

AN INTERACTIVE FIELD THEORY APPROACH TO THE STOCHASTIC SINE-GORDON MODEL

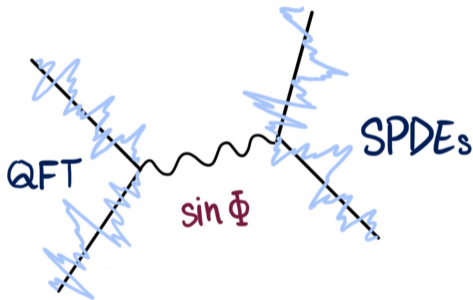


UNIVERSITÀ
DI PAVIA



LABORATOIRE
DE PROBABILITÉS
STATISTIQUE
& MODÉLISATION

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Decorated tree-like structures for singular dynamics
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Joint work with



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Singular Stochastic PDEs

$$Lu = F(u) + H(u)\xi$$

Rough source ξ : proper meaning to the nonlinear contributions?
 \Rightarrow need for **renormalization**

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Present frameworks:

- Regularity structures
- Paracontrolled calculus
- Renormalization group techniques

Pros 

Well-posedness results
Widely applicable

Cons 

Few information on the solution

Sine-Gordon QFT

- **Geometry:** 2-dim Minkowski (\mathbb{R}^2, η)
- **Field theory:** $a \in \mathbb{R}, g \in \mathcal{D}(\mathbb{R}^2)$

$$(\square + m^2)\psi + \lambda g a \sin(a\psi) = 0$$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - \frac{1}{2} m^2 \psi^2 - \lambda g \cos(a\psi)$$

Finite ultraviolet regime: $a^2 < 4\pi/\hbar$

pAQFT framework

2-step approach

1. construction of an **algebra of observables** \mathcal{A}
 - dynamics
 - causality
 - CCR/CAR ...

pAQFT framework

2-step approach

1. construction of an **algebra of observables** \mathcal{A}
 - dynamics
 - causality
 - CCR/CAR ...
2. **state**: positive, normalized, linear functional $\omega : \mathcal{A} \rightarrow \mathbb{C}$
→ Expectations

Quantization? → Deformation of the algebraic structure

Functional-based approach

\mathcal{A} : functionals on the space of field configurations $\mathcal{E}(\mathbb{R}^2)$

$$\Phi_f(\varphi) = \int_{\mathbb{R}^2} f(t, x)\varphi(t, x) dt dx, \quad \forall f \in \mathcal{D}(\mathbb{R}^2), \varphi \in \mathcal{E}(\mathbb{R}^2)$$

$$\Phi_f^2(\varphi) = \int_{\mathbb{R}^2} f(t, x)\varphi^2(t, x) dt dx$$

The theory goes **quantum** by switching to deformed products

$$F \star_{\hbar K} G := \mathcal{M} \circ e^{D_{\hbar K}}(F \otimes G)$$

$$D_{\hbar K} := \left\langle \hbar K, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle = \int_{\mathbb{R}^2} dx dy \hbar K(x, y) \frac{\delta}{\delta\varphi(x)} \otimes \frac{\delta}{\delta\varphi(y)}$$

Functional-based approach

Examples: We choose K to be

- $\Delta = \Delta_R - \Delta_A$ causal propagator \rightarrow CCR
- $\omega = \frac{i}{2}\Delta + H$ Hadamard parametrix \rightarrow Wick-ordered observables
- Δ_F Feynman propagator \rightarrow time ordering

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Issues:


1. Singular structure of the kernel = clash with the one of functional derivatives of the observables
2. $e^{D_{\hbar}K}$ yields a formal power series in \hbar

Interacting theory 

- S -matrix

$$S(\lambda V) := \exp_{\star_{\hbar\Delta_F}} \left(\frac{i}{\hbar} \lambda V \right) := \sum_{n \geq 0} \frac{1}{n!} \left(\frac{i}{\hbar} \lambda \right) \underbrace{V \star_{\hbar\Delta_F} \dots \star_{\hbar\Delta_F} V}_n$$

$$V_g := \frac{V_{a,g} + V_{-a,g}}{2}, \quad V_{a,f}(\varphi) := \int_{\mathbb{R}^2} dx f(x) e^{ia\varphi(x)}$$

Interacting theory 

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- Bogoliubov map**: interacting version of $F \in \mathcal{F}((\mathbb{R}^2)^{\otimes m})$

$$R_{\lambda V}(F) = \sum_{n \geq 0} \frac{\lambda^n}{n!} R_{n,m}(V^{\otimes n}, F),$$

where $R_{n,m}(V^{\otimes n}, F)$ involves Δ_F and Δ_{AF}

Algebraic/microlocal approach to SPDEs

Idea: singular SPDEs as nonlinear QFTs

state ω \longleftrightarrow Covariance of the random field

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$$\Delta u = \lambda u^3 + \xi$$

ξ spacetime white noise: a Gaussian **random distribution** with

$$\mathbb{E}[\xi(f)] = 0, \quad \mathbb{E}[\xi(f)\xi(h)] = \langle f, h \rangle_{L^2}, \quad f, h \in \mathcal{D}(\mathbb{T}^d)$$

\longrightarrow perturbation of the linear equation

$$\Delta u_0 = \xi$$

Algebraic/microlocal approach to SPDEs

$G \in \mathcal{D}'(\mathbb{T}^d \times \mathbb{T}^d)$ fundamental solution

$$u_0 = G * \xi$$

$$\mathbb{E}[u_0(f)u_0(g)] = \int_{(\mathbb{T}^d)^2} \left(\int_{\mathbb{T}^d} G(\bar{z}, z) G(\bar{z}, z') d\bar{z} \right) f(z)g(z') dzdz' = Q(f, g)$$

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Perturbative study of the solution

$$u[\lambda] = \sum_{n=0}^{\infty} \lambda^n u_n$$

$$u_0 = G * \xi = \text{!}$$

$$u_1 = G * (G * \xi)^3 = \text{Y}$$

$$u_n = \sum_{k_1+k_2+k_3=n-1} G * (u_{k_1} u_{k_2} u_{k_3})$$

Algebra of observables^{1,2}

Inspired by [Algebraic Quantum Field Theory](#)

1. Promote the random field $u_0 = G * \xi$ to a functional-valued distribution Φ defined via

$$\Phi_f(\varphi) = \int_{\mathbb{R}^2} dt dx f(t, x) \varphi(t, x), \quad \forall f \in \mathcal{D}(\mathbb{T}^d), \varphi \in \mathcal{E}(\mathbb{T}^d)$$

2. [algebra of observables](#): polynomial (multi)local functionals + condition on the **wavefront set**

¹C. Dappiaggi, N. Drago, P. Rinaldi, L. Zambotti, CCM (2022).

²A. B., C. Dappiaggi, P. Rinaldi, AHP (2023)

Deformation of the algebra structure

Functionals don't know about the stochastic nature of the problem
⇒ **deformation** of the tensor product

$$D_Q(F)(f; \varphi) := \left\langle Q, \frac{\delta}{\delta \bar{\varphi}} F(f; \varphi) \right\rangle \quad F \in \mathcal{P}_{Loc}$$

$$\Gamma_Q(F) = e^{D_Q(F)}$$

$$\Gamma_Q(F_1 \cdot F_2) = \Gamma_Q(F_1) \star_Q \Gamma_Q(F_2) \quad \text{Algebra endomorphism}$$

Remark

These are formal expressions. One should either regularize G or perform a renormalization procedure (Epstein-Glaser)

Expectation values

The deformation map allows to **compute expectation values** of polynomial expressions in the random distribution u_0 :

$$\mathbb{E}[P(u_0)(f)] = \Gamma_Q(P(\Phi))(f; 0)$$

Example

$$\Gamma_Q(\Phi^2)(f, 0) = Q(f\delta_{Diag_2}) \equiv \mathbb{E}[u_0^2(f)]$$

$$\Gamma_Q(\Phi^2) = \text{V} + 0$$

Renormalization

$Q \sim G^2$ ill-defined product of distributions

Theorem

The Fourier transform of a compactly supported smooth function is rapidly decreasing.

Wavefront set: singular points as well as singular directions

→ products of distributions, composition...

Scaling degree: local behaviour

→ extension over the singular support (renormalization)

Algebras of functional-valued distributions

- **Microcausal functionals**: condition on the WF
→ Analogy with the problem of **Wick renormalization**

$$\mathcal{D}'_C(M^k; Pol) := \left\{ F \in \mathcal{D}'(M^k; Pol) \mid WF(F^{(n)}) \subseteq C_{k+n} \forall n \geq 0 \right\}$$

C_n sort of Cartesian product of diagonals

$$\mathcal{A}_0 := \mathcal{E}\{\mathbf{1}, \Phi\}, \quad \mathcal{A}_j := \mathcal{E}\{\mathcal{A}_{j-1} \cup G * \mathcal{A}_{j-1}\} \quad \mathcal{A} := \varinjlim \mathcal{A}_j$$

Algebraic structure: **pointwise product** $[\tau_1 \tau_2](f; \eta) := \tau_1 \otimes \tau_2(f \delta_{Diag_2}; \eta)$

Algebra of functional-valued distributions

\mathcal{A} is a positively bi-graded algebra over the ring $\mathcal{E}(M)$

$$\mathcal{A} = \bigoplus_{l,k \in \mathbb{N}_0} \mathcal{M}_{l,k} \quad \text{Ex. : } G * (\Phi^2 G * \Phi^3) \in \mathcal{M}_{2,5}$$

Then we construct a map $\Gamma_Q : \mathcal{A} \rightarrow \mathcal{D}'(M; Pol)$ with the following properties

- $\Gamma_Q(\tau) = \tau \quad \forall \tau \in \mathcal{M}_1$
- $\Gamma_Q(P * \tau) = P * \Gamma_Q(\tau) \quad \forall \tau \in \mathcal{A}$
- $\Gamma_Q \circ \delta_\psi = \delta_\psi \circ \Gamma_Q, \quad \Gamma_Q(\psi\tau) = \psi\Gamma_Q(\tau) \quad \forall \psi \in \mathcal{E}(M)$
- bound on the scaling degree of $\Gamma_Q(\tau)$ w.r.t. the one of τ

Inductive construction of Γ_Q

$\mathcal{M}_j^l := \mathcal{M}_j \cap \mathcal{A}^l \Rightarrow$ double induction on j and l to cover all of \mathcal{A}

- at every step we **renormalize** by choosing an extension

Remark: Renormalization is performed at the level of the deformation

Theorem

Let $\Gamma_Q : \mathcal{A} \rightarrow \mathcal{D}'_C(M; Pol)$ be the deformation map constructed. Then $(\mathcal{A}_{\cdot_Q} := \Gamma_Q(\mathcal{A}), \cdot_Q)$ is a unital, commutative and associative algebra w.r.t. the product

$$\tau_1 \cdot_Q \tau_2 := \Gamma_Q[\Gamma_Q^{-1} \tau_1 \Gamma_Q^{-1} \tau_2]$$

Back to stochastic sine-Gordon

Recent results³ on convergence in the AQFT framework
⇒ we adapted them to our setting

$$(\square + m^2)u + \lambda g a \sin(au) = \chi \xi$$

- Algebra of functionals $(\mathcal{F}^V \subset \mathcal{F}_{\mu c, \star \hbar K, *}) \supset$ exponentials of the field

Standard approach: classical Möller map + action of Γ_Q to get expectation values

³D. Bahns, N. Pinamonti, K. Rejzner, JMAA (2021)

Strategy⁴ 

→ Cannot address the question on convergence of the perturbative series defining the expectation values.

We divide our approach in two steps:

1. **stochastic information within the quantum theory**
2. recovering expectations via **classical limit**

$$\Gamma_Q[r_\lambda V_g(\varphi)]|_{\varphi=0} = \lim_{\hbar \rightarrow 0^+} \Gamma_Q[R_\lambda V_g(\varphi)]|_{\varphi=0}$$

⁴A. B., C. Dappiaggi, P. Rinaldi, Arxiv (2023)


Interplay between quantum and stochastic 

$$F \in \mathcal{F}_{loc}(\mathbb{R}^2) \rightarrow R_{\lambda V_g}(F) = S(\lambda V_g)^{\star \hbar \omega^{-1}} \star_{\hbar \omega} (S(\lambda V_g) \star_{\hbar \Delta_F} F)$$

$$\Gamma_Q[R_{\lambda V_g}(F)] = \Gamma_Q[S(\lambda V_g)^{\star \hbar \omega^{-1}}] \star_{\hbar \omega + Q} [\Gamma_Q[S(\lambda V_g)] \star_{\hbar \Delta_F + Q} \Gamma_Q[F]]$$

We introduce the so called **$Q - S$ matrix**


$$\Gamma_Q[S(\lambda V_g)] = \sum_{n \geq 0} \frac{1}{n!} \left(\frac{i\lambda}{2\hbar} \right)^n \sum_{k=0}^n \binom{n}{k} \mathcal{T}^{\hbar \Delta_F + Q} \left[\Gamma_Q[V_{a,g}]^{\otimes k} \otimes \Gamma_Q[V_{-a,g}]^{\otimes n-k} \right]$$

Interplay between quantum and stochastic 

$$\Gamma_Q(V_{\pm a, g}) = \dots = \int_{\mathbb{R}^2} dx g(x) e^{-\frac{a^2}{2} Q(x, x)} e^{\pm ia\varphi(x)} := V_{\pm a, g_Q}$$

$$g_Q(x) := g(x) e^{-\frac{a^2}{2} Q(x, x)} \in \mathcal{D}(\mathbb{R}^2)$$

$\Rightarrow \Gamma_Q$ simply modifies the localization

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$\Rightarrow \Gamma_Q$ simply modifies the localization

$$|[\Gamma_Q[S(\lambda V_g)]]_n| \leq \frac{1}{n!} \left(\frac{\lambda}{\hbar}\right)^n \text{ev}_0[\mathcal{T}_n^{\hbar H+Q}(V_g \otimes \dots \otimes V_g)], \quad H = \text{Re}(\Delta_F)$$

Remark | H and Q are symmetric \Rightarrow we switch to a commutative algebra

Convergence of the Q-S matrix

- **Conditioning and inverse conditioning** (Euclidean QFT⁵):
controlling the massive theory via the massless one

Remark | to obtain positivity, one has to restrict to spacetime diamonds

$$D_\mu := \{(t, x) \in \mathbb{R}^2 \mid -\mu < t-x < \mu, -\mu < t+x < \mu\} \Rightarrow \text{supp}(g) \subseteq D_\mu$$

- **Cauchy determinant**: specific form of the propagators in 1 + 1 dimensions

⁵J. Fröhlich, CMP (1976)

Convergence of the Q-S matrix

Theorem

Setting $\alpha := \frac{a^2 \hbar}{4\pi}$ and for $0 < \alpha < 1$, there exist positive constants \tilde{C} , $C_Q(\mu)$ and K such that

$$|[\Gamma_Q[S(\lambda V_g)]]_n| \leq \frac{2(2\mu)^{n\alpha} (C_Q)^{n^2}}{(n!)^{1-1/p}} \left(\frac{2\lambda e^{2^{-1}a^2K}}{\hbar} \right)^n \|g\|_{L^q} C^{n/p},$$

for $p \in [1, \alpha^{-1})$, $\frac{1}{p} + \frac{1}{q} = 1$.

As a corollary, the series $\Gamma_Q[S(\lambda V_g)](\varphi) = \sum_{n \geq 0} [\Gamma_Q[S(\lambda V_g)]]_n$ is **absolutely convergent** for all $\varphi \in \mathcal{E}(\mathbb{R}^2)$

Other convergence results

- Stochastic interacting field

$$\Gamma_Q[\Phi_{I,f}] := \Gamma_Q[R_{\lambda V_g}(\Phi_f)]$$

- n -point functions

$$\Gamma_Q[R_{\lambda V_g}(\Phi_{f_1} \dots \Phi_{f_n})]$$

In both cases we get absolute convergence of the power series in λ as a generalization of the $Q - S$ matrix case

Classical limit $\hbar \rightarrow 0^+$

We must get rid of the quantum side

For $\lim_{\hbar \rightarrow 0^+} R_{\lambda V_g}(F)$ to exist, **the argument must not contain negative powers of \hbar** .

\Rightarrow combining it with absolute convergence of the series ensures existence of the **non-perturbative momenta** of the solution

Classical limit $\hbar \rightarrow 0^+$

Via combinatorial arguments we obtain that:

- all the non-vanishing contributions to $R_{n,m}(V_g^{\otimes n}, F)$ are such that any V_g is connected with one of the entries of F .
- for any $n \geq 0$,

$$R_{n,m}(V_g^{\otimes n}, F) = \mathcal{O}(\hbar^0)$$

Non-trivial task:

$$\lim_{\hbar \rightarrow 0^+} R_{\lambda V_g}(F) \stackrel{?}{=} r_{\lambda V_g}(F)$$

→ we are studying the actual solution to the stochastic sine-Gordon equation

Hopf-algebraic renormalization in AQFT

1. Adaptation of the **Connes-Kreimer scheme** to
 - the present framework
 - AQFT

2. **Bosonization:**

massless sine-Gordon model \longleftrightarrow massive Thirring model

Question: does it survives at the stochastic level?

Convergence results for the SG model + algebraic approach to the stochastic Dirac equation⁶

- Convergence for Dirac?
- Similar results for SHE \longleftrightarrow KPZ equation

⁶A. B., B. Costeri, C. Dappiaggi, P. Rinaldi, Arxiv (2023)

If you have any questions, I would be glad to
(try to) answer them

Nonlocal algebra - correlations

$$\mathcal{T}(\mathcal{A}_{\bullet_Q}) := \mathcal{E}(M) \oplus \bigoplus_{n \geq 1} \mathcal{A}_{\bullet_Q}^{\otimes n}, \quad \mathcal{T}'_C(M; Pol) := \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathcal{D}'_C(M^n, Pol)$$

We look for a map $\Gamma_{\bullet_Q} : \mathcal{T}(\mathcal{A}_{\bullet_Q}) \rightarrow \mathcal{T}'_C(M; Pol)$ implementing the covariance of the noise

\Rightarrow we just remove the pullback on the diagonal from \cdot_Q .

Theorem

Given $\Gamma_{\bullet_Q} : \mathcal{T}(\mathcal{A}_{\bullet_Q}) \rightarrow \mathcal{T}'_C(M; Pol)$, $(\mathcal{A}_{\bullet_Q} := \Gamma_{\bullet_Q}(\mathcal{T}(\mathcal{A}_{\bullet_Q})), \cdot_Q)$ is a unital, commutative and associative algebra w.r.t. the product

$$\tau_1 \bullet_Q \tau_2 := \Gamma_{\bullet_Q}[\Gamma_{\bullet_Q}^{-1} \tau_1 \otimes \Gamma_{\bullet_Q}^{-1} \tau_2], \quad \tau_1, \tau_2 \in \mathcal{A}_{\bullet_Q}$$

Renormalized φ_n^4 equation

Applying Γ_Q to the mild equation we get

$$\Psi_{\cdot, Q} = \Phi - \lambda G * (\Psi_{\cdot, Q} \cdot_Q \Psi_{\cdot, Q} \cdot_Q \Psi_{\cdot, Q})$$

To express it in terms of pointwise products, there is a price to pay

Theorem

There exists a sequence of functional-valued linear operators $\{M_n\}_{n \in \mathbb{N}}$ such that

- $M_n(\varphi) : \mathcal{E}(M) \rightarrow \mathcal{E}(M)$ for all $\varphi \in \mathcal{E}(M)$, $\forall n \in \mathbb{N}$,
- $M_n(\varphi)$ has an even polynomial dependence on φ ,
- defining $M := \sum_{n \geq 0} \lambda^n M_n$, $\Psi_{\cdot, Q}$ solves the equation

$$\Psi_{\cdot, Q} = \Phi - \lambda G * \Psi_{\cdot, Q}^3 - G * M \Psi_{\cdot, Q}$$