Regularity Structures

August 13, 2023

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Abstract

We introduce the theory of Regularity Structures via the (generalised) KPZ equation which is a singular stochastic partial differential equation (SPDE). Its solution is described by a local expansion whose monomials are recentered stochastic iterated integrals. This is the main motivation for introducing abstract definitions that are at the core of the theory of Regularity Structures. What has made this theory so successful for treating a large class of models is the use of various Hopf algebras that systematise the computations for renormalising and recentering the main iterated integrals that describe the solution.

Keywords: Decorated trees, Generalised KPZ equation, Hopf algebras, Regularity Structures, Rough paths, Singular SPDEs, Stochastic analysis.

Contents

1 Introduction

The theory of Regularity Structures has been introduced in [Hairer , 2014] by Martin Hairer. The main motivation was to develop a systematic approach for getting notion of solution, as well as existence and uniqueness results, which had been open problems for decades. It is called a theory as it covers a broad class of problems and models. The main equations considered are Stochastic Partial Differential Equations (SPDEs) which are Partial Differential Equations (PDEs) where space-time noise is added. This noise is a random distribution. The crucial idea idea of the theory draws its inspiration from Rough Paths introduced by Terry Lyons at the end of the 90' in Lyons [1998] where it has been understood that for singular Stochastic Diffrential Equations (SDEs) the solution map must depend on some iterated stochastic integrals built from the main features of the equation such as the noise considered and the integration map used in the integral formulation of the equation. In Gubinelli [2004] and Gubinelli [2010], one uses this integrals for giving a Taylor type expansion of the solution. Regularity Structures is an extension of this approach to a PDE context. We will present this theory via examples such as the (generalised) KPZ (Kadar-Parisi-Zhang) equation.

2 A first example: The KPZ equation

We present the main idea of the theory via a famous example: The KPZ equation named after Kadar, Parisi and Zhang. This equation is given below by:

$$\partial_t u = \partial_x^2 u + (\partial_x u)^2 + \xi, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{2.1}$$

where ξ is a space-time white noise which is a Gaussian process whose covariance is given by $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$. The KPZ equation is widely used for describing the fluctuation of random growing interfaces that share the same features: a smoothing effect given by the Laplacian term $\partial_x^2 u$, a tendancy to grow with the quadratic term $(\partial_x u)^2$ and the independence given by the space-time white noise ξ.

When one tries to predict the regularity of the solution of the KPZ equation, a scaling argument allows to say that it should behave as the solution of the linear equation given below by:

$$\partial_t v = \partial_x^2 v + \xi, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}$$

which has an explicit solution when the initial solution is zero given by $v = K * \xi$. Here K is the heat kernel and $*$ is a space-time convolution. This convolution between K and ξ is called the stochastic convolution. The solution v is not differentiable since it is only $\frac{1}{2}$ Hölder. Therefore, u is not differentiable and the product $(\partial_x u)^2$ is a distributional product which makes the KPZ equation ill-posed. The strategy chosen by Martin Hairer in Hairer [2013] was to proceed with a perturbative expansion by looking at solutions of the form $u = v + R$ where R is a remainder that should have better regularity than v in the sense that it will be Hölder with a bigger exponent. Substituting this ansatz in (2.1), one can compute a (formal) global expansion of the solution given by

$$u = \sum_{\tau \in \mathcal{T}} c_\tau \, K * u_\tau + R$$

where c_τ are constant coefficients coming from the perturbative expansion, u_τ are stochastic iterated integrals indexed by $\mathcal{T}$ which is a finite set of symbols. Below, we present expressions of some of the previous terms ($u_\tau \equiv \tau$):

$$\xi \equiv \circ, \quad (\partial_x K * \xi)^2 \equiv \ , \quad (\partial_x K * \xi)(\partial_x K * (\partial_x K * \xi)^2) \equiv$$

where we have encoded convolution with $\partial_x K$ by a brown edge and space-time white noises are denoted by a white dot. These symbols are decorated trees which are rooted trees with decorations on the edges and the vertices. The last two symbols needed in the expansion are two trees with four leaves: and . The previous sum is formal in the sense that the u_τ contain the ill-defined products coming from the equation. Indeed, $(\partial_x K * \xi)^2$ is not well-defined as its expectation is infinite:

$$\mathbb{E}((\partial_x K * \xi)^2) = (\partial_x K * \partial_x K)(0) = +\infty.$$

In order to be rigourous, one has to smooth out the noise. This is performed by replacing ξ by $\xi_\varepsilon = \varrho_\varepsilon * \xi$ where $(\varrho_\varepsilon)_{\varepsilon>0}$ is a family of even space-time mollifier such that ξ_ε converges to ξ when ε is sent to zero. If we denote by u_ε the solution of the KPZ equation with the smooth noise ξ_ε, we would like to find a topology on the noises such that

1. the solution map $\Phi : \xi_\varepsilon \to u_\varepsilon$ is continuous.
2. $\xi_\varepsilon \to \xi$ when $\varepsilon \to 0$.

The first point requires a sufficiently strong topology, while the second requires a sufficiently weak topology. In fact, no solution seems possible if the regularity of ξ is too low, and even in the simplest case of stochastic ordinary differential equations it is a theorem that it is impossible to find a Banach space containing samples of the noise ξ and making the solution map continuous Lyons [1991]. The remedy is to include in the dependency of the solution map the stochastic iterated integrals appearing in the perturbative expansion of the solution. We obtained a new solution map denoted by Ψ such that $\Psi(\mathbf{u}^\varepsilon) = \Phi(\xi_\varepsilon)$ where $\mathbf{u}^\varepsilon = (u_\tau^\varepsilon)_{\tau \in \mathcal{T}}$ corresponds to the iterated integrals u_τ where all occurence of ξ are replaced by ξ^ε. The term $\mathbf{u}_\tau^\varepsilon$ is called a model and belongs to a metric space $(\mathscr{M}, \mathrm{d})$ called space of models.

A natural question is to know where to stop in the pertubative expantion. Due to the smoothing effect of the Laplacian, one expects to produce terms that are more regular. This is encoded into some power counting depending on the Hölder regularity of the noise. We call this power counting a degree $\deg : \mathcal{T} \to \mathbb{R}$ and it is quite similar to the power counting in Feynman diagrams for computing its degree of divergence. The space-time white noise has the degree $-\frac{d+2}{2}$ where d is the space dimension. Integration against the heat kernel increases degrees by 2, it corresponds to the Schauder estimate given by $f \mapsto K * f$ that maps α Hölder functions to $\alpha + 2$ Hölder functions. Differentiation lowers degree by 1, and degrees are additive under multiplication. Using these rules, one computes easily the following degrees: $\deg(\circ) = -\frac{3}{2}$, $\deg(\ ) = -1$ and $\deg(\ ) = -\frac{1}{2}$. In the expansion of the solution is sufficient to stop with degree 0, this corresponds to the two decorated trees with four noises listed above. One can notice that for the KPZ equation, one gets only a finite number of trees with negative degree. This is one limitation of the theory of Regularity Structures as one can only treat subcritical models which are models providing a finite number of decorated trees with negative degree. This corresponds to superrenormalisable theory in quantum field theory where only a finite number of diagrams has to be considered for renormalisation.

3 Renormalisation

The remaining main task is to construct the stochastic iterated integrals $(u^{\varepsilon}_{\tau})_{\varepsilon>0}$ by sending ε to zero for getting a limiting object. We have noticed that we do not get a limit as the expectation of these terms could be divergent. The major problem that occurs in the stochastic iterated integrals is that the products appearing in their expressions may diverge in the limit $\varepsilon \to 0$ so that we do not expect in general that $\mathbf{u}^{\varepsilon}$ converges in $(\mathscr{M}, \mathrm{d})$ as $\varepsilon \to 0$. To overcome this problem, it is necessary to modify/renormalise some components of $\mathbf{u}^{\varepsilon}$, and define a new lift $\hat{\mathbf{u}}^{\varepsilon} \in \mathscr{M}$ of ξ_{ε}. For example, the pointwise product $(\partial_x K * \xi_{\varepsilon})^2$, which diverges when $\varepsilon \to 0$, can be replaced by

$$\mathbf{u}^{\varepsilon} = (\partial_x K * \xi_{\varepsilon})^2 \quad \mapsto \quad (\partial_x K * \xi_{\varepsilon})^2 - \mathbb{E}[(\partial_x K * \xi_{\varepsilon})^2] = \hat{\mathbf{u}}^{\varepsilon} \tag{3.1}$$

We want to build a lift $\hat{\mathbf{u}}^{\varepsilon}$ of ξ_{ε} such that

- we respect the non-linear constraints that define the space of models $\mathscr{M}$,
- the lift respects the meaning of edges as convolution operators for decorated trees, so one imposes for example that $\hat{u}^{\varepsilon} = \partial_x K * \hat{\mathbf{u}}^{\varepsilon}$,
- we get a converging family in $(\mathscr{M}, \mathrm{d})$ when $\varepsilon \to 0$,

The continuity of the solution map Ψ together with a family $\hat{u}_{\varepsilon} := \Psi(\hat{\mathbf{u}}^{\varepsilon})$ converging in $\mathscr{D}'(\mathbb{R}^{d+1})$, space of Schwartz distributions, to some limit $\hat{u}$ provides the solution to the singular SPDEs we have started with. At this stage, the renormalisation procedure of the u^{ε}_{τ} is more complex than just removing the expectation and one has to apply a renormalisation type procedure that we will describe algebraically in the last section. Let us briefly explain what are the analytical/stochastic tools used for dealing with these stochastic iterated integrals. First, one can notice that the u^{ε}_{τ} are polynomial in the Gaussian noise ξ_{ε} which encourrages us to use Gaussian Calculus. Indeed, the key quantity to control is the second moment of this process $\mathbb{E}[|u^{\varepsilon}_{\tau}|^2]$ which can be computed using the Wick formula given for a product of random Gaussian variables $\prod_{i\in I} \xi_i$, where I is a finite set, by

$$\mathbb{E}[\prod_{i\in I} \xi_i] = \sum_{\pi\in\mathscr{P}(I)} \prod_{(i,j)\in\pi} \mathbb{E}[\xi_i \xi_j]$$

where $\mathscr{P}(I)$ are all the possible pairings of I and a pairing π is a partition of disjoint pairs of I. One wants to get uniform bounds over ε on this quantity. Using some Isometry properties, this amounts to having uniform bounds on deterministic iterated integrals in L^2 which is similar to bound Feynman diagrams. The terms u^{ε}_{τ} could be considered as half Feynman diagrams. Divergencies appear inside u^{ε}_{τ} where two noises from the same tree are paired but not from pairing of two noises from the two copies of u^{ε} in $\mathbb{E}[|u^{\varepsilon}_{\tau}|^2]$. The degree map gives us the correct power counting for detecting subdivergencies that appear in the half Feynman diagrams u^{ε}_{τ}. Below, we provide a non-trivial renormalisation of the KPZ stochastic iterated integral with three noises:

$$u^{\varepsilon} \quad \mapsto \quad u^{\varepsilon} - C^{\varepsilon} u^{\varepsilon}, \quad C^{\varepsilon} = \mathbb{E}[(\partial_x K * \xi_{\varepsilon})(\partial_x K * \partial_x K * \xi_{\varepsilon})] \tag{3.2}$$

where $u^{\varepsilon} = \partial_x K * \xi_{\varepsilon}$, the renormalisation constant C^{ε} is zero if ϱ_{ε} is an even mollifier and otherwise it diverges in $\log(\varepsilon)$. The subtree corresponds to a subdivergence when two internal nodes of are paired. One can notice that we do not need to subtract the expectation as we have an odd number of noises. The changes in the components of $\mathbf{u}^{\varepsilon}$ are constrained by the non-linear structure of the model. The constansts C^{ε}_{τ} introduces a parametrisation of the renormalisation, this is encoded via a *renormalisation group* $\mathscr{G}_-$ that has been described in Bruned *et al.* [2019]. It is precisely the group of transformations of $\mathscr{M}$ that respect this structure.

After the renomalisation procedure and the passage to the limit, we obtain a renormalised solution given by $\hat{u} := \Psi(\hat{\mathbf{u}})$, which is also the unique solution of a fixed point problem. This works for very general noises, well beyond the Gaussian case. Indeed the computation of the second moment via Wick formula could be replaced by writing the moments via a cumulant expansion.

One can notice that we do not have anymore the relation $\Psi(\mathbf{X}^{\varepsilon}) = \Phi(\xi_{\varepsilon})$ as the renormalisation has changed the integrals considered. So in the end, what happened? Which equation are we solving? In fact, we have to consider solutions to a family of equations indexed by a number of constants. For example, in the case of the KPZ equation, we could consider the family of equations

$$\partial_t u = \partial_x^2 u + \lambda_1 (\partial_x u)^2 - \lambda_2 + \xi \;,$$

parametrised by $\lambda \in \mathbb{R}^2$. We should then add a dependency in λ for the solution maps Φ and Ψ. It was then shown in Bruned *et al.* [2021] that the renormalisation group $g_{\varepsilon} \in \mathscr{G}_-$ already mentioned earlier does not only come with an action R on the space of models, but also with an action S on the parameter space of our class of equations, and these actions are intertwined in such a way that

$$\Psi(\lambda, \hat{\mathbf{u}}^{\varepsilon}) = \Psi(\lambda, R^{g_{\varepsilon}} \mathbf{u}^{\varepsilon}) = \Psi(S^{g_{\varepsilon}} \lambda, \mathbf{u}) \;.$$

One way of interpreting this is that the renormalisation procedure is nothing but a change in parametrisation for the family of solutions $\lambda \mapsto \Psi(\lambda, \xi_{\varepsilon})$. This situation arises in quantum field theory, when one considers renormalised coupling constants. The renormalised version of the KPZ equation is then

$$\partial_t \hat{u}_{\varepsilon} = \partial_x^2 \hat{u}_{\varepsilon} + \lambda_1 (\partial_x \hat{u}_{\varepsilon})^2 + \lambda_2 - \lambda_1^2 C^{\varepsilon} + \xi_{\varepsilon}, \tag{3.3}$$

with the ε-dependent change of parameters $(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2 - \lambda_1^2 C^{\varepsilon}_{\vee})$ and $C^{\varepsilon}_{\vee} = \mathbb{E}[(\partial_x K * \xi_\varepsilon)^2]$. Let us finish this section by an important remark. There are two main approaches to the action of renormalisation to the equation:

- *A top-down approach* that puts the correct parameters λ at the level of the equation. This corresponds to S^{g_ε}. Then, from this equation, one generates the stochastic iterated integrals with their renormalisation and check that they converge. This is equivalent to the procedure in quantum field theory when one renormalises the Lagrangian of a model and then perform the perturbative expansion giving the renormalised Feynman diagrams.
- *A bottom-up approach* which starts by expanding the stochastic iterated integrals and then renormalises them. The difficult task is to understand how the modification of an expansion implies changes at the level of the equation that generates this expansion. This is the approach advocated in the context of Reguarity Structures in Bruned *et al.* [2021]. More recently, the top-down approach has been used in Linares *et al.* [2021] with multi-indices for describing the expansion of the solution.

4 A second example: The generalised KPZ equation

The resolution of the KPZ equation is based on a global expansion of the solution via stochastic iterated integrals indexed by decorated trees. This approach was possible as one has to consider only a polynomial non-linearity: $(\partial_x u)^2$. Such an approach could potentially work for many models coming from quantum field theory such as the φ^4_d given by:

$$\partial_t u = \Delta u - u^3 + \xi, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

where the 4 in the exponent of φ^4_d corresponds to the degree minus one of the non-linearity u^3 which is 3. These models are quite singular as the solution is a distribution for $d \geq 2$ and it is critical for $d = 4$. By critical, it means that we are not able to solve this equation via the expansion described above as it will produces an infinite number of stochastic iterated integrals. One main equation that needs a different perspective is the geometric KPZ equation. It has been treated via Regularity Structures in Bruned *et al.* [2022]. It is a natural random dynamic on the space of loops in a Riemannian manifold with metric g. This evolution can be viewed as the solution to the SPDE given in local coordinates by

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\, \partial_x u^\beta \partial_x u^\gamma + \sum_{i=1}^m \sigma_i^\alpha(u)\, \xi_i \,, \tag{4.1}$$

see Figure 1. Here, Γ denotes the Christoffel symbols of the metric g while the σ_i are any finite collection of smooth vector fields such that

$$\sum_i \sigma_i^\alpha(u) \sigma_i^\beta(u) = g^{\alpha\beta}(u) \,. \tag{4.2}$$

The main challenge is that if we consider an expansion of the form $u = \sum_{\tau \in \mathfrak{T}} c_\tau K * u_\tau + R$ how can we proceed with a term $f(u)$ for continuing the expansion? This is one of the key ingredient introduced by the theory of Regularity Structures that draws its inspiration from numerical analysis. Indeed, we need to *linearise* $f(u)$ and we proceed via a Taylor-type expansion. The idea is now to have a local expansion:

$$u(z) = u(z') + \sum_{\tau \in \hat{\mathfrak{T}}} c_{\tau,z'} u_{\tau,z'}(z) + R_{z'}(z), \quad z, z' \in \mathbb{R}_+ \times \mathbb{R} \tag{4.3}$$

where $u_{\tau,z'}$ is a localised version of u_τ around the based point z' and $R_{z'}$ is a Taylor remainder such that

$$|u_{\tau,z'}(z)| \lesssim |z - z'|^{\deg \tau}, \quad |R_{z'}(z)| \lesssim |z - z'|^{\alpha}, \, \forall \tau \in \hat{\mathfrak{T}}, \deg \tau \leq \alpha.$$

where $|z - z'|$ is the Euclidean norm with the scaling $(2, 1)$ that makes count the time double in comparison to space. This is coming from the equation where one has two spatial derivatives for one time derivative. If the quantities above are not smooth, one can replace the evaluation at the point z' by a test function $\varphi_{z'}^{\lambda}$ localised at the point z' which can be seen as a scale λ approximation of a Dirac δ-distribution centred at z'. The bounds we obtain are of the form $|u_{\tau,z'}(\varphi_{z'}^{\lambda})| \lesssim \lambda^{\deg \tau}$. The coefficents $c_{\tau,z'}$ could be understood as derivatives. The set $\hat{\mathfrak{T}}$ contains many new elements such as the classical monomials of a Taylor expansion that are denoted by X^k for $k = (k_1, k_2) \in \mathbb{N}^2$ with $\deg X^k = 2k_1 + k_2$. One has $u_{X^k,z'}(z) = (z - z')^k = (\prod_{i=1}^{2} (z_i - z_i')^{k_i})$ for $k = (k_1, k_2), z = (z_1, z_2)$, and $z' = (z_1', z_2')$. Then, one defines the composition of (4.3) with a smooth function f as

$$\hat{f}_\gamma(u(z)) = P_\gamma \sum_{k} \frac{1}{k!} (\sum_{\tau \in \hat{\mathfrak{T}}} c_{\tau,z'} u_{\tau,z'}(z) + R_{z'}(z))^k f^{(k)}(u(z'))$$

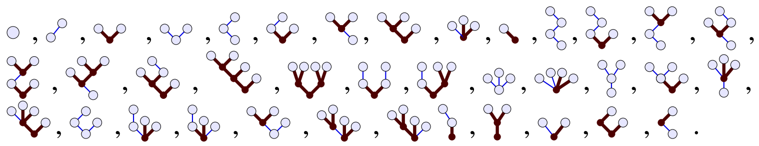
where P_γ is a projection up to the order γ that disregards terms of order greater than $|z - z'|^\gamma$. This definition is inspired from numerical analysis where one defines composition of numerical methods. The first step is to define the composition of a numerical method represented as a tree series (Butcher series) with a smooth function. Then, having products of tree series on the right hand side of a singular SPDEs implies that one has:

$$u_{\tau,z'}(z) u_{\tau',z'}(z) = u_{\tau\tau',z'}(z)$$

where $\tau\tau'$ is the merging root product that identifies the roots of τ and τ'. For terms of the form $X^k \tau$ and $X^{k'} \tau'$, the merging product gives $X^{k+k'} \tau\tau'$ where X^k can be interpreted as a node decoration that is additive for the merging tree product. For the convolution with the kernels K and $\partial_x K$, one has to subtract a Taylor expansion in order to get the correct bound in $|z - z'|$. This depends on the degree of the tree considered:

$$u_{\mathcal{I}(\tau),z'}(z) = (K * u_{\tau,z'})(z) - \sum_{|\ell| \leq \deg \mathcal{I}(\tau)} \frac{(z - z')^\ell}{\ell!} (\partial^\ell K * u_{\tau,z'})(z') \tag{4.4}$$

where $\ell = (\ell_1, \ell_2) \in \mathbb{N}^2$, $|\ell| = 2\ell_1 + \ell_2$, and $\mathcal{I}(\tau)$ is the tree obtained by connecting τ to a new root via a blue edge that encodes the space-time convolution with the kernel K. The model $\mathscr{M}$ is given by the collection of the $u_{\tau,z'}$ equipped with a suitable distance d. The list of trees indexing the components of the model associated to this class of equations is much longer. For example, the most relevant trees of negative degree are the following:



(4.5)

The space $\hat{\mathcal{T}}$ is given by the $\mathcal{I}(\tau)$ where the τ belong to the previous list. In Bruned *et al.* [2022], natural geometric quantities such as the scalar curvature play an important and fascinating role in the study of the equation (4.1). It was shown there that it is possible to perform the renormalisation of this equation in such a way that solutions perform under changes of variables as expected from the naïve application of the rules of calculus and such that the law of these solutions is independent of the choice of vector fields σ_i satisfying (4.2). This is like having Itô isometry and chain rule property at the same time which is not possible in the context of stochastic differential equations when one has to choose between Itô and Stratonovich for the stochastic integration.

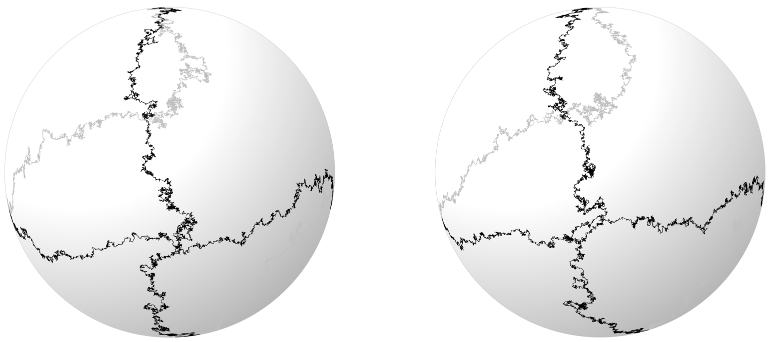


Figure 1: The solution of (4.1) on the sphere at two successive times.

5 Abstract definitions of Regularity Structures

From the previous examples, we are able to provide the main abstract definition of a Regularity Stuctures (A, T, G) as it was introduced in Hairer [2014]. It consists of the following elements:

- a set of degrees, which is a set $A \subset \mathbb{R}$ bounded from below and locally finite.
- a model space, which is a graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$.

- a structure group G, which is a group of linear endomorphisms of T such that, for every $\Gamma \in G$, every $\alpha \in A$, and every $\tau \in T_\alpha$ one has $\Gamma\tau - \tau \in \bigoplus_{\beta<\alpha} T_\beta$.

In the examples we have seen in the previous sections, A is the set of degrees of the decorated trees T, and T_α is a space of decorated trees of degree α. Given a Regularity Structure, a model consists of a collection of linear maps $\Pi_x : T \to \mathcal{D}'(\mathbb{R}^{d+1})$, with $\mathcal{D}'(\mathbb{R}^{d+1})$ space of Schwartz distributions, and of elements of the structure group $\Gamma_{xy} \in G$ such that they satisfy the algebraic properties

$$\Pi_y = \Pi_x \Gamma_{xy}, \quad \Gamma_{xy} = \Gamma_{xz}\Gamma_{zy}$$

as well as the estimates for all compact subsets $\mathfrak{K}$ of $\mathbb{R}^{d+1}$

$$\langle \Pi_x \tau, \varphi_x^\lambda \rangle \lesssim \lambda^\alpha, \quad |\Gamma_{xy}\tau|_\beta \lesssim |y-x|^{\alpha-\beta},$$

uniformly over all $\tau \in T_\alpha$, $x, y \in \mathfrak{K}$, $\lambda \in (0,1)$ and all localized test functions φ_x^λ. Here $|\cdot|_\beta$ means that we consider the coefficients in front of elements in T_β. Let us give a motivation about the notion of the structure group and the Γ_{xy} on polynomials. A smooth function $f : \mathbb{R}^{d+1} \to \mathbb{R}$ is well described locally via a polynomial:

$$f(z) \approx \sum_n f_n(z')(z-z')^n, \text{ near } z'$$

where $f_n(z')$ are some coefficients. Then, one wants to know the behaviour of f around another point y via the transformation: $(z-z')^n = (z-y+y-z')^n = \sum_{\ell \le n} \binom{n}{\ell}(y-z')^{n-\ell}(z-y)^\ell$. We obtain the following local description:

$$f(z) \approx \sum_\ell \left(\sum_{n, \ell \le n} f_n(z')\tbinom{n}{\ell}(y-z')^{n-\ell} \right)(z-y)^\ell$$

which is an illustration of the identity $\Pi_{z'} = \Pi_y \Gamma_{yz'}$. The map Γ_{xy} is a re-expansion map that allows to move a local expansion from a point to another. It is essential for defining the topology of the models $(\mathcal{M}, \mathrm{d})$ in which one sets the fixed point that provides a notion of solution to the singular SPDE we started with. Regularity Structures applied to singular SPDEs reach a high degree of generality thanks to four steps realised in a systematic way:

- *Analytical step*: Construction of the space of models $(\mathcal{M}, \mathrm{d})$ and continuity of the solution map $\Psi : \mathcal{M} \to \mathcal{D}'(\mathbb{R}^{d+1})$, Hairer [2014].
- *Algebraic step*: Description of a group action on the space of models describing the transformation $\mathcal{M} \ni \mathbf{u}^\varepsilon \mapsto \hat{\mathbf{u}}^\varepsilon \in \mathcal{M}$ from the canonical model to the renormalised model, Bruned *et al.* [2019].
- *Probabilistic step*: Convergence in probability of the renormalised model $\hat{\mathbf{u}}^\varepsilon$ to a limit model $\hat{\mathbf{u}}$ in $(\mathcal{M}, \mathrm{d})$, Chandra and Hairer [2016].
- *Second algebraic step*: Identification of the renormalised equation satisfied by $\hat{u}_\varepsilon := \Psi(\hat{\mathbf{u}}^\varepsilon)$, Bruned *et al.* [2021].

6 Algebraic structures

The main success of Regularity Structures is to have introduced Hopf algebras on decorated trees which make it possible to carry out the analytical operations necessary to construct the renormalised model:

- Recentering for constructing the main components of the model that allows us to move from u^{ε}_{τ} to $u^{\varepsilon}_{\tau,x}$.
- Renormalisation for dealing with the ill-defined products of the equations and allows us to move from u^{ε}_{τ} to $\hat{u}^{\varepsilon}_{\tau}$.

What is quite complex is that one wants to perform the two operations as the same time meaning the construction of $\hat{u}^{\varepsilon}_{\tau,x}$. This construction corresponds to a nice cointeraction between two Hopf algebras.

We have already noticed that the renormalisation procedure is more complicated than the simple subtraction of a constant. It is shown in Bruned *et al.* [2019] that as long as the collection of trees $\mathcal{T}$ generating $\mathcal{H}$ has some properties natural in this context, there is a single (deterministic) $g_\varepsilon \in \mathcal{G}_-$, element of the renormalisation group such that if we set $\hat{u}^{\varepsilon}_{\tau} = u^{\varepsilon}_{g_\varepsilon \tau}$, then all components of $\hat{\mathbf{u}}^{\varepsilon}$ have zero expectation. This is very similar to the 'BPHZ renormalisation' prescription found in the physics literature Bogoliubov and Parasiuk [1957], Hepp [1969], Zimmermann [1969], so we call this particular choice of $\hat{\mathbf{u}}^{\varepsilon}$ the 'BPHZ lift' of the noise.

To describe the renormalisation group $\mathcal{G}_-$, we consider the algebra with unit $(\mathcal{H}_-, \cdot, \mathbf{1}_-)$ generated by the trees $\mathcal{T}_- \subset \mathcal{T}$ of negative degree and we realise $\mathcal{G}_-$ as the space of *characters* of $\mathcal{H}_-$, which are the algebra morphisms $g : \mathcal{H}_- \to \mathbb{R}$. To describe the group product in $\mathcal{G}_-$, we endow $\mathcal{H}_-$ with a structure of *coalgebra* with a *coproduct* $\Delta^- : \mathcal{H}_- \to \mathcal{H}_- \otimes \mathcal{H}_-$ which satisfies a property of *coassociativity*

$$(\Delta^- \otimes \mathrm{id})\Delta^- = (\mathrm{id} \otimes \Delta^-)\Delta^- \tag{6.1}$$

and a *counit* $\eta_- \in \mathcal{H}_-^*$ such that

$$(\eta_- \otimes \mathrm{id})\Delta^- = (\mathrm{id} \otimes \eta_-)\Delta^- = \mathrm{id} \tag{6.2}$$

on $\mathcal{H}_-$. The space $(\mathcal{H}_-, \cdot, \mathbf{1}_-, \Delta^-, \eta_-)$ is a *Hopf algebra*. The product in $\mathcal{G}_-$ is the dual of the coproduct in $\mathcal{H}_-$:

$$\mathcal{G}_- \times \mathcal{G}_- \ni (g_1, g_2) \mapsto g_1 \cdot g_2 \in \mathcal{G}_-, \qquad (g_1 \star g_2)(h) := (g_1 \otimes g_2)\Delta^- h$$

for every $h \in \mathcal{H}_-$. The coassociativity (6.1) of Δ^- implies that this product is *associative*:

$$(g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3),$$

and the counit η_- is the neutral element such that $\eta_- \star g = g \star \eta_- = g$ for every $g \in \mathcal{G}_-$, thanks to (6.2). Moreover, it is possible to show that every element of $\mathcal{G}_-$ has an inverse.

If $\mathbf{u} \in \mathscr{M}$ is a model and $g \in \mathscr{G}_-$ is an element of the renormalisation group, we can define a new model $\mathbf{u}^g = R^g \mathbf{u} \in \mathscr{M}$ by

$$\mathbf{u}^g : \mathscr{H} \to \mathscr{D}'(\mathbb{R}^{d+1}), \qquad u^g_\tau := (g \otimes \mathbf{u})\Delta^- \tau \,, \tag{6.3}$$

where $\Delta^- : \mathscr{H} \to \mathscr{H}_- \otimes \mathscr{H}$ is defined very similarly to the coproduct of $\mathscr{H}_-$.

The renormalisation group $\mathscr{G}_-$ must preserve another underlying algebraic structure, described by another group called $\mathscr{G}_+$ and which allows to describe the topology of the space $\mathscr{M}$. For describing this group, we use another Hopf algebra $(\mathscr{H}_+, \cdot, \mathbf{1}_+, \Delta^+, \eta_+)$ generated by a collection $\mathscr{T}_+$ of trees, this time of *positive* degree, and the group $\mathscr{G}_+$ is described as the character group of $\mathscr{H}_+$. The group $\mathscr{G}_+$ acts on $\mathscr{H}$ similarly to above by $\Gamma_g : \tau \mapsto (\mathrm{id} \otimes g)\Delta^+ \tau$ with $\Delta^+ : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}_+$ given by a formula very similar to that defining Δ^+. This gives us a nice parametrisation of the structure group $G = \{\Gamma_g,\, g \in \mathscr{G}_+\}$.

A linear map $\mathbf{u} : \mathscr{H} \to \mathscr{D}'(\mathbb{R}^{d+1})$ then defines a model if there exists a $\mathscr{G}_+$-valued function $\mathbb{R}^{d+1} \ni x \mapsto f_x \in \mathscr{G}_+$ such that, for every $x \in \mathbb{R}^{d+1}$, the 'recentred' model $\mathbf{u}_x = (\mathbf{u} \otimes f_x)\Delta^+$ satisfies a bound of the type $|u_{\tau,x}(\varphi^\lambda_x)| \lesssim \lambda^{\deg \tau}$. Then, the map Γ_{xy} is just given by $\Gamma_{xy} = \Gamma_{(f_x)^{-1}} \circ \Gamma_{f_y}$.

The fact that this topology is preserved by $\mathscr{G}_-$ is encoded in an action of $\mathscr{G}_-$ on $\mathscr{G}_+$, that is, a group morphism of $\mathscr{G}_-$ into the (outer) automorphisms of $\mathscr{G}_+$. The action of $\mathscr{G}_-$ on $\mathscr{G}_+$ is described by a map $\Delta^- : \mathscr{H}_+ \to \mathscr{H}_- \otimes \mathscr{H}_+$ which satisfies a property called *cointeraction*:

$$\mathscr{M}^{(13)(2)(4)}(\Delta^- \otimes \Delta^-)\Delta^+ = (\mathrm{id} \otimes \Delta^+)\Delta^- \,, \tag{6.4}$$

where $\mathscr{M}^{(13)(2)(4)}(\tau_1 \otimes \tau_2 \otimes \tau_3 \otimes \tau_4) := (\tau_1 \cdot \tau_3 \otimes \tau_2 \otimes \tau_4)$.

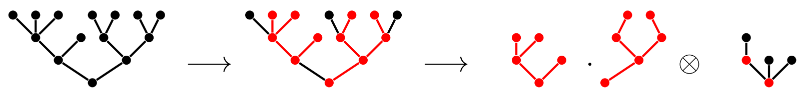
We now define the action of $\mathscr{G}_-$ on $\mathscr{G}_+$ like this: for $g_- \in \mathscr{G}_-$ and $g_+ \in \mathscr{G}_+$, $g_- \bullet g_+ \in \mathscr{G}_+$ is given by

$$(g_- \bullet g_+)(h_+) = (g_- \otimes g_+)\Delta^- h_+, \qquad \forall\, h_+ \in \mathscr{H}_+.$$

We can easily see that the cointeraction property (6.4) defines an action:

$$g_- \bullet (\bar g_- \bullet g_+) = (g_- \star \bar g_-) \bullet g_+, \qquad g_-, \bar g_- \in \mathscr{G}_-, \quad g_+ \in \mathscr{G}_+.$$

Let us conclude by giving a simplified description of the operations $\Delta^\pm$. Recall that the spaces $\mathscr{H}$, $\mathscr{H}_+$ and $\mathscr{H}_-$ are realised as vector spaces generated by decorated trees obtained from the perturbative expansion of the solution of a singular SPDE. The operations Δ^- and Δ^+ on such trees are both constructed by using an operation *of extraction and contraction* of subforests:



where

- we start from a tree or forest, drawn here in black on the left

- in the center, we select a subforest, colored in red
- on the right, the selected subforest is *extracted* in the left term of the tensor product, and *contracted* on the right. Note that in particular the total number of edges is always preserved by such operations.

The main difference between Δ^- and Δ^+ is in the selection of the subforests which are extracted: in the case of Δ^+, we extract only subforests consisting of a single tree that contains the root of the initial tree; in the case of Δ^-, we extract arbitrary subforests. In addition, the operation Δ^- only extracts subtrees of *negative* degree while Δ^+ only extracts those subtrees such that each 'trunk' adjacent to the root of the tree remaining on the right after contraction determines a subtree of *positive* degree. In fact due to the degree, the cointeraction (6.4) is not quite true. The main issue is that one wants the coproduct Δ^- to be degree preserving in the sense that the contracted tree has the same degree for applying Δ^+. A main reason is that one does not want the recentering to depend on the renormalisation, that is the length of the Taylor expansion around a fixed base point. To solve this problem, extended decorations are added to the trees in Bruned *et al.* [2019], they are just the degree of the decorated trees which have been contracted via Δ^-.

Alternatively, in Bruned [2018], one can define the renormalisation map using a variant of Δ^+ by extracting only trees with negative degree at the root. This map is then recursively iterated within the tree. This is enough for defining a renormalised model and also to treat the second algebraic steps by providing a short proof of the renormalised equation in Bailleul and Bruned [2021].

We have described how the two coproducts $\Delta^\pm$ act on shapes but they are quite complex as they also encode Taylor expansions by changing decorations on the edges and the vertices which corresponds to add monomials of the form X^k and the corresponding derivatives on some kernels. Algebraically, this can be interpreted as a deformation that could be seen directly on a pre-Lie product in Bruned and Manchon [2023]. One can actually go further in the algebraic interpretation by seeing that these coproducts come from a post-Lie product in Bruned and Katsetsiadis [2023].

Regarding the action of Δ^+, consider for example again the case of the generalised KPZ equation where $\deg \circ = -\frac{3}{2}$. In this case, one has for example

$$\Delta^+ \; = \; \otimes \mathbf{1} + \circ \otimes \; , \qquad \Delta^+ \; = \; \otimes \mathbf{1} + \mathbf{1} \otimes \; ,$$

since $\deg \; = \frac{1}{2}$. A model $\mathbf{u}$ then must be such that there exists a function $x \mapsto f_{\cdot,x}$ with the property that

$$|(u + f_{\cdot,x} u_\circ)(\varphi_x^\lambda)| \lesssim \lambda^{-1} \, , \qquad |(u + f_{\cdot,x})(\varphi_x^\lambda)| \lesssim \lambda^{1/2} \, .$$

Note now that, one must have $u = K * u_\circ = K * \xi$, which is a Hölder continuous function. Since the exponent $\frac{1}{2}$ appearing in the second bound above is positive, this forces to have $f_{\cdot,x} = -(K * \xi)(x)$ which corresponds exactly to the Taylor expansion in (4.4).

Regarding the action of Δ^-, still in the same context, one has for example

$$\Delta^- \; = \mathbf{1} \otimes \; + \; \otimes \mathbf{1} \, , \qquad \Delta^- \; = \mathbf{1} \otimes \; + \; \otimes \mathbf{1} + 2 \; \otimes \; .$$

One can disregard extraction of the noise $\circ$ as we will always consider centred noise: $\mathbb{E}[\xi] = 0$. We then see that if we want to construct the BPHZ lift of a noise ξ_ε, the first identity, combined with the BPHZ prescription that $\mathbb{E}[u^\varepsilon_\tau] = 0$ for $\deg \tau < 0$, forces us to choose a character g_ε such that $g_\varepsilon(\,) = -\mathbb{E}[u^\varepsilon]$, while the second identity then forces us to choose $g_\varepsilon(\,) = -\mathbb{E}[u^\varepsilon]$, yielding

$$\hat{u}^\varepsilon = u^\varepsilon - \mathbb{E}[u^\varepsilon] - 2u^\varepsilon \cdot \mathbb{E}[u^\varepsilon] \ .$$

The coassociativity and cointeraction properties seen above have a natural interpretation in terms of combinatorial operations on these trees and forests. Note that an algebraic structure very similar to this construction is known to arise in the numerical analysis of ordinary differential equations. There, this approach was pioneered by J. Butcher Butcher [1972] who pointed out that the natural composition operation for Runge-Kutta methods can be described by a Hopf algebra very similar to $\mathscr{H}_+$. More recently, it was pointed out by E. Hairer and his collaborators Chartier *et al.* [2010] that an analogue of the Hopf algebra $\mathscr{H}_-$ has a natural interpretation as a 'substitution operation' for Runge-Kutta methods. The Hopf algebra $\mathscr{H}_+$ is also a generalisation of the so-called *Connes-Kreimer algebra* which was introduced in the 1990s to describe algebraically renormalisation in quantum field theory Connes and Kreimer [1998]. In Calaque *et al.* [2011], the cointeraction at the Hopf algebraic structure is given. Among the future directions of Regularity Structures one can mention

- The study of the discrete counterpart of these continuous dynamics for determining the invariant measures. Many recent preprints build upon the discrete version of the theory of Regularity Structures given in Erhard and Hairer [2019].
- A better understanding of the symemtries as in Bruned *et al.* [2022] or in Chandra *et al.* [2022a,b] where the authors study the Yang-Mills SPDE in 2D and 3D.
- Developing the algebraically part of the theory which has been recently understood from a post-Lie point of view in Bruned and Katsetsiadis [2023]. Also decorated trees are not the only combinatorial objects that could be used for describing the expansion of the solution. In Otto *et al.* [2021], the authors introduce multi-indices and one has in Linares *et al.* [2023] a construction of the structure group based on this approach. A recursive proof of the convergence of the renormalised model is given in Linares *et al.* [2021].
- Apply the main ideas of Regularity Structures to other fields beyong singular SPDEs. This has been initiated in numerical analysis for proposing efficient low regularity schemes for a large class of dispersive equations in Bruned and Schratz [2022]. Also the main analytical properties of Regularity Structures could be re-express in a more general setting see Caravenna and Zambotti [2020].

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