1, k]\!]$, let λ^i be the partition obtained by removing the inner dot in row i. If t is a tableau of shape λ where n is at the last entry of row i, let t^i be the tableau of shape λ^i obtained by removing n.

According to the previous lemma, it is sufficient to find a chain of subspace

$$\{\mathbf{0}\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = S^\lambda$$

such that $V_i/V_{i-1} \cong S^{\lambda^i}$ for each $i \in [\![1,k]\!]$. For this, we choose V_i to be the vector spanned by the polytabloids $\boldsymbol{e}_t$ where n appears in a row $j \in [\![r_1, r_i]\!]$. Let $\theta_i : M^\lambda \longrightarrow M^{\lambda^i}$ be the map defined by $\theta_i(\boldsymbol{\{t\}}) = \boldsymbol{\{t_i\}}$ if n is in row r_i of $\{t\}$, and $\mathbf{0}$ otherwise. Let $\pi \in \mathcal{S}_{n-1}$ and let $\{t\}$ be a tabloid contaning n in its r_i-th row. Then,

$$\pi\theta_i(\boldsymbol{\{t\}}) = \pi\boldsymbol{\{t_i\}} = \boldsymbol{\{\pi t_i\}}.$$

As π fixes n (seen as an element of $\mathcal{S}_n$), n is still in row r_i of πt and so $\boldsymbol{\{\pi t_i\}} = \theta_i(\pi\boldsymbol{\{t\}})$. This means θ_i is a $\mathcal{S}_{n-1}$-homomorphism.

Furthermore, if t is standard, $\theta(\boldsymbol{e}_t) = \boldsymbol{e}_{t_i}$ if n is in row r_i of t and $\theta(\boldsymbol{e}_t) = 0$ if n is in row r_j with $j < i$. As standard polytabloids form a basis of their corresponding Specht modules, we have

$$\theta_i(V_i) = S^{\lambda^i} \quad \text{and} \quad V_{i-1} \subset \ker\theta_i.$$

This means we have the chain

$$\{\mathbf{0}\} = V_0 \subset V_1 \cap \ker\theta_1 \subset V_1 \subset V_2 \cap \ker\theta_2 \subset V_2 \subset \cdots \subset V_k = S^\lambda,$$

but as $\theta_i(V_i) = S^{\lambda^i}$, we have $V_i/(V_i \cap \ker\theta_i) = \dim\theta_i(V_i) = f^{\lambda^i}$. As $f^\lambda = \sum_{\lambda^-} f^{\lambda^-}$, the inclusions $V_i \subset V_{i+1} \cap \ker\theta_{i+1}$ must be equalities. We have a chain

$$\{\mathbf{0}\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = S^\lambda$$

with

$$V_i/V_{i-1} \cong V_i/(V_i \cap \ker\theta_i) \cong S^{\lambda^i}.$$

— *Restriction.* We will use Frobenius reciprocity and the first statement of the branching rule: let χ^λ be the character of S^λ. There exists coefficients $(m_\mu)_{\mu\vdash n+1}$ such that $S^\lambda\uparrow^{\mathcal{S}_{n+1}} \cong \bigoplus_{\mu\vdash n+1} m_\mu S^\mu$. Consider their characters: we then have $\chi^\lambda\uparrow^{\mathcal{S}_{n+1}} = \sum_{\mu\vdash n+1} m_\mu\chi^\mu$. Furthermore,

$$\begin{aligned} m_\mu &= \left\langle \chi^\lambda \uparrow^{\mathcal{S}_{n+1}}, \chi_\mu \right\rangle \\ &= \left\langle \chi^\lambda, \chi^\mu \downarrow_{\mathcal{S}_{n-1}} \right\rangle \\ &= \left\langle \chi^\lambda, \sum_{\mu^-} \chi^{\mu^-} \right\rangle \\ &= \sum_{\mu^-} \left\langle \chi^\lambda, \chi^{\mu^-} \right\rangle \\ &= \delta_{\lambda^+,\mu}. \end{aligned}$$

□

Remark 2.5.5. We also could have first prove the induction rule, and then use FROBENIUS reciprocity to prove the restriction rule.

2.6 The LITTLEWOOD-RICHARDSON rule

We know want to study the coefficients $m_{\lambda,\mu}$ appearing in the decomposition

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda,\mu} S^\lambda.$$

Note that we already know $m_{\mu,\mu} = 1$. For this we need to introduce a new kind of tableaux and we will then give a combinatorial interpretation to those tableaux.

Definition 2.6.1. *Let $n \geq 1$ be an integer. A composition of n is a sequence $\alpha = (\alpha_1, \dots, \alpha_k)$ such that*

$$n = \sum_{i=1}^{k} \alpha_i.$$

In opposition to partitions, composition are not needed to be non-decreasing sequences.

Example 2.6.2. $(2, 0, 2, 1)$ an $(1, 2, 0, 1, 1)$ are compositions of 5.

Definition 2.6.3. *A generalized* YOUNG *tableau of shape $\lambda \vdash n$ is a* FERRERS *diagram (of shape λ) where nodes have been replaced with integers in $[\![1, n]\!]$ (with repetitions allowed). If T is a tableau and μ_i is the numbers of $i \in [\![1, n]\!]$ in T, then the composition $\mu = (\mu_1, \dots, \mu_m)$ is called the* content *of T.*

Example 2.6.4. If $n = 7$,

$$T = \begin{array}{cccc} 1 & 2 & 4 & 4 \\ 3 & 2 & 2 & \end{array}$$

is a tableau of shape $(4,3)$ and content $(1,3,1,2)$.

Notation. If λ is partition of n and μ a composition of n, $\mathcal{T}_{\lambda,\mu}$ denotes the set of tableaux of shape λ and content μ.

Remark 2.6.5. Let T be a tableau of shape λ and content μ. If $i \in [\![1,n]\!]$, let us denote by $T(i)$ the i-th entry of the tableau line after line. For example if $\lambda = (4,2,2,1)$,

$$T = \begin{array}{cccc} T(1) & T(2) & T(3) & T(4) \\ T(5) & T(6) & & \\ T(7) & T(8) & & \\ T(9) & & & \end{array}.$$

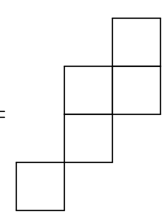
With this notation, we can give a $\mathcal{S}_n$-module structure to $\mathbb{K}[\mathcal{T}_{\lambda,\mu}]$ for every $\lambda, \mu \vdash n$: if $\pi \in \mathcal{S}_n$ and $T \in \mathcal{T}_{\lambda,\mu}$, set $(\pi T)(i) := T(\pi^{-1} i)$ for every $i \in [\![1,n]\!]$. With this definition, M^μ and $\mathbb{K}[\mathcal{T}_{\lambda,\mu}]$ are isomorphic $\mathcal{S}_n$-modules.

Semistandard tableaux. A tableau T of shape λ and content μ is said to be *semi-standard* if its columns are increasing and its rows are weakly increasing. Let $\mathcal{T}^0_{\lambda,\mu}$ be the set of semistandard tableaux of shape λ and content μ, and $K_{\lambda,\mu} = \operatorname{card} \mathcal{T}^0_{\lambda,\mu}$. It can be proven that a basis of $\operatorname{Hom}(S^\lambda, M^\mu)$ is indexed by $\mathcal{T}^0_{\lambda,\mu}$. This means that

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} K_{\lambda,\mu} S^\lambda.$$

The $K_{\lambda,\mu}$ are called the Kotska *numbers*. This decomposition in terms of Specht modules and Kotska numbers is called Young*'s rule*.

Skew tableaux. Let λ and μ be partitions such that $\mu \subset \lambda$ (as Ferrers diagrams). We define λ/μ to be the diagram of points that are in λ and not in μ. For example with $(3,3,2,1)$ and $(2,1,1)$,

$\lambda/\mu =$ 

Skew tableaux will helps us find coefficients when considering tensor products of Specht modules.

Definition 2.6.6. *Let μ and ν be partitions of two integers such that $|\mu|+|\nu| = n$. Then,*

$$(S^\mu \otimes S^\nu)\uparrow^{\mathcal{S}_n} = \bigoplus_\lambda c^\lambda_{\mu,\nu} S^\lambda.$$

We call Littlewood-Richardson coefficients *the numbers $c^\lambda_{\mu,\nu}$.*

To understand those coefficients, we need to introduce some combinatorial concepts.

Definition 2.6.7. *A* lattice permutation *of positive integers $\pi = (i_1\ i_2\ \cdots\ i_n)$ is a sequence such that for every prefix $\pi_k = (i_1\ i_2\ \cdots\ i_k)$ and $\ell \geq 1$, the number of ℓ's in in π_k is greater or equal to the number of $(\ell+1)$'s. We call* inverse lattice permutation *a sequence π such that the reversed word π^r is a lattice permutation.*

Example 2.6.8. The sequence $(1\ 1\ 2\ 1\ 2\ 3)$ is a lattice permutation.

Definition 2.6.9. *If T is a tableau of any shape with rows $R_1, \ldots, R_\ell$, the* row word *of T is the permutation*

$$\pi_T = R_\ell R_{\ell-1} \cdots R_1.$$

Theorem 2.6.10 (Littlewood-Richardson rule)**.** *Let λ, μ, ν be partitions such that $|\mu| + |\nu| = |\lambda|$. The* Littlewood-Richardson *coefficient $c^\lambda_{\mu,\nu}$ is equal to the number of semistandard tableaux T such that*

— *T has shape λ/μ and content ν ;*

— *the permutation word π_T is a reverse permutation lattice.*

Proof. We omit the proof for clarity reasons. To prove this, we would need to show that Littlewood-Richardson coefficients appear in some formal series when studying symmetric functions.

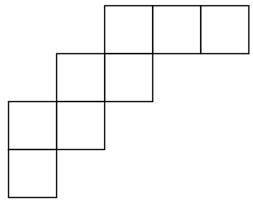
□

2.7 The Murnaghan-Nakayama rule

Definition 2.7.1. *A* skew hook *(or* rim hook*) is a skew diagram that is edgewise connected and contains no 2×2 subset of cells.*

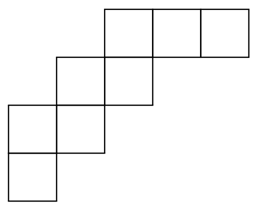
Basically, a skew hook is a skew diagram where we have started from the bottom, and then a cell is added by choosing to go North or East. We call *leg length* of a skew hook ξ its number of rows minus one denotated by $\ell\ell(\xi)$.

Example 2.7.2. The following skew diagram

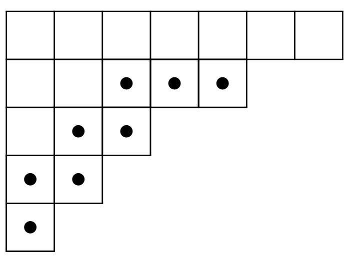
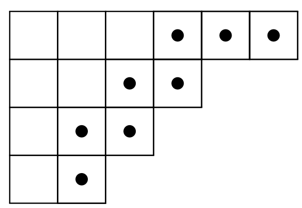
$\xi =$ 

has leg length $\ell\ell(\xi) = 4 - 1 = 3$.

Remark 2.7.3. A skew hook ξ is said to be the rim hook of a partition λ if ξ is an edgewise connected subdiagram of the right border of λ seen as a Ferrers' diagram, such that the cells remaining in λ after removing ξ is still a partition. For example,

$\xi =$ 

can be the rim hook of

$\lambda =$  or $\mu =$ 

If ξ is the rim hook of a partition λ, we write $\lambda \backslash \xi$ for the diagram remaining after removing the cells of ξ. With this notations if $\mu = \lambda \backslash \xi$, then $\xi = \lambda / \mu$.

Notations. We will the following notations.

— If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a composition, we let $\alpha \backslash \alpha_1 = (\alpha_2, \ldots, \alpha_k)$.

— If $\lambda \vdash n$ and α is a composition of n, χ^λ_α is the restriction of the irreducible character χ^λ on the conjugacy class given by α.

— if $\lambda \vdash n$ and $m \geq 1$, let $\mathrm{Rh}(\lambda, m)$ be the set of rim hooks ξ of λ having m cells.

Theorem 2.7.4 (Murnaghan-Nakayama rule)**.** *Let λ be a partition of n and let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of n. Then,*

$$\chi_\alpha^\lambda = \sum_{\xi \in \mathrm{Rh}(\lambda, \alpha_1)} (-1)^{\ell\ell(\xi)} \chi_{\alpha \backslash \alpha_1}^{\lambda \backslash \xi}.$$

Remark 2.7.5. The branching rule is a special case of the Murnaghan-Nakayama rule with $\alpha_1 = 1$.

3 Modular characters

3.1 Definition

We have seen that characters are extremely convenient tools to describe $\mathbb{K}[G]$-modules when $\mathbb{K}$ has characteristic 0. Unfortunately, the proofs and reasoning behind this theory do not carry well to positive characteristic field. Characters as they have been defined before cannot determine representations of a group G: if $p > 0$ is the characteristic of $\mathbb{K}$ and $(\mathbb{K}^n, \rho)$ is a representation of G of character χ, then the representation $\rho \oplus \cdots \oplus \rho : G \longrightarrow (\mathbb{K}^n)^p$ (added to itself p-times) has character $p\chi = 0$. For this reason, the theory behind *modular* characters is different.

In this section, G denotes a finite group. $\mathbb{K}$ is a field of characteristic 0 complete with respect to a discrete valuation ν. This means that we consider $\mathbb{K}$ with a function $\nu : \mathbb{K} \longrightarrow \mathbb{Z} \cup \{\infty\}$ such that

(*i*) for all $x \in \mathbb{K}$, $\nu(x) = \infty \iff x = 0$;

(*ii*) for all $x, y \in \mathbb{K}$, $\nu(xy) = \nu(x) + \nu(y)$;

(*iii*) for all $x, y \in \mathbb{K}$, $\nu(x + y) \geq \min(\nu(x), \nu(y))$.

$\mathbb{K}$ is complete in the sense that it is complete with respect to the metric

$$d : \left| \begin{array}{ccc} \mathbb{K} \times \mathbb{K} & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & 2^{-\nu(x-y)} \end{array} \right. .$$

With this discrete valuation, we can consider the subring $A = \{x \in \mathbb{K} \mid \nu(x) \geq 0\}$ (called a *discrete valuation ring*). A is a local ring, which means it contains a unique maximal ideal $\mathfrak{m}$. Here,

$$\mathfrak{m} = \{x \in \mathbb{K} \mid \nu(x) > 0\} \subset A.$$

This means that $k := A/\mathfrak{m}$ is a field. For the purpose of this presentation, we will suppose k has characteristic $p > 0$.

Let G_{reg} be the set of p-regular elements of G ($i.e$ the set of elements whose order is not divisible by p). Finally, let m be the LCM of the orders of elements of G and let m' be the LCM of the orders of elements of G_{reg}. We will suppose $\mathbb{K}$ is *sufficiently large*, $i.e$ it contains all m-th roots of unity.

Proposition 3.1.1. *Let $\mu_{\mathbb{K}}$ and μ_k be the groups of m'-th roots of unity of $\mathbb{K}$ and k respectively. The function*

$$\Gamma : \left| \begin{array}{ccc} \mu_{\mathbb{K}} & \longrightarrow & \mu_k \\ \lambda & \longmapsto & \lambda \mod \mathfrak{m} \end{array} \right.$$

is a group isomorphism.

Proof. We only show Γ is bijective. First, notice that $\mu_{\mathbb{K}}$ is contained in A. Let $\pi : A \longrightarrow k = A/\mathfrak{m}$ be the canonical map and let $x \in \mu_{\mathbb{K}}$ be a primitive m'-th root of unity, $i.e$ $\mu_{\mathbb{K}} = \langle x \rangle$. Consider the following polynomial

$$P = \prod_{j=1}^{m'-1} (X - x^j) \in \mathbb{K}[X].$$

Its image by π is

$$Q = \prod_{j=1}^{m'-1} (X - \Gamma(x^j)) \in k[X].$$

We know that $P(1) = m'$, but since m' and p are coprime, this means that m' is invertible (more specifically nonzero) in k and so $\pi(P(1)) \neq 0$, that is to say $Q(1) \neq 0$. We have showed that $\Gamma(x)^j \neq 1$ for all $j \in [\![1, m'-1]\!]$. This shows that Γ is injective. Furthermore, Γ is surjective because the set $\{1, x, x^2, \dots, x^{m'-1}\}$ has cardinality m and there cannot be more m'-th roots of unity in k.

□

Notation. If $\lambda \in \mu_k$, we will write $\tilde{\lambda} := \Gamma^{-1}(\lambda)$. This way, $\tilde{\lambda}$ is the unique element of $\mathbb{K}$ such that $\lambda = \tilde{\lambda} \mod \mathfrak{m}$.

Proposition 3.1.2. *Let E be a $k[G]$-module of dimension n, let $s \in G_{\text{reg}}$ and let s_E be its associated endomorphism. Then, s_E is diagonalizable.*

Proof. Since G is finite, s_E has finite order j prime to p (as $s \in G_{\text{reg}}$). This means that $X^j - 1$ annihilates s_E. In particular, j divise m' so $X^j - 1$ has j distinct roots, and so s_E is diagonalizable.

□

Definition 3.1.3. *Let E be a $k[G]$-module of dimension n. Let $s \in G_{\mathrm{reg}}$ and $\lambda_1, \ldots, \lambda_n$ be its eigenvalues. We define*

$$\varphi_E(s) = \sum_{j=1}^{n} \tilde{\lambda}_j.$$

The mapping $\varphi_E : G_{\mathrm{reg}} \longrightarrow A$ is called the modular character *(or* Brauer *character) of E.*

Remark 3.1.4. Notice that φ_E takes its values in $\mathbb{K}$ (more specifically A) and not in k.

3.2 Properties and main theorem

We first detail a few basic properties that will convince us that this definition of modular character is adapted to what we already know in characteristic 0.

Proposition 3.2.1. *If E is a $k[G]$-module of finite dimension n, then*

$$\varphi_E(1_G) = n.$$

Proof. If $s = 1_G$, $s_E = \mathrm{id}_E$, $\lambda_1 = \cdots = \lambda_n = 1_k$ and $\tilde{\lambda}_1 = \cdots = \tilde{\lambda}_n = 1_{\mathbb{K}}$. □

Proposition 3.2.2. *Consider an exact sequence of $k[G]$-modules*

$$0 \longrightarrow E \longrightarrow E' \longrightarrow E'' \longrightarrow 0.$$

Then,

$$\varphi_{E'} = \varphi_E + \varphi_{E''}.$$

In other words, modular characters are compatible with the group structure of Grothendieck groups over categories of finitely generated $k[G]$-modules.

Remark 3.2.3. Consequently, if $E = E_1 \oplus E_2$ is a direct sum of $k[G]$-modules, then $\varphi_E = \varphi_{E_1} + \varphi_{E_2}$.

Theorem 3.2.4 (Brauer)**.** *Let S_k be the set of isomorphism classes of simple $k[G]$-modules. The set $\{\varphi_E \,|\, E \in S_k\}$ forms a basis of the $\mathbb{K}$-vector space of class functions $G_{\mathrm{reg}} \longrightarrow \mathbb{K}$.*

Example 3.2.5. With this theorem, we know we can study character tables in positive characteristics. A few examples are given in appendix B.

References

[Ful96] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry.* London Mathematical Society Student Texts. Cambridge University Press, 1996.

[Sag01] B. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions.* Graduate Texts in Mathematics. Springer New York, 2001.

[Ser98] Jean-Pierre Serre. *Linear representation of finite groups.* Hermann, fifth edition, 1998.

Appendices

A Examples of characters table in characteristic 0

A.1 Cyclic group of order 3

Let $j = e^{\frac{2i\pi}{3}}$.

	$\bar{0}$	$\bar{1}$	$\bar{2}$
χ_1	1	1	1
χ_j	1	j	j^2
χ_{j^2}	1	j^2	j

A.2 Cyclic group of order 4

	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
χ_1	1	1	1	1
χ_{-1}	1	-1	1	1
χ_i	1	i	-1	$-i$
χ_{-i}	1	$-i$	-1	i

A.3 Symmetric group of order 3

	$\mathrm{id}_{[\![1,3]\!]}$	$(1\,2)$	$(1\,2\,3)$
χ^{triv}	1	1	1
χ^{sgn}	1	-1	1
$\chi^{\perp}$	2	0	-1

Taken from [Sag01], chapter 1 section 1.9. The calculations are given throughout the first chapter of the book.

A.4 Symmetric group of order 4

	$\mathrm{id}_{[\![1,4]\!]}$	$(1\,2)$	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(1\,2\,3\,4)$
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

Taken from [Ser98], chapter 18 section 18.5.

B Examples of characters table in positive characteristic

In this appendix, we will present a few characters table of $\mathcal{S}_4$ in positive characteristic. Contrary to characteristic 0 character tables, we only have to consider the p-regular classes.

Both tables are (also) taken from [Ser98], chapter 18 section 18.5.

B.1 $p = 2$

	$\mathrm{id}_{[\![1,4]\!]}$	$(1\,2\,3)$
φ_1	1	1
φ_2	2	-1

B.2 $p = 3$

	$\mathrm{id}_{[\![1,4]\!]}$	$(1\,2)$	$(1\,2)(3\,4)$	$(1\,2\,3\,4)$
φ_1	1	1	1	1
φ_2	1	-1	1	-1
φ_4	3	1	-1	-1
φ_5	3	-1	-1	1