Almost global existence for nonresonant Hamiltonian PDEs on compact manifolds

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June 3, 2025

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Joint work with J.Bernier,B. Grébert, R. Imekratz

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The problem

- Model equations

$$u_{tt} = \Delta u - mu + f(x,u) \ , \qquad x \in M \ , \qquad (1)$$

$$i\psi_t = -\Delta\psi + V(x)\psi + f(x,|\psi|^2)\psi \ , \qquad x \in M \ , \qquad (2)$$

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and similarly for the NLS (2).

Motivation

- Meaning of Sobolev norms:
  - an example $M = \mathbb{T}^d$

$$u = \sum_{k \in \mathbb{Z}^d} \hat{u}_k e^{ik \cdot x} \ , \quad \|u\|_{H^s}^2 \equiv \sqrt{\sum_{k \in \mathbb{Z}^d} (1 + |k|^{2s}) \, |\hat{u}_k|^2}$$

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  - since energy is conserved, growth of high Sobolev norms means that energy flows to high frequency modes, namely to small scales, it is a measure of development of turbulence.
- Numerical computations: If the solution is smooth, one can use large discretization steps.

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Main Result: the nonlinear Klein Gordon equation

Let M be a C^∞ compact Riemannian manifold without boundary. Consider the Klein Gordon Equation

$$u_{tt} - \Delta u + mu = f(x,u) , \quad x \in M$$

with $f(x,0) = 0$ and initial datum $(u_0, \dot u_0)$.

Theorem db+Bernier+Grébert+Imekratz (2025)

Let $s_0 > d/2$. For all $r \geq 1$, almost all $m > 0$, $\exists C > 0$, $\varepsilon_0 > 0$ s.t. given $s \geq Cs_0$, assume

$$\|(u_0, \dot u_0)\|_{H^s \times H^{s-1}} \leq 1 , \quad \varepsilon := \|(u_0, \dot u_0)\|_{H^{s_0} \times H^{s_0-1}} < \varepsilon_0$$

then one has

$$u(t) \in C^0((-\varepsilon^{-r}, \varepsilon^{-r}); H^s(M)) \cap C^1((-\varepsilon^{-r}, \varepsilon^{-r}); H^{s-1}(M)) .$$

Furthermore, as long as $|t| \leq \varepsilon^{-r}$, one has

$$\|(u(t), \dot u(t))\|_{H^{s_0} \times H^{s_0-1}} \lesssim \varepsilon.$$

Main Result, continued

Consider now initial data of the form $u_0 = \varepsilon v_0$, $\dot{u}_0 = \varepsilon \dot{v}_0$.

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Corollary

Assume $v_0, \dot{v}_0 \in C^\infty(M)$, then for any $\varepsilon \in (0,1)$ $\exists T_\varepsilon$, with

$$\lim_{\varepsilon \to 0} \varepsilon^r T_\varepsilon = +\infty$$

s.t. $u(.) \in C^\infty((-T_\varepsilon, T_\varepsilon); C^\infty(M))$, furthermore $\forall s \geq 1$, one has

$$\|(u(t), \dot{u}(t))\|_{H^s \times H^{s-1}} \leq C_{s,v} \|(u_0, \dot{u}_0)\|_{H^s \times H^{s-1}}$$

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- multilinear estimates on the nonlinearity.

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- Existence with loss of derivatives: NLW on tori: Bernier+Faou+Grebert (2020) tori.

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The Hamiltonian Structure

- The case of cubic NLKG (assume for simplicity $-\partial_u f(x,0)=0$):
  - Let (λ_j, e_j), be the eigenvalues-eigenvectors of the linearized problem: $-\Delta$ on M:
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  - The energy is also the Hamiltonian:

$$H = \int_M \frac{\dot u^2 + u(-\Delta + m)u}{2}dx + \frac{1}{4}\int_M u^4 dx$$
$$= \sum_j \frac{p_j^2+\omega_j^2 q_j^2}{2} + \sum_{j_1,\dots,j_4} c_{j_1,\dots,j_4} q_{j_1}\dots q_{j_4} \ , \qquad \omega_j := \sqrt{\lambda_j + m} \ .$$

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$$\dot p_j = -\frac{\partial H}{\partial q_j} \ , \quad \dot q_j = \frac{\partial H}{\partial p_j} \iff NLKG$$

  - Infinitely many Harmonic oscillators plus nonlinear perturbation.

Finite dimensional case: the problem of normal form

Consider

$$H = H_2 + H_3 + H_4 + \dots.$$

where

$$H_2 = \sum_{j=1}^{N} \frac{p_j^2 + \omega_j^2 q_j^2}{2}$$

and H_r is a homogeneous polynomial of degree r.

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$$H \circ \mathcal{T}_1 = H + \{H; g_1\} + h.o.t. = H_2 + \{H_2; g_1\} + H_3 + \mathcal{O}_4$$

Homological equation

- Find g_1 s.t. the red part vanishes. Rewrite it as

$$L_{H_2} g_1 = -H_3 \ , \quad L_{H_2} := \{H_2 ; .\} \tag{3}$$

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- Eigenvectors $z^\alpha \bar{z}^\beta$, eigenvectors $i(\alpha - \beta) \cdot \omega$.

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provided $(\alpha - \beta)\cdot\omega \neq 0$.
In the nonresonant case ($\omega \cdot k \neq 0 \ \forall k \neq 0$) one gets

$$L_{H_2} g_r + H_{r+2} = \sum_\alpha H_{\alpha,\alpha} |z|^{2\alpha} \ .$$

The normal form depends only on the actions $|z|^2$.

Weaker normal form

One can decide to "keep" more terms:

- fix a set $\mathcal{R} \subset \mathbb{N}^{2N}$ and define

$$g_r(z, \bar{z}) = \sum_{(\alpha,\beta) \notin \mathcal{R}} \frac{H_{\alpha,\beta}}{-i(\alpha - \beta) \cdot \omega} z^\alpha \bar{z}^\beta$$

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- Gain: One has to consider only small denominators

$$\omega \cdot (\alpha - \beta) \ , \quad \text{with } (\alpha, \beta) \notin \mathcal{R} \ .$$

- Price: one gets a weaker "normal form":

$$\sum_{(\alpha,\beta)\in\mathcal{R}} H_{\alpha,\beta} z^\alpha \bar{z}^\beta$$

Take home message

To eliminate a monomyal

$$z^\alpha \bar{z}^\beta$$

you have to control a denominator

$$(\alpha - \beta) \cdot \omega$$

Nonresonance condition

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then, to go to order r one considers

$$\inf_{\substack{k \neq 0 \\ |k| \leq r}} \left| \sum_{j=1}^{n} \omega_j k_j \right| = \gamma_r(n) \ .$$

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Classical nonresonant situation: assume

$$\omega \cdot k \neq 0 \ , \quad \forall k \in \mathbb{Z}^n \setminus \{0\} \ ,$$

then, to go to order r one considers

$$\inf_{\substack{k \neq 0 \\ |k| \leq r}} |\sum_{j=1}^{n} \omega_j k_j| = \gamma_r(n) \ .$$

What happens in the case of PDEs, when $n \to \infty$?

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1. Know your enemy

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The denominators go to zero as the index increases.

Tame estimate and cutoffs

- A powerfull tool: in PDEs the nonlinearity is not general, but, due to Leibnitz formula

$$\left\|u^{r+1}\right\|_{H^s} \leq C \left\|u\right\|^r_{H^{\frac{d}{2}+}} \left\|u\right\|_{H^s}\ , \quad s \geq \frac{d}{2}+ \qquad (4)$$

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- One can use this estimate to show that polynomials quadratic in high modes are small and do not count. This means terms which in the Hamiltonian are cubic.

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$$\sum_{|j|\leq N} \omega_j k_j \pm \omega_i \ , \quad i > N \ . \tag{6}$$

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In $d = 1$ the 1st and 2nd-Melnikov conditions typically hold due to

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This is also true in some particular higher dimensional situations (Zoll Manifolds and so on).

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By Weyl law the eigenvalues of the Lapalcian $\lambda_j \sim j^{2/d}$ in NLKG

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so the inequality is authomatic for $\ell > 2r^{1/\rho} N$. For smaller values apply the estimate with $N' = 2r^{1/\rho} N$;

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A very weak normal form

Lemma

For any r and any s_0 large enough there exists a canonical transformation conjugating to

$$H_2 + Z_0 + Z_2 + R \ ,$$

with Z_k homogeneous of degree k in $z^{\perp}$ and in normal form:

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Dynamical consequence:

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Dynamics of high modes

$$\dot{z}^{\perp} = A z^{\perp} + \Pi^{\perp} X_R(z)\,. \tag{10}$$

with $A := \Lambda + A_1(z^{\leq})$, skewsymmetric $A = -A^*$.

The main computation

Denote $D := \sqrt{1-\Delta}$ so that $\|z\|_{s_0}^2 = \|D^{s_0} z\|_0^2$. Denote also $\langle z; w\rangle := \sum_j \bar{z}_j w_j$
Compute

$$\frac{d}{dt}\left\|z^{\perp}\right\|_{s_0}^2 = \langle D^{s_0}\dot{z}^{\perp}; D^{s_0}z^{\perp}\rangle + \langle D^{s_0}z^{\perp}; D^{s_0}\dot{z}^{\perp}\rangle$$
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Main result: the abstract theorem

$$H = H_2 + P \ , \quad P = \mathcal{O}(|z|^3) \ , \quad H_2 = \sum_{j \geq 1} \omega_j |z_j|^2 \ .$$

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Assumptions

- Frequencies:
  - Weyl law: $\exists \beta > 0$: $\# \{ j \ : \ \omega_j < \lambda \} \sim \lambda^{\beta}$
  - Clustering (corollary of Weyl law) Exists a sequence of disjont orderd segments $[a_n, b_n]$ with $a_n \sim n$, $b_n \sim n$ s.t.

$$\bigcup_j \left\{ \omega_j^{1/\alpha} \right\} \subset \bigcup_n [a_n, b_n] \ ,$$

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  - Weyl law: $\exists \beta > 0$: $\# \{j \ : \ \omega_j < \lambda\} \sim \lambda^\beta$
  - Clustering (corollary of Weyl law) Exists a sequence of disjont orderd segments $[a_n, b_n]$ with $a_n \sim n$, $b_n \sim n$ s.t.

$$\bigcup_j \left\{ \omega_j^{1/\alpha} \right\} \subset \bigcup_n [a_n, b_n] \ ,$$

Define $\mathcal{C}_n := \left\{ j \ : \ \omega_j^{1/\alpha} \in [a_n, b_n] \right\}$

Main result: the abstract theorem

$$H = H_2 + P \ , \quad P = \mathcal{O}(|z|^3) \ , \quad H_2 = \sum_{j\geq 1} \omega_j |z_j|^2 \ .$$

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  - Nonresonance condition:

for all $r \geq 1$, there exists $\tau > 0$ such that $\forall j \in \mathbb{N}^r$, $\forall \sigma \in \{-1, 1\}^r$

$$\text{if } \exists k \in \mathbb{N}, \quad \sum_{i \ s.t. \ j_i \in \mathcal{C}_k} \sigma_i \neq 0 \quad \text{then} \quad |\sigma_1 \omega_{j_1} + \cdots + \sigma_q \omega_{j_q}| \geq \frac{\gamma}{|\max j_i|^\tau},$$

Assumptions on P

- Tame estimate: Assume that the vector field of P is smooth and tame: there exists s_0 and $\mathcal{U} \subset H^{s_0}$ bounded, s.t. $\forall s$ large enough $X_P \in C^\infty(\mathcal{U} \cap H^s; H^s)$ and

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- Multilinear estimate (following Delort-Szeftel). Denote

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We assume that forall choices of $z^{(k)}$ s.t. $z^{(k)} = \Pi_k z^{(k)}$, one has

$$\left|\mathrm{d}^q P(0)(z^{(1)}, \cdots, z^{(q)})\right| \lesssim \frac{(k_3^\star)^{\nu+n}(k_4^\star)^\nu \cdots (k_q^\star)^\nu}{(k_1^\star - k_2^\star + k_3^\star)^n} \prod_{\ell=1}^{q} \|z^{(\ell)}\|_{\ell^2}.$$

where $k_j^\star$ is the decreasing reordering of k_j

Abstract Theorem

Theorem db+Bernier+Grebert+Imekratz (2025)

For all $r \geq 1$, $s \gg 1$, $\exists \varepsilon_0$ s.t. $\forall z^{(0)} \in H^s$ with $\varepsilon := \|z^{(0)}\|_{H^s} \leq \varepsilon_0$, there exists a unique solution

$$z \in C^0\big((-\varepsilon^{-r}, \varepsilon^{-r}); H^s\big) \cap C^1\big((-\varepsilon^{-r}, \varepsilon^{-r}); H^{s-\frac{1}{\beta}}\big) \ .$$

Furthermore, as long as $|t| \leq \varepsilon^{-r}$, one has $\|u(t)\|_{H^{s_0}} \lesssim \varepsilon$.

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Other examples: stability H^s of the ground state of NLS in arbitray manifolds, Klein Gordon type equation on $\mathbb{R}^d$ with a quadratic potential.

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  - For the smoothness of solutions there are compatibility conditions which are different in the linear and in the nonlinear case: It is difficult to consider the nonlinear case as a perturbation of the linear one.

THANKS

THANK YOU