The partition preLie algebra of graphs and Maurer-Cartan elements associated to E_n-operads

Benoit Fresse

Université de Lille

Nancy, December 3, 2025

Introduction

- Review of: GC_n, graph complex introduced by Kontsevich (in the case $n = 2$),
  defined over $\mathbb{Q}$,
  and equipped with
  - a preLie algebra structure
  - a differential $\delta_\mu(\alpha) = [\mu, \alpha]$ determined by a Maurer-Cartan element $\mu \in \mathtt{MC}(\mathsf{GC}_n)$
- Theorem (Willwacher, BF-Willwacher):

$$\mathbb{Q}^\times \ltimes \exp \mathrm{Z}^0(\mathsf{GC}_n^\mu \hat{\otimes} \Omega^*(\Delta^\bullet)) \sim \mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_n)^\wedge_{\mathbb{Q}} \tag{1}$$

  where
  - E_n = model of an E_n-operad

- Remark: In the case $n = 2$:

$$\mathbb{Q}^{\times} \ltimes \exp \mathtt{Z}^0(\mathsf{GC}_2^{\mu} \hat{\otimes} \Omega^*(\Delta^{\bullet})) \sim \mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_2)^{\wedge}_{\mathbb{Q}}, \qquad (1)$$

we have

$$\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_2)^{\wedge}_{\mathbb{Q}} \sim \mathtt{GT}(\mathbb{Q}) \ltimes (\mathtt{SO}_2)^{\wedge}_{\mathbb{Q}} \qquad \text{(BF)}$$

and (1) is related to:

$$H^0(\mathsf{GC}_2^{\mu}) \simeq \mathfrak{grt}_1 \oplus \mathbb{Q}[1] \qquad \text{(Willwacher)}$$

- Remark:

$$\mathbb{Q}^{\times} \ltimes \exp \mathrm{Z}^0(\mathsf{GC}_n^{\mu} \hat{\otimes} \Omega^*(\Delta^{\bullet})) \sim \mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_n)^{\wedge}_{\mathbb{Q}} \quad (1)$$

$$\Leftrightarrow \quad \mathtt{MC}_{\bullet}(\mathsf{GC}_n)_{\mu} \sim \mathrm{hofib}\big(\mathrm{B}\,\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_n)^{\wedge}_{\mathbb{Q}} \to \mathrm{B}\,\mathtt{Aut}_{\mathcal{O}p}\,H_*(\mathsf{E}_n,\mathbb{Q})\big) \quad (1')$$

where

- $\mathtt{MC}_{\bullet}(\mathfrak{g}) = \mathtt{MC}(\mathfrak{g}\,\hat{\otimes}\Omega^*(\Delta))$
- $\mathtt{MC}_{\bullet}(\mathfrak{g})_{\mu}$ = connected component based at $\mu \in \mathtt{MC}(\mathfrak{g})$

- Goal: EGC_n, graph complex *with coefficients in an injective resolution of the trivial representation of permutations/graph automorphisms*, defined over any ring $\mathbb{K}$, and equipped with
  - a partition preLie algebra structure
- Conjecture ($\mathbb{K} = \bar{\mathbb{F}}_\ell$ with ℓ odd):

$$\mathtt{MC}_\bullet(\mathsf{EGC}_n)_\lambda \sim \operatorname{hofib}\big(\mathtt{B}\,\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_n)^\wedge_\ell \to \mathtt{B}\,\mathtt{Aut}_{\mathcal{O}p}\,H_*(\mathsf{E}_n, \mathbb{Z}_\ell)\big) \qquad (2)$$

  where
  - $\mathtt{MC}_\bullet(\mathfrak{g}) = \mathtt{MC}(\mathfrak{g}\,\hat{\otimes} C^*(\Delta))$
  - $\lambda \in \mathtt{MC}(\mathsf{EGC}_n)$ is a well-determined Maurer-Cartan element associated to the operad E_n

- Remark: In the case $n = 2$:

$$\mathtt{MC}_\bullet(\mathsf{EGC}_2)_\lambda \sim \mathrm{hofib}\big(\mathtt{B}\,\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_2)^\wedge_\ell \to \mathtt{B}\,\mathtt{Aut}_{\mathcal{O}p}\,H_*(\mathsf{E}_2,\mathbb{Z}_\ell)\big) \tag{2}$$

we have

$$\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_2)^\wedge_\ell \sim \mathtt{GT}(\mathbb{Z}_\ell) \ltimes (\mathtt{SO}_2)^\wedge_\ell \qquad \text{(Horel)}$$

so that (2) gives:

$$\mathtt{MC}_\bullet(\mathsf{EGC}_2)_\lambda \sim \mathrm{hofib}\big(\mathtt{B}(\mathtt{GT}(\mathbb{Z}_\ell) \ltimes (\mathtt{SO}_2)^\wedge_\ell) \to \mathtt{B}(\mathbb{Z}^\wedge_\ell)\big) \qquad \text{(conjecture)}$$

Plan

1. Quick recollections on operads and on E_n-operads
2. Survey of the definition of the graph complex GC_n
3. The definition of the graph complex EGC_n
4. Results and conjectures

§0. Recollections on operads and on E_n-operads

- An operad P in a symmetric monoidal category $(\mathcal{M}, \otimes) = (\mathcal{T}op, \times), (dg\,\mathcal{M}od, \otimes), \dots$ consists of:
  - a collection of objects $\mathsf{P}(r) \in \mathcal{M}$, $r \in \mathbb{N}$, so that the elements $p \in \mathsf{P}(r)$ represent operations on r variables

$$p = p(x_1, \dots, x_r)$$

  together with:
  - an action of the symmetric groups $\Sigma_r \curvearrowright \mathsf{P}(r)$ so that

$$\sigma p = p(x_{\sigma(1)}, \dots, x_{\sigma(r)}), \quad \text{for } \sigma \in \Sigma_r,$$

  - composition products $\circ_i : \mathsf{P}(k) \otimes \mathsf{P}(l) \to \mathsf{P}(k+l-1)$, $i = 1, \dots, k$, so that

$$p \circ_i q = p(x_1, \dots, x_{i-1}, q(x_i, \dots, x_{i+l-1}), x_{i+l}, \dots, x_{k+l-1})$$

- Definition: an E_n-operad in $\mathcal{T}op$ is an operad $\mathsf{E}_n \in \mathcal{T}op$ such that:

$$\mathsf{E}_n \sim \mathsf{C}_n$$

  where $\mathsf{C}_n =$ operad of little n-cubes.

- Idea: E_n governs operations associated to multiplicative structures that are ho-commutative up to some degree, measured by n.
- Example: $\mathsf{E}_n = \mathsf{FM}_n$, where

$$\mathsf{FM}_n(r) = \text{Fulton-MacPherson compactification of } \mathsf{F}(\mathbb{R}^n, r)/\mathbb{R}_{>0} \ltimes \mathbb{R}^n$$

  with

$$\mathsf{F}(\mathbb{R}^n, r) = \{(v_1, \dots, v_r) \in (\mathbb{R}^n)^{\times r} | v_i \neq v_j \, (\forall i \neq j)\}$$

- Remark: $\mathsf{FM}_n(2) = S^{n-1}$ and $\mathsf{E}_n(2) \sim S^{n-1}$ in general.

§1. The graph complex GC_n

- Reminder: $\mathsf{FM}_n(2) = S^{n-1}$ and in general $\mathsf{E}_n(2) \sim S^{n-1}$.
- Observations:
  1. For $r \in \mathbb{N}$, we have a map:

$$\psi : \mathsf{E}_n(r) \to \bigtimes_{1 \leqslant i,j \leqslant r} S^{n-1}_{ij}$$

     such that

$$\psi(v_1, \dots, v_r) = \frac{v_i - v_j}{||v_i - v_j||} \quad (\text{in the case } \mathsf{E}_n = \mathsf{FM}_n)$$
$$\text{and} \quad \psi(p) = p(*, \dots, x_i, \dots, x_j, \cdots *) \quad (\text{in general})$$

  2. The collection $\mathsf{K}^S_n(r) = \bigtimes_{ij} S^{n-1}_{ij}$ inherits the structure of an operad and ψ defines an operad morphism.

- Construction:
  - In $(gr\,\mathrm{Com}\,\mathcal{A}lg, \otimes)$ we form:

$$\mathrm{Gra}_n^c(r) = \bigotimes_{1 \leqslant i,j \leqslant r} S(\omega_{ij})$$

    where $S(-)$ = symmetric algebra and with $\deg^*(\omega_{ij}) = n-1$, $\omega_{ji} = (-1)^n \omega_{ij}$.
  - The elements $\gamma \in \mathrm{Gra}_n^c(r)$ are represented by graphs

$$\gamma = \gamma(\circ_1, \dots, \circ_r),$$

    based at vertices $\circ_1, \dots, \circ_r$, with an edge $\circ_i - \circ_j$ for each factor ω_{ij}.
  - The collection $\mathrm{Gra}_n^c(r)$ is equipped with a cooperad structure with composition coproducts:

$$\circ_{\underline{k}}^* : \mathrm{Gra}_n^c(\underline{r}) \to \mathrm{Gra}_n^c(\underline{r}\,/\,\underline{k}) \otimes \mathrm{Gra}_n^c(\underline{k})$$

    such that

$$\circ_{\underline{k}}^* : \gamma \mapsto \gamma/(\gamma_{|\underline{k}} \equiv \circ_*) \otimes \gamma_{|\underline{k}}$$

    where $\gamma_{|\underline{k}}$ = induced subgraph based at the vertices $\circ_i$, $i \in \underline{k}$

- Recollections:
  - Com = operad of commutative algebras with $\mathrm{Com}(r) = \mathbb{K}$ $(\forall r)$, and Com^c = dual cooperad.
  - We have

$$\mathtt{B}^c(\Lambda^{-n}\,\mathrm{Com}^c) \xrightarrow{\sim} \mathrm{Lie}_n \quad \Leftrightarrow \quad \mathrm{Lie}_n^c \xrightarrow{\sim} \mathtt{B}(\Lambda^n\,\mathrm{Com}),$$

    where:
    - $\mathtt{B}(-)/\mathtt{B}^c(-)$ = operadic bar/cobar construction
    - Λ^k = operadic suspension with $\Lambda^k\,\mathrm{P}(r) = \mathrm{P}(r)^{\pm}_{*+k(r-1)}$
    - Lie_n = graded operad of Lie algebras with $\deg[-,-] = n-1$ and Lie_n^c = dual cooperad.

    so that:

$$\mathrm{holie}_n = \mathtt{B}^c(\Lambda^{-n}\,\mathrm{Com}^c)$$

    is an operad govering homotopy Lie_n-algebras.

- Recollections:
  - In general, for C a dg-cooperad and P a dg-operad, the hom-object:

$$\mathtt{Dfm}(\mathsf{C},\mathsf{P}) = \mathtt{Hom}_{\Sigma}(\mathsf{C}_{\geqslant 2},\mathsf{P}_{\geqslant 2})$$

is equipped with a preLie-algebra structure such that:

$$u\{v\}(c) = \sum_{(c)} u(c') \circ_{*} v(c''),$$

where we take the sum over all cooperadic coproduct shapes $\circ_{\underline{k}} : c \mapsto c' \otimes c''$ in C,
  - and we have:

$$\mathtt{Mor}_{dg\,\mathcal{O}p}(\mathrm{B}^{c}(\mathsf{C}),\mathsf{P}) = \mathtt{MC}(\mathtt{Dfm}(\mathsf{C},\mathsf{P})) = \mathtt{Mor}_{dg\,\mathcal{O}p^{c}}(\mathsf{C},\mathrm{B}(\mathsf{P}))$$

- Definitions:

$$\begin{aligned} \mathsf{fGC}_n &= \mathtt{Dfm}(\mathsf{Gra}_n^c, \Lambda^n \mathsf{Com}) \\ &= \widehat{\mathtt{Span}}\{\gamma^{\vee} | \gamma = \gamma(\bullet,\dots,\bullet) \text{ graph with undistinguishable vertices}\} \\ \mathsf{GC}_n &= \text{subcomplex spanned by connected graphs in } \mathsf{fGC}_n \end{aligned}$$

§2. The graph complex EGC_n

- Recollections (the surjection operad):
  - For $r \in \mathbb{N}$, we form the dg-module $\mathsf{E}(r)$ such that

$$\mathsf{E}(r)_p = \mathtt{Span}\{\underline{u} = (u(1), \dots, u(r+p)) | u(t) \in \underline{\mathrm{r}}\}/\equiv,$$

where we take

$$\underline{u} \equiv 0 \quad \begin{cases} \text{if } t \mapsto u(t) \text{ does not surject over } \underline{\mathrm{r}} \\ \text{or if } u(t) = u(t+1) \text{ for some } t \end{cases}$$

together with the differential such that

$$\delta(\underline{u}) = \sum_{t=1}^{r+p} \pm (u(1), \dots, \widehat{u(t)}, \dots, u(r+p)).$$

  - The collection $\mathsf{E}(r)$ inherits an operad structure such that $\mathsf{E} \xrightarrow{\sim} \mathsf{Com}$.

- Construction:
  - In $dg\,\mathsf{E}\,\mathcal{A}lg$ we form:

$$\mathsf{EGra}_n^c(r) = \bigvee_{1\leqslant i,j\leqslant r} \mathsf{E}(\omega_{ij})$$

where $\mathsf{E}(-)$ = free E-algebra, $\bigvee$ = coproduct in $dg\,\mathsf{E}\,\mathcal{A}lg$, and with $\deg^*(\omega_{ij}) = n-1$, $\omega_{ji} = (-1)^n\omega_{ij}$.
  - The elements in $\mathsf{EGra}_n^c(r)$ are represented by tensors $\underline{u}\otimes\gamma$, where

$$\gamma = \gamma(\circ_1,\dots,\circ_r),$$

is a graph based at vertices $\circ_1,\dots,\circ_r$ and edge set $\underline{\mathsf{e}} = \{\omega_{i_t j_t} | t = 1,\dots,m\}$ and $\underline{u}\in\mathsf{E}(\underline{\mathsf{e}})$.
  - The collection $\mathsf{EGra}_n^c(r)$ is equipped with a cooperad structure with composition coproducts:

$$\circ_{\underline{\mathsf{k}}}^* : \mathsf{EGra}_n^c(\underline{\mathsf{r}}) \to \mathsf{EGra}_n^c(\underline{\mathsf{r}}/\underline{\mathsf{k}}) \otimes \mathsf{EGra}_n^c(\underline{\mathsf{k}})$$

such that $\circ_{\underline{\mathsf{k}}}^* : \underline{u}\otimes\gamma \mapsto \pm(\underline{u}'\otimes\gamma/(\gamma_{|\underline{\mathsf{k}}}\equiv\circ_*))\otimes(\underline{u}''\otimes\gamma_{|\underline{\mathsf{k}}})$

where $\begin{cases} \underline{u}'\subset\underline{u}, \text{terms associated to edges } \omega\notin\gamma_{|\underline{\mathsf{k}}}, \\ \underline{u}''\subset\underline{u}, \text{terms associated to edges } \omega\in\gamma_{|\underline{\mathsf{k}}}, \end{cases}$

- Definition:

$$\begin{aligned} \mathrm{fEGC}_n &= \mathtt{Dfm}(\mathrm{EGra}_n^c, \Lambda^n \,\mathrm{Com}) \\ &= \widehat{\mathtt{Span}}\{\underline{u}^\vee \otimes \gamma^\vee | \gamma = \gamma(\bullet, \dots, \bullet)\} \end{aligned}$$

$$\mathrm{EGC}_n = \text{subcomplex spanned by connected graphs in } \mathrm{fEGC}_n$$

- Observation: The complexes fEGC_n and EGC_n inherit a Γ preLie-algebra structure with:

$$\begin{aligned} & \underline{u}^\vee \otimes \alpha^\vee \{\underline{v}_1^\vee \otimes \beta_1^\vee, \dots, \underline{v}_r^\vee \otimes \beta_r^\vee\}_{m_1,\dots,m_r} \\ &\quad = (u \sqcup \underbrace{v_1 \sqcup \cdots \sqcup v_1}_{m_1} \sqcup \cdots \sqcup \underbrace{v_r \sqcup \cdots \sqcup v_r}_{m_r})^\vee \\ &\qquad \otimes \alpha\{\underbrace{\beta_1, \dots, \beta_1}_{m_1}, \dots, \underbrace{\beta_r, \dots, \beta_r}_{m_r}\}/m_1! \cdots m_r! \end{aligned}$$

and so that:

$$\mathtt{MC}(\mathrm{fEGC}_n) \simeq \mathtt{Mor}_{dg\,\mathcal{O}p^c}(\mathrm{EGra}_n^c, \underbrace{B(\Lambda^n \,\mathrm{Com})}_{\mathrm{hoLie}_n^c}).$$

- Recollections:
  - The collection $\mathsf{E}(r)^\vee = \mathtt{Hom}(\mathsf{E}(r),\mathbb{K})$ is equipped with an operad structure so that we have:

$$\mathsf{Com} \xrightarrow{\sim} \mathsf{E}^\vee \to \mathsf{Zin},$$

    where Zin = Zinbiel operad,
    with $\mu \in \mathsf{Com}(2)$ carried to $\mu = \underline{12}^\vee + \underline{21}^\vee \in \mathsf{E}(2)^\vee$ (Brantner et al.)
  - For $K \in s\mathcal{S}et$, we have $\mathsf{E} \rightharpoondown C^*(K)$ (Berger-BF)
- Observation: We have $\mathsf{E}^\vee \otimes \mathsf{preLie} \rightharpoondown \mathsf{EGC}_n$, giving to EGC_n the structure of a partition preLie algebra, and as a consequence

$$\Gamma\,\underbrace{\mathtt{B}^c(\Lambda^{-1}\mathsf{Perm})}_{=\mathsf{hopreLie}} \to \mathsf{E}^\vee \otimes \mathsf{preLie} \otimes \mathsf{E} \rightharpoondown \mathsf{EGC}_n \hat{\otimes} C^*(\Delta^\bullet)$$

so that (by Verstraete thesis) we can form

$$\mathtt{MC}_\bullet(\mathsf{EGC}_n) = \mathtt{MC}(\mathsf{EGC}_n \hat{\otimes} C^*(\Delta^\bullet)).$$

§3. Results and conjectures

- Construction:
  1. For $\alpha \in \mathtt{MC}(\mathsf{fGC}_n)$, we can form:

$$\mathtt{Tw}^{\alpha}\,\mathsf{Gra}_n^c(r) = \mathtt{Span}\{\gamma(\circ_1, \ldots, \circ_r, \bullet, \ldots, \bullet)\}$$

     with a differential twisted by the action of α.
  2. For $\alpha \in \mathtt{MC}(\mathsf{GC}_n)$, we can form:

$$\mathsf{Graphs}_n^{\alpha} = \mathtt{Tw}^{\alpha}\,\mathsf{Gra}_n^c(r)/\equiv$$

     where we take $\gamma \equiv 0$ if the graph γ contains a connected component of internal vertices $\bullet$
  3. We have $\mathsf{GC}_n^{\alpha} \rightharpoonup \mathsf{Graphs}_n^{\alpha}$.

- Recollections:
  - Take $\mu = \bullet - \bullet \in \mathtt{MC}(\mathsf{GC}_n)$. We have
$$\mathsf{Pois}_n^c = H^*(\mathsf{FM}_n, \mathbb{R}) \xleftarrow{\sim} \mathsf{Graphs}_n^\mu \xrightarrow{\sim} \Omega^*_\sharp(\mathsf{FM}_n, \mathbb{R}) \qquad \text{(Kontsevich)}$$
    where $\Omega^*_\sharp(-, \mathbb{R})$ = real operadic Sullivan model, and as a follow-up:
$$\langle \mathsf{Graphs}_n^\mu \rangle \sim (\mathsf{FM}_n)^\wedge_{\mathbb{R}},$$
    where $\langle - \rangle$ = Sullivan realization.
  - These equivalences can be defined over $\mathbb{Q}$ (BF-Willwacher), so that we can take
$$(\mathsf{E}_n)^\wedge_{\mathbb{Q}} = \langle \mathsf{Graphs}_n^\mu \rangle.$$
- Theorem (BF-Willwacher): The action $\mathsf{GC}_n^\mu \rightharpoondown \mathsf{Graphs}_n^\mu$ integrates to the equivalence:
$$\mathbb{Q}^\times \ltimes \exp \mathrm{Z}^0(\mathsf{GC}_n^\mu \hat{\otimes} \Omega^*(\Delta^\bullet)) \sim \mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_n)^\wedge_{\mathbb{Q}}$$

- Construction:
  - In the case $char\,\mathbb{K} = \ell$ odd, we can form the map:

$$\bigvee_{1 \leqslant i,j \leqslant r} \mathsf{E}(\omega_{ij}) \to \bigvee_{1 \leqslant i,j \leqslant r} C^*(S^{n-1}_{ij}) \xrightarrow{\sim} C^*(\bigtimes_{1 \leqslant i,j \leqslant r} S^{n-1}_{ij})$$

    by fixing a representative of the fundamental class $\omega_{ij} = \omega \in C^*(S^{n-1})$ such that $\tau\omega = (-1)^n\omega$.
  - Then we get a morphism of dg-cooperads:

$$\underbrace{\bigvee_{1 \leqslant i,j \leqslant r} \mathsf{E}(\omega_{ij})}_{=\mathsf{EGra}^c_n} \to C^*(\underbrace{\bigtimes_{1 \leqslant i,j \leqslant r} S^{n-1}_{ij}}_{=\mathsf{K}^S_n}) \xrightarrow{\psi^*} C^*(\mathsf{E}_n(r))$$

    which we can compose with the Koszul duality map of E_n-operads:

$$C^*(\mathsf{E}_n) \xrightarrow{\sim} \mathtt{B}(\Lambda^n C_*(\mathsf{E}_n)) \to \mathtt{B}(\Lambda^n \mathsf{Com})$$

    to get a morphism of dg-cooperads

$$\mathsf{EGra}^c_n \to \mathtt{B}(\Lambda^n \mathsf{Com})$$

    corresponding to some $\lambda \in \mathtt{MC}(\mathsf{fEGC}_n)$.

- Claim: This element $\lambda \in \mathtt{MC}(\mathsf{fEGC}_n)$, corresponding to:

$$\mathsf{EGra}_n^c \to C^*(\mathsf{E}_n) \xrightarrow{\sim} \mathtt{B}(\Lambda^n C_*(\mathsf{E}_n)) \to \mathtt{B}(\Lambda^n \mathsf{Com}),$$

consists of connected graphs, and hence satisfies $\lambda \in \mathtt{MC}(\mathsf{fEGC}_n)$.

- Conjecture ($\mathbb{K} = \bar{\mathbb{F}}_\ell$ with ℓ odd):

$$\mathtt{MC}_\bullet(\mathsf{EGC}_n)_\lambda \sim \mathrm{hofib}\big(\mathtt{B}\,\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_n)^\wedge_\ell \to \mathtt{B}\,\mathtt{Aut}_{\mathcal{O}p} H_*(\mathsf{E}_n, \mathbb{Z}_\ell)\big)$$

Thank you for your attention