Yamaguti algebras and noncrossing partitions

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Plan of the talk

Connections: torsion and curvature

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and the Bianchi identities:

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- $\nabla T = 0$, $\nabla R = 0$: M locally looks like a reductive homogeneous space (think of G/H, where $Lie(H)$ acts semisimply on $Lie(G)$)

Connections and algebras

We can consider the tangent bundle with the operations T and R only, or with operations ∇ and T (the operation R is then a "secondary operation" expressed via ∇ and T via the Ricci identity). The first choice leads to the "tensorial algebra", while the second one gives the "connection algebra".

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For $R = T = 0$, the tensorial algebra structure is trivial, and the connection algebra is a pre-Lie algebra: for $X \circ Y := \nabla_Y X$, we have

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{\nabla_X Y - \nabla_Y X} Z \Leftrightarrow (Z \circ Y) \circ X - (Z \circ X) \circ Y = Z \circ (Y \circ X - X \circ Y).$$

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Of course, combinatorics of free pre-Lie algebras is by now well known (Ermolaev 1994, Chapoton–Livernet 2000).

Connections and algebras

For $\nabla T = 0$, $R = 0$, the tensorial algebra structure is a Lie algebra, and the connection algebra is a post-Lie algebra (Vallette 2004): for $X \circ Y := \nabla_Y X$, $\{X, Y\} := T(X, Y)$ we have

$$\{X,\{Y,Z\}\}+\{Y,\{Z,X\}\}+\{Z,\{X,Y\}\}=0,$$
$$\{X,Y\}\circ Z=\{X\circ Z,Y\}+\{X,Y\circ Z\},$$
$$X\circ\{Y,Z\}=(X\circ Y)\circ Z-(X\circ Z)\circ Y-X\circ(Y\circ Z)+X\circ(Z\circ Y).$$

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$$X \circ \{Y,Z\} = (X \circ Y) \circ Z - (X \circ Z) \circ Y - X \circ (Y \circ Z) + X \circ (Z \circ Y).$$

A description of free post-Lie algebras follows from work of Vallette, who proved that the operad of post-Lie algebras looks like $\mathrm{Lie} \circ \mathrm{Mag}$.

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and the connection algebra is a "Lie-admissible triple algebra" (Munthe-Kaas and Stava 2023):

$$[Z,Y,X] = (X \circ Y) \circ Z - (X \circ Z) \circ Y - X \circ (Y \circ Z) + X \circ (Z \circ Y),$$
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Free Lie triple systems and free Lie-admissible triple algebras are described by Munthe-Kaas and Stava; in fact, the description of free Lie triple systems follows from my work with Markl and Remm (2017) where we had shown that the operad of Lie triple systems is the Veronese square of the Lie operad. For Lie-admissible triple algebras, the operad looks like $LTS \circ \mathrm{Mag}$, mimicking the description of the post-Lie operad of Vallette.

Connections and algebras

For $\nabla T = 0$, $\nabla R = 0$, the tensorial algebra structure is a "Lie-Yamaguti algebra": for $\{X, Y\} := T(X, Y)$, $[X, Y, Z] := R(X, Y, Z)$, we have

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$$\{\{X, Y\}, Z\} + \{\{Y, Z\}, X\} + \{\{Z, X\}, T\} - [X, Y, Z] - [Y, Z, X] - [Z, X, Y] = 0,$$
$$[\{X, Y\}, Z, W + [\{Y, Z\}, X, W] + [\{Z, X\}, Y, W] = 0,$$
$$[X, Y, \{Z, W\}] = \{[X, Y, Z], W\} + \{Z, [X, Y, W]\},$$
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$$\{\{X, Y\}, Z\} + \{\{Y, Z\}, X\} + \{\{Z, X\}, T\} - [X, Y, Z] - [Y, Z, X] - [Z, X, Y] = 0,$$
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The connection algebra was considered by Munthe-Kaas and Stava (2023) under the name "post-Lie-Yamaguti algebra", but will not be discussed today.

Lie-Yamaguti algebras

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Free algebras are not well understood, for example, the operation $\{-,-\}$ satisfies a nontrivial identity of degree 8 (and not less; Bremner 2013):

$$\operatorname*{Alt}_{S_8}\Big(\{\{\{\{a,b\},c\},\{d,e\}\},\{\{f,g\},h\}\}-\frac{3}{2}\{\{\{\{a,b\},c\},\{\{d,e\},f\}\},\{g,h\}\}$$
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Stava (2024) claimed to have described free Lie-Yamaguti algebras, but in his description the binary operation satisfies no identities (besides being skew-symmetric)...

Yamaguti algebras

In September 2025, Das in arXiv:2509.03648 introduced a structure he called "(associative-)Yamaguti algebras": they have a binary operation $a \cdot b$ and two ternary operations $\{a, b, c\}$ and $\{\!\{a, b, c\}\!\}$ satisfying

$$(a \cdot b) \cdot c - a \cdot (b \cdot c) + \{a, b, c\} - \{\!\{a, b, c\}\!\} = 0,$$
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Note that the first relation is not set-theoretic, so it is not a linearization of a set operad.

Yamaguti algebras

If $(A, - \cdot -, \{-,-,-\}, \{\!\{-,-,-\}\!\})$ is a Yamaguti algebra, then

$$\{a, b\} = a \cdot b - b \cdot a,$$
$$[a, b, c] = \{a, b, c\} - \{b, a, c\} - \{\!\{c, a, b\}\!\} + \{\!\{c, b, a\}\!\}$$

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Why more tractable?

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Why more tractable? Because they are algebras over a nonsymmetric operad, which can be handled in a much more efficient way.

Yamaguti algebras

We have

$$\{\!\{a, b, c\}\!\} = (a \cdot b) \cdot c - a \cdot (b \cdot c) + \{a, b, c\},$$

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$$\{a,b\cdot c,d\} = \{a\cdot b,c,d\},$$
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and

$$a\cdot(b\cdot(c\cdot d)) = a\cdot((b\cdot c)\cdot d) + (a\cdot b)\cdot(c\cdot d) - ((a\cdot b)\cdot c)\cdot d + a\cdot\{b,c,d\} - \{a\cdot b,c,d\}.$$

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These relations form a Gröbner basis (for "weighted `pathrevlex`").

Noncrossing partitions appear

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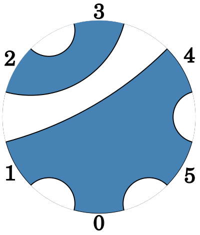
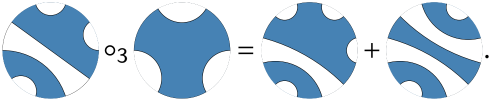


An operad structure, informally

Merge-or-cut:



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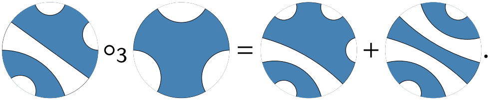

Note: this is not a set-operad!

An operad structure, formally

We denote by $B(n)$ the set of noncrossing partitions of $\{0,1,\ldots,n\}$, and by $\mathscr{B}(n)$ the vector space with basis $B(n)$.

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$$\circ_i\colon \mathscr{B}(m)\otimes\mathscr{B}(n)\to\mathscr{B}(m+n-1),\quad 1\le i\le m$$

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Note: it can happen that the second case creates a singleton block, in which case this term is omitted. This happens exactly when either the block in ν containing 0 has 2 elements or the block in π containing i has 2 elements.

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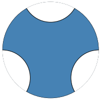
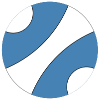
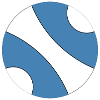
 $\in \mathscr{B}(2)$ and ,  $\in \mathscr{B}(3)$.

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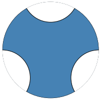
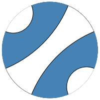
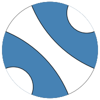
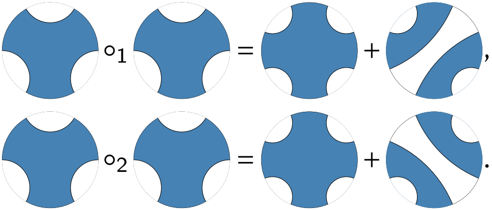
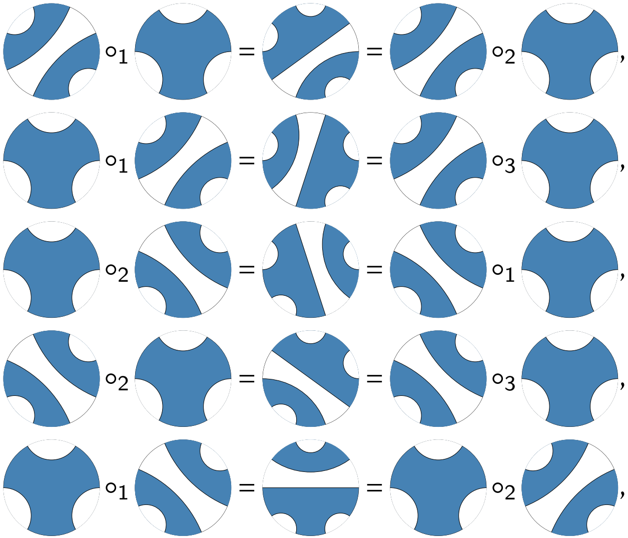
 $\in \mathscr{B}(2)$ and ,  $\in \mathscr{B}(3)$.

Furthermore, we have the "hardest" Yamaguti identity, since



An operad structure, continued

The other Yamaguti identities also hold, for example:



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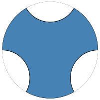
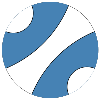
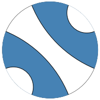
$- \cdot - \longmapsto$  ,

$\{-,-,-\} \longmapsto -$  , $\{\!\{-,-,-\}\!\} \longmapsto -$  .

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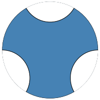
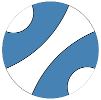
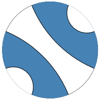
$- \cdot - \longmapsto$  ,

$\{-,-,-\} \longmapsto -$  , $\{\!\{-,-,-\}\!\} \longmapsto -$  .

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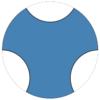
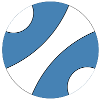
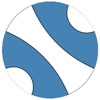
$- \cdot - \longmapsto$  ,

$\{-,-,-\} \longmapsto -$  , $\{\!\{-,-,-\}\!\} \longmapsto -$  .

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Corollary. The free Yamaguti algebra on k generators has a combinatorial basis of noncrossing partitions whose segments except for the "zero" segment at the bottom are labelled by elements of $\{1,\ldots,k\}$, with a simple "combinatorial" rule for computing operations.

Some questions

Question: our computations show that an analogue of the result of Bremner holds: the binary operation of the Yamaguti operad satisfies nontrivial identities. The lowest one is

$$a(b((cd)e)) + (a((bc)d))e + ((ab)c)(de) = \\ a((b(cd))e) + (ab)(c(de)) + ((a(bc))d)e.$$

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Question: do Lie-Yamaguti algebras embed into their universal Yamaguti envelopes? Is there a PBW theorem for universal Yamaguti envelopes of Lie-Yamaguti algebras?

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Conjecture: exceptional sequences $\{m_n\}$ are proportional to the one with

$$m_0 = 1, \quad m_1 = -\frac{9\sqrt{3}}{4\pi + 6\sqrt{3}}$$

(guessed using the "Inverse Equation Solver" of Robert Munafo, `https://mrob.com/pub/ries/index.html`).