Yamaguti algebras and noncrossing partitions

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arXiv:2510.03148 (joint with F. Chapoton)

Nancy, December 3, 2025

Plan of the talk

1. Nonassociative algebras in differential geometry
2. Lie-Yamaguti algebras and Yamaguti algebras
3. An operad of noncrossing partitions
4. The main theorem

Connections: torsion and curvature

M a manifold, ∇ an affine connection (for the tangent bundle).
Two classical objects: tensors of torsion and curvature:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

These satisfy various identities, e.g. the skew-symmetry identities $T(X,Y) = -T(Y,X)$, $R(X,Y,Z) = -R(Y,X,Z)$, the Ricci identity

$$R(X,Y,Z) - \nabla_{T(X,Y)} Z = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z - \nabla_Y \nabla_X Z + \nabla_{\nabla_Y X} Z,$$

and the Bianchi identities:

$$\sum_{\text{cyclic perm.} X,Y,Z} (T(T(X,Y),Z) + \nabla_X T(Y,Z) - R(X,Y,Z)) = 0,$$

$$\sum_{\text{cyclic perm.} X,Y,Z} ((\nabla_X R)(Y,Z,W) - R(X,T(Y,Z),W)) = 0,$$

Nomizu work on affine connections

Nomizu (1954): four important particular cases of affine connections:

- $R = T = 0$: M locally looks like an abelian Lie group,
- $\nabla T = 0$, $R = 0$: M locally looks like a Lie group,
- $T = 0$, $\nabla R = 0$: M locally looks like a symmetric homogeneous space (think of G/H, where H is the fixed subgroup of an involution of G)
- $\nabla T = 0$, $\nabla R = 0$: M locally looks like a reductive homogeneous space (think of G/H, where $Lie(H)$ acts semisimply on $Lie(G)$)

Connections and algebras

We can consider the tangent bundle with the operations T and R only, or with operations ∇ and T (the operation R is then a "secondary operation" expressed via ∇ and T via the Ricci identity). The first choice leads to the "tensorial algebra", while the second one gives the "connection algebra".

For $R = T = 0$, the tensorial algebra structure is trivial, and the connection algebra is a pre-Lie algebra: for $X \circ Y := \nabla_Y X$, we have

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{\nabla_X Y - \nabla_Y X} Z \Leftrightarrow (Z \circ Y) \circ X - (Z \circ X) \circ Y = Z \circ (Y \circ X - X \circ Y).$$

Of course, combinatorics of free pre-Lie algebras is by now well known (Ermolaev 1994, Chapoton–Livernet 2000).

Connections and algebras

For $\nabla T = 0$, $R = 0$, the tensorial algebra structure is a Lie algebra, and the connection algebra is a post-Lie algebra (Vallette 2004): for $X \circ Y := \nabla_Y X$, $\{X, Y\} := T(X, Y)$ we have

$$\{X, \{Y, Z\}\} + \{Y, \{Z, X\}\} + \{Z, \{X, Y\}\} = 0,$$
$$\{X, Y\} \circ Z = \{X \circ Z, Y\} + \{X, Y \circ Z\},$$
$$X \circ \{Y, Z\} = (X \circ Y) \circ Z - (X \circ Z) \circ Y - X \circ (Y \circ Z) + X \circ (Z \circ Y).$$

A description of free post-Lie algebras follows from work of Vallette, who proved that the operad of post-Lie algebras looks like $\mathrm{Lie} \circ \mathrm{Mag}$.

Connections and algebras

For $T = 0$, $\nabla R = 0$, the tensorial algebra structure is a Lie triple system (Jacobson 1949): if $[X, Y, Z] := R(X, Y, Z)$, then

$$[X, Y, Z] = -[Y, X, Z],$$
$$[X, Y, Z] + [Y, Z, X] + [Z, X, Y] = 0,$$
$$[U, V, [X, Y, Z]] = [[U, V, X], Y, Z] + [X, [U, V, Y], Z] + [X, Y, [U, V, Z]],$$

and the connection algebra is a "Lie-admissible triple algebra" (Munthe-Kaas and Stava 2023):

$$[Z, Y, X] = (X \circ Y) \circ Z - (X \circ Z) \circ Y - X \circ (Y \circ Z) + X \circ (Z \circ Y),$$
$$[X, Y, Z] \circ W = [X \circ W, Y, Z] + [X, Y \circ W, Z] + [X, Y, Z \circ W].$$

Free Lie triple systems and free Lie-admissible triple algebras are described by Munthe-Kaas and Stava; in fact, the description of free Lie triple systems follows from my work with Markl and Remm (2017) where we had shown that the operad of Lie triple systems is the Veronese square of the Lie operad. For Lie-admissible triple algebras, the operad looks like $LTS \circ \mathrm{Mag}$, mimicking the description of the post-Lie operad of Vallette.

Connections and algebras

For $\nabla T = 0$, $\nabla R = 0$, the tensorial algebra structure is a "Lie-Yamaguti algebra": for $\{X, Y\} := T(X, Y)$, $[X, Y, Z] := R(X, Y, Z)$, we have

$$\{X, Y\} = -\{Y, X\},$$
$$[X, Y, Z] = -[Y, X, Z],$$
$$\{\{X, Y\}, Z\} + \{\{Y, Z\}, X\} + \{\{Z, X\}, Y\} - [X, Y, Z] - [Y, Z, X] - [Z, X, Y] = 0,$$
$$[\{X, Y\}, Z, W] + [\{Y, Z\}, X, W] + [\{Z, X\}, Y, W] = 0,$$
$$[X, Y, \{Z, W\}] = \{[X, Y, Z], W\} + \{Z, [X, Y, W]\},$$
$$[U, V, [X, Y, Z]] = [[U, V, X], Y, Z] + [X, [U, V, Y], Z] + [X, Y, [U, V, Z]].$$

The connection algebra was considered by Munthe-Kaas and Stava (2023) under the name "post-Lie-Yamaguti algebra", but will not be discussed today.

Lie-Yamaguti algebras

Lie-Yamaguti algebras are already present in the work of Nomizu (1954), axiomatized by Yamaguti (1957) as "general Lie triple systems", renamed into "Lie triple algebras" by Kikkawa (1975), and finally got their present name from Kinyon and Weinstein (2001) who studied them in relation with Leibniz algebras and Courant algebroids.

Free algebras are not well understood, for example, the operation $\{-,-\}$ satisfies a nontrivial identity of degree 8 (and not less; Bremner 2013):

$$\begin{aligned}\operatorname*{Alt}_{S_8}\Big(&\{\{\{\{a,b\},c\},\{d,e\}\},\{\{f,g\},h\}\}-\frac{3}{2}\{\{\{\{a,b\},c\},\{\{d,e\},f\}\},\{g,h\}\}\\&-\{\{\{\{\{a,b\},c\},d\},\{e,f\}\},\{g,h\}\}+\{\{\{\{\{a,b\},c\},\{d,e\}\},f\},\{g,h\}\}\\&+2\{\{\{\{\{a,b\},c\},d\},\{\{e,f\},g\},h\}+3\{\{\{\{\{a,b\},c\},\{\{d,e\},f\}\},g\},h\}\\&+2\{\{\{\{\{\{a,b\},c\},d\},\{e,f\}\},g\},h\}-2\{\{\{\{\{\{a,b\},c\},\{d,e\}\},f\},g\},h\}\Big)=0,\end{aligned}$$

Stava (2024) claimed to have described free Lie-Yamaguti algebras, but in his description the binary operation satisfies no identities (besides being skew-symmetric)...

Yamaguti algebras

In September 2025, Das in `arXiv:2509.03648` introduced a structure he called "(associative-)Yamaguti algebras": they have a binary operation $a \cdot b$ and two ternary operations $\{a, b, c\}$ and $\{\!\{a, b, c\}\!\}$ satisfying

$$(a \cdot b) \cdot c - a \cdot (b \cdot c) + \{a, b, c\} - \{\!\{a, b, c\}\!\} = 0,$$

$$\{a \cdot b, c, d\} = \{a, b \cdot c, d\}, \quad \{a, b, c \cdot d\} = \{a, b, c\} \cdot d,$$

$$\{\!\{a \cdot b, c, d\}\!\} = a \cdot \{\!\{b, c, d\}\!\}, \quad \{\!\{a, b \cdot c, d\}\!\} = \{\!\{a, b, c \cdot d\}\!\},$$

$$a \cdot \{b, c, d\} = \{\!\{a, b, c\}\!\} \cdot d,$$

$$\{\{a, b, c\}, d, e\} = \{a, \{\!\{b, c, d\}\!\}, e\} = \{a, b, \{c, d, e\}\},$$

$$\{a, \{b, c, d\}, e\} = \{\{\!\{a, b, c\}\!\}, d, e\},$$

$$\{\!\{\{\!\{a, b, c\}\!\}, d, e\}\!\} = \{\!\{a, \{b, c, d\}, e\}\!\} = \{\!\{a, b, \{\!\{c, d, e\}\!\}\}\!\},$$

$$\{\!\{a, \{\!\{b, c, d\}\!\}, e\}\!\} = \{\!\{a, b, \{c, d, e\}\}\!\},$$

$$\{a, b, \{\!\{c, d, e\}\!\}\} = \{\!\{\{a, b, c\}, d, e\}\!\}.$$

Note that the first relation is not set-theoretic, so it is not a linearization of a set operad.

Yamaguti algebras

If $(A, - \cdot -, \{-,-,-\}, \{\!\{-,-,-\}\!\})$ is a Yamaguti algebra, then

$$\{a,b\} = a \cdot b - b \cdot a,$$
$$[a,b,c] = \{a,b,c\} - \{b,a,c\} - \{\!\{c,a,b\}\!\} + \{\!\{c,b,a\}\!\}$$

defines a Lie-Yamaguti algebra on A. Therefore, there is hope to view these Yamaguti algebras as more tractable envelopes of Lie-Yamaguti algebras.

Why more tractable? Because they are algebras over a nonsymmetric operad, which can be handled in a much more efficient way.

Yamaguti algebras

We have

$$\{\!\{a,b,c\}\!\} = (a\cdot b)\cdot c - a\cdot(b\cdot c) + \{a,b,c\},$$

so the generator $\{\!\{-,-,-\}\!\}$ is redundant. A minimal set of relations between the minimal set of generators $-\cdot-$ and $\{-,-,-\}$:

$$\{a,b\cdot c,d\} = \{a\cdot b,c,d\},$$

$$\{a,b,c\cdot d\} = \{a,b,c\}\cdot d,$$

$$\{a,b,\{c,d,e\}\} = \{\{a,b,c\},d,e\},$$

$$(a\cdot(b\cdot c))\cdot d = ((a\cdot b)\cdot c)\cdot d - a\cdot\{b,c,d\} + \{a,b,c\}\cdot d,$$

$$\{a\cdot(b\cdot c),d,e\} = \{(a\cdot b)\cdot c,d,e\} - \{a,\{b,c,d\},e\} + \{\{a,b,c\},d,e\},$$

and

$$a\cdot(b\cdot(c\cdot d)) = a\cdot((b\cdot c)\cdot d) + (a\cdot b)\cdot(c\cdot d) - ((a\cdot b)\cdot c)\cdot d + a\cdot\{b,c,d\} - \{a\cdot b,c,d\}.$$

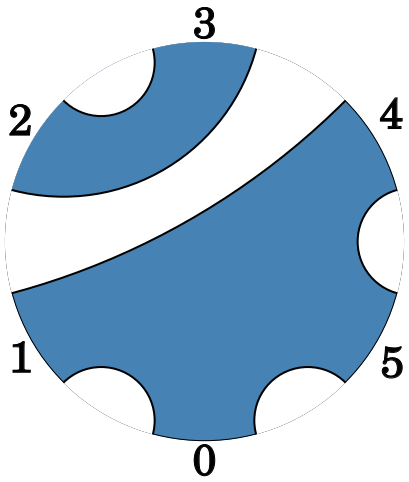
These relations form a Gröbner basis (for "weighted `pathrevlex`").

Noncrossing partitions appear

Using this Gröbner basis, it is easy to determine the dimensions of components of the Yamaguti operad:

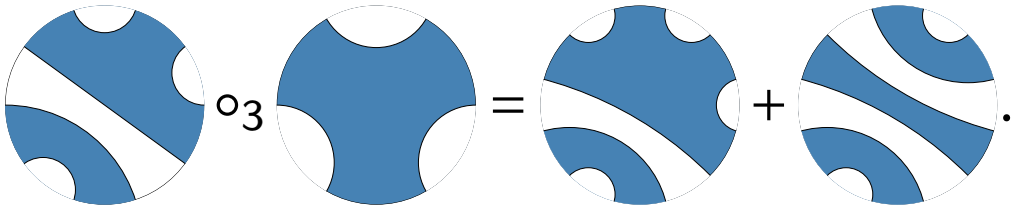
$$1, 1, 3, 6, 15, 36, 91, 232, 603, 1585, \ldots$$

These are the "Riordan numbers" (A005043) that count noncrossing partitions of $\{0, 1, \ldots, n\}$ without singleton blocks.



An operad structure, informally

Merge-or-cut:



Note: this is not a set-operad!

An operad structure, formally

We denote by $B(n)$ the set of noncrossing partitions of $\{0,1,\ldots,n\}$, and by $\mathscr{B}(n)$ the vector space with basis $B(n)$. The composition maps

$$\circ_i : \mathscr{B}(m) \otimes \mathscr{B}(n) \to \mathscr{B}(m+n-1), \quad 1 \leq i \leq m$$

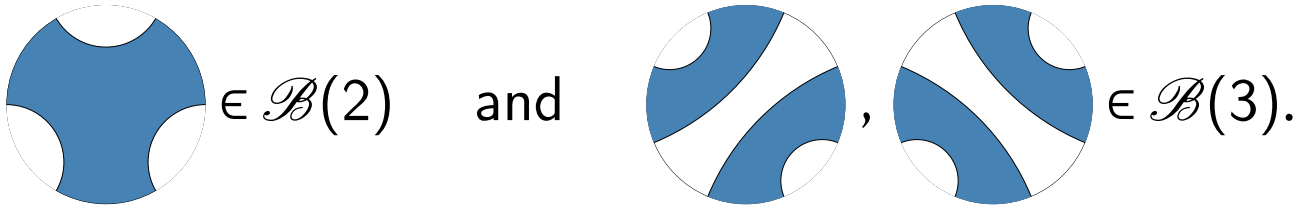
are defined on the basis as follows. Let $\pi \in B(m)$, $\nu \in B(n)$. Then $\pi \circ_i \nu$ is the sum of two terms:

- the noncrossing partition obtained by juxtaposition of the two disks and identification of the segment i of π with the segment 0 of ν,
- the noncrossing partition obtained from the previous one by cutting the new block along the gluing line made from the identified segments, thereby creating two blocks.

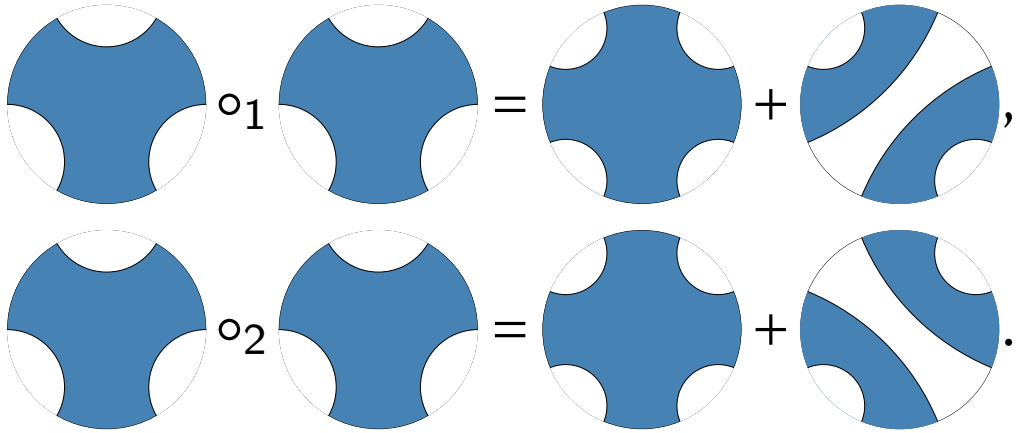
Note: it can happen that the second case creates a singleton block, in which case this term is omitted. This happens exactly when either the block in ν containing 0 has 2 elements or the block in π containing i has 2 elements.

An operad structure, continued

With some care, one checks that this is a nonsymmetric operad. (In fact, a nonsymmetric cyclic operad.) Moreover, this operad is generated by the elements

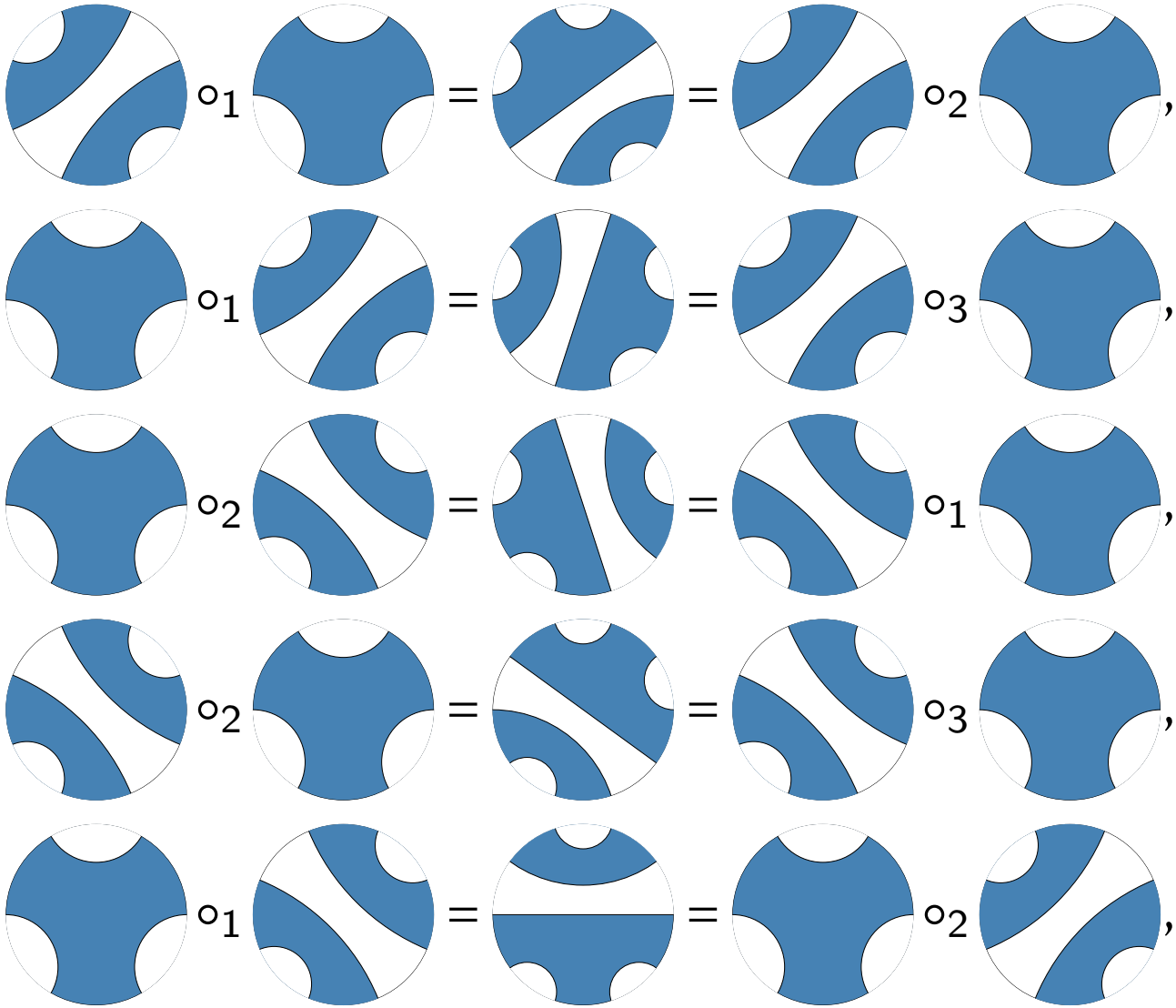


Furthermore, the following leads to the "hardest" Yamaguti identity:



An operad structure, continued

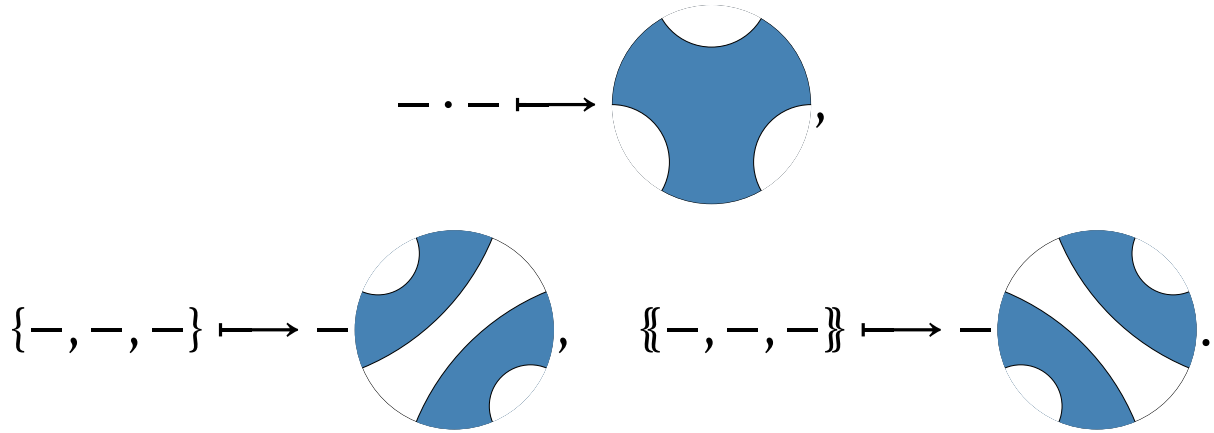
The other Yamaguti identities also hold, for example:



The main theorem

Theorem. The thus defined nonsymmetric operad $\mathscr{B}$ of noncrossing partitions without singleton blocks is isomorphic to the Yamaguti operad.

Proof: we have a well defined map from the Yamaguti operad to $\mathscr{B}$:



It is surjective, and the dimensions of components of the two operads are the same, hence this map is an isomorphism.

Corollary. The free Yamaguti algebra on k generators has a combinatorial basis of noncrossing partitions whose segments except for the "zero" segment at the bottom are labelled by elements of $\{1, \ldots, k\}$, with a simple "combinatorial" rule for computing operations.

Some questions

Question: our computations show that an analogue of the result of Bremner holds: the binary operation of the Yamaguti operad satisfies nontrivial identities. The lowest one is

$$a(b((cd)e)) + (a((bc)d))e + ((ab)c)(de) = a((b(cd))e) + (ab)(c(de)) + ((a(bc))d)e.$$

Can one describe all such identities? (There is one new identity of degree six, one more of degree seven, and one more of degree nine.)

Question: is the Yamaguti operad with its original quadratic-linear presentation inhomogeneous Koszul?

Question: do Lie-Yamaguti algebras embed into their universal Yamaguti envelopes? Is there a PBW theorem for universal Yamaguti envelopes of Lie-Yamaguti algebras?

A small puzzle

It is known that the Riordan numbers that we encountered satisfy the recurrence

$$(n+2)r_n = 2nr_{n-1} + 3nr_{n-2}, \quad r_0 = 0, r_1 = 1.$$

Poincaré: keep only the leading terms of the polynomial coefficients:

$$r'_n = 2r'_{n-1} + 3r'_{n-2}.$$

The vector space of "approximate solutions" $\{r'_n\}$ has a basis $\{3^n\}$, $\{(-1)^n\}$, so we expect a generic $\{r_n\}$ to resemble the first one, and a one-dimensional vector space of exceptional slowly growing $\{r_n\}$ that resemble the second one.

Conjecture: exceptional sequences $\{m_n\}$ are proportional to the one with

$$m_0 = 1, \quad m_1 = -\frac{9\sqrt{3}}{4\pi + 6\sqrt{3}}$$

(guessed using the "Inverse Equation Solver" of Robert Munafo, `https://mrob.com/pub/ries/index.html`).