Superdiffusive Central Limit Theorem for the Critical Stochastic Burgers Equation

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Singular SPDEs, invariant measures & discrete models - Nancy
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The Stochastic Burgers Equation

$$\partial_t \eta = \frac{1}{2} \Delta \eta + \lambda (w \cdot \nabla) \eta^2 + \nabla \cdot \vec{\xi} \qquad \text{(SBE)}$$

- $\eta = \eta(t,x)$ scalar, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$
- $\vec{\xi} = (\xi_1, \dots, \xi_d)$ space-time Gaussian white noise

$$\mathbb{E}\left[\xi_i(t,x) \xi_j(s,y) \right] = \delta_{i,j} \, \delta(s-t, x-y)$$

- $w \in \mathbb{R}^d$ a given vector, $|w| = 1$
- $\lambda > 0$ is the COUPLING CONSTANT

$\rightsquigarrow$ $\lambda = 0 \Rightarrow \eta$ is Gaussian

Motivation

▷ Mesoscopic model for Driven Diffusive Systems : ASEP
[van Beijren, Kutner, Spohn, PRL '85]

$$\partial_t \eta = \underbrace{\frac{1}{2}\Delta\eta}_{\text{Diffusion}} + \underbrace{\lambda\,(w\cdot\nabla)\,\eta^2}_{\text{Drift (non-linear)}} + \underbrace{\nabla\cdot\vec{\xi}}_{\text{Noise}}$$

- Divergence-Form $\longrightarrow$ local scalar conservation law

Noise :

- White in time $\longrightarrow$ Markovian

▷ Random Growth (in 1d only) : $\eta = \partial_x h$

KPZ Equation : $\partial_t h = \frac{1}{2}\Delta h + \lambda\,(\partial_x h)^2 + \xi$

Goal & Conjectures

ROTATION INVARIANCE $w = e_1$

$$\partial_t \eta = \frac{1}{2}\Delta\eta + \lambda\,\partial_1 \eta^2 + \nabla\cdot\vec{\xi} \qquad \text{(SBE)}$$

- How does the non-linearity affect the behaviour of SBE?
- What are its scaling exponents ?
- What is its large-scale behaviour ?

Heuristics: $\eta_\tau(t,x) := \tau^{d/4}\,\eta\left(\tau t, \sqrt{\tau}\, x\right)$

$$\partial_t \eta = \frac{1}{2}\Delta\eta + \underbrace{\lambda\,\tau^{-\frac{d-2}{4}}}_{\lambda_\tau}\,\partial_1 \eta^2 + \nabla\cdot\vec{\xi}$$

$d = 1$	$\lambda_\tau \uparrow \infty$	RELEVANT	SUB-CRITICAL
$d \geq 3$	$\lambda_\tau \downarrow 0$	IRRELEVANT	SUPER-CRITICAL
$d = 2$	$\lambda_\tau = \lambda$	MARGINAL	CRITICAL

Challenge & Regularization

Problem: the SBE is ILL-POSED $\forall\, d$

$$\partial_t \eta = \frac{1}{2}\Delta\eta + \lambda\, \partial_1 \eta^2 + \nabla\cdot\vec{\xi}$$

$$\vec{\xi} \sim -\frac{d+1}{2} \quad \Rightarrow \quad \eta \sim -\frac{d-2}{2} \quad \Rightarrow \quad \eta^2 \sim ?$$

$$(\ \lambda = 0\ )$$

Regularization: $\varrho \in C_c^\infty(\mathbb{R}^d)$, $\int \varrho(x)\,dx = 1$

- Noise $\quad \vec{\xi} \longmapsto \varrho * \vec{\xi} = (\varrho * \xi_1, \dots \varrho * \xi_d) \in C^\infty$
- Non-linearity $\quad \eta^2 \longmapsto \varrho^{*2} * \eta^2$

$$\partial_t \eta_1 = \frac{1}{2}\Delta\eta_1 + \lambda\, \partial_1 \varrho^{*2} * \eta_1^2 + \nabla\cdot \varrho * \vec{\xi}$$

$\Rightarrow$ η_1 is SMOOTH!

Main Properties of the Regularized SBE

$$\partial_t \eta_1 = \frac{1}{2}\Delta \eta_1 + \lambda\, \partial_1 \varrho^{*2} * \eta_1^2 + \nabla \cdot \varrho * \vec{\xi} \qquad (SBE_1)$$

Lemma [C.- Moulard - Toninelli '24+]

(1) (SBE_1) admits a UNIQUE GLOBAL solution

(2) $t \longmapsto \eta_1(t,\cdot)$ is a MARKOV process

(3) (SBE_1) has INVARIANT MEASURE

$$\mu_1 = \mu * \varrho \,, \qquad \mu \text{ GAUSSIAN WHITE NOISE}$$

(4) $\forall\, \nu > 0$ $\quad \mathbb{E}\left[\eta_1(t,x)\,\eta_1(0,0)\right] \lesssim |x|^{-\nu}$

A good observable : DIFFUSIVITY MATRIX

$$\partial_t \eta_1 = \frac{1}{2}\Delta\eta_1 + \lambda\, \partial_1 \left(g^{*2} * \eta_1^2\right) + \nabla\cdot g * \vec{\xi}\,, \qquad \eta_1(0,\cdot) = \mu_1$$

$$D_{ij}(t) := \frac{1}{t}\int_{\mathbb{R}^d} x_i x_j\, \mathbb{E}\left[\eta_1(t,x)\,\eta_1(0,0)\right] dx$$

- Decay of Correlation $\Rightarrow$ $D(t)$ well-defined
- Relevance: $\lambda = 0 \Rightarrow D(t) = \Theta(1)$ ($\rightsquigarrow$ Diffusive)

  as $t\uparrow\infty$ $\quad D(t) = \begin{cases} \uparrow & (\rightsquigarrow \text{Super diffusive}) \\ \downarrow & (\rightsquigarrow \text{Subdiffusive}) \end{cases}$

- Scaling Exponents :

$$\eta^\tau(t,x) = \tau^{d/4}\, r_\tau\, \eta\left(\tau t, \tau^{1/2} R_\tau^{-1/2} x\right)$$

Known Results : Out of Criticality

$d = 1$ SUB-CRITICAL

- ✓ SBE well-posed even with $\xi = \xi_0$
  - ⇝ [Bertini-Cancrini '96, Hairer '13, Gubinelli-Perkowski '16, Gonçalves-Jara-Gubinelli-Perkowski '13-'18, Kuppiainen '17, Chandra-Ferdinand '23, ...]
- ✓ Superdiffusive, $D(t) \sim C\, t^{1/3}$
  - ⇝ [Balázs-Quastel-Kuppiainen '11, Gu-Komorowski '22-'24]
- ✓ Scaling limit, KPZ Fixed Point NOT GAUSSIAN!
  - ⇝ [Quastel-Sarkar '23, Virag '20]

$d \geq 3$ SUPER-CRITICAL

- ✓ SBE ILL-posed ⇝ RS / PC / ES / RG DON'T APPLY!
  - [A solution theory is NOT expected]
- ✓ Diffusive, $D(t) \sim C$
  - ⇝ SBE [C.-Gubinelli-Toninelli '24] ASEP [Esposito-Marra-Yau '94, Landim-Yau '96]
- ✓ Scaling limit, SHE GAUSSIAN
  - ⇝ SBE [C.-Gubinelli-Toninelli '24] ASEP [Chang-Landim-olla '01, Landim-olla-Varadhan '04]

Known Results : At Criticality

$d = 2$
CRITICAL

✓ SBE ILL-posed ⇝ RS / PC / ES / RG DON'T APPLY!
[A solution theory is NOT expected]

✓ Superdiffusive, Conj: $D(t) \sim (\log t)^{2/3}$
⇝ [van Beijeren-Kutner-Spohn '85]

$$(\log\log\log t)^{-3-\delta} (\log t)^{2/3} \lesssim D(t) \lesssim (\log t)^{2/3} (\log\log\log t)^{3+\delta}$$

(In Tauberian sense)
⇝ SBE [De Gasperi-Haunschmid '24]
ASEP [Yau '04]

✓ Scaling Limit, OPEN!!

Main Results: Logaritmic Superdiffusivity

$$(SBE_1)\quad \partial_t \eta_1 = \tfrac{1}{2}\Delta \eta_1 + \lambda\, \partial_1 \varrho^{*2} \eta_1^2 + \nabla\cdot \varrho * \vec{\xi}, \qquad \eta_1(0,\cdot) = \mu_1(\cdot)$$

Theorem [C.- Moulard-Toninelli '24+]

$$D(t) = \begin{pmatrix} c_{\text{eff}}\, \lambda^{4/3} (\log t)^{2/3} (1+o(1)) & 0 \\ 0 & 1 \end{pmatrix}$$

- [vBKS85]'s conjecture is PROVED.
- FIRST EXACT result for CRITICAL off-equilibrium models
- SCALING EXPONENTS

$$R_\gamma = \begin{pmatrix} (\log \gamma)^{-2/3} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\eta_\gamma(t,x) = \gamma^{1/2} (\log \gamma)^{1/6}\, \eta\left(\gamma t, \gamma^{1/2} R_\gamma^{-1/2} x\right)$$

Main Results : Superdiffusive CLT

$$(SBE_1)\quad \partial_t \eta_1 = \tfrac{1}{2}\Delta \eta_1 + \lambda\, \partial_1 \big(\varrho^{*2} * \eta_1^2\big) + \nabla\cdot \varrho * \vec{\xi}\,, \qquad \eta_1(0,\cdot) = \mu_1(\cdot)$$

- Set $\eta_\tau(t,x) := \tau^{1/2}\, \nu_\tau^{-1/4}\, \eta\big(\tau t,\ \tau^{1/2} R_\tau^{-1/2} x\big)$

$$\begin{cases} R_\tau = \begin{pmatrix} \nu_\tau & 0 \\ 0 & 1 \end{pmatrix} \\ \nu_\tau = \dfrac{1}{(\log \tau)^{2/3}} \end{cases}$$

$$(SBE_\tau)\quad \begin{cases} \partial_t \eta_\tau = \dfrac{1}{2}\,\nabla\cdot R_\tau \nabla \eta_\tau + \lambda \nu_\tau^{3/4}\, \partial_1 \big(\varrho_\tau^{*2} * \eta_\tau^2\big) + \nabla\cdot \sqrt{R_\tau}\, \varrho_\tau * \vec{\xi} \\ \eta_\tau(0,\cdot) = \mu_\tau(\cdot) \end{cases}$$

Remarks :

- $\varrho_\tau(x) := \tau^{1/2}\, \nu_\tau^{-1/4}\, \varrho\big(\tau^{1/2} R_\tau^{-1/2} x\big) \longrightarrow \delta_0$
- All the terms in e_1 are formally VANISHING !!

Main Results: Superdiffusive CLT

$$\eta_\tau(t,x) := \tau^{1/2}\nu_\tau^{-1/4}\,\eta\left(\tau t,\ \tau^{1/2}R_\tau^{-1/2}x\right), \qquad R_\tau = \begin{pmatrix} \nu_\tau & 0 \\ 0 & 1 \end{pmatrix} \ \&\ \nu_\tau = \frac{1}{(\log\tau)^{2/3}}$$

$$(SBE_\tau)\quad \partial_t\eta_\tau = \frac{1}{2}\nabla\cdot R_\tau\nabla\eta_\tau + \lambda\nu_\tau^{3/4}\partial_1 S_\tau^{*2}\eta_\tau^2 + \nabla\cdot\sqrt{R_\tau}\,S_\tau^*\vec{\xi}, \qquad \eta_\tau(0,\cdot)=\mu_\tau$$

Theorem [C.- Moulard-Toninelli '24+]

$$\eta_\tau \xrightarrow{\text{fdd}} \eta_{\text{eff}} \quad \text{as} \quad \tau\uparrow\infty$$

where η_{eff} solves

$$\begin{cases} \partial_t\eta_{\text{eff}} = \frac{1}{2}\nabla\cdot D_{\text{eff}}\nabla\eta_{\text{eff}} + \nabla\cdot\sqrt{D_{\text{eff}}}\,\vec{\tilde{\xi}} \\ \eta_{\text{eff}}(0,\cdot) = \mu(\cdot) \end{cases} \qquad D_{\text{eff}} = \begin{pmatrix} c_{\text{eff}}\lambda^{4/3} & 0 \\ 0 & 1 \end{pmatrix}$$

Comments

✓ Large-scale behaviour of SBE is FULLY characterized.
[FIRST critical SPDE for which this is the case]

✓ The Theorem holds beyond stationarity, $\eta(0,\cdot) \ll \mu$

✓ The constant $\lambda^{4/3}$ is dimension-independent
$\rightsquigarrow$ universal for anisotropic systems.

✓ Result beyond Weak Coupling / Intermediate Disorder

$$\partial_t u^\tau = \tfrac{1}{2}\Delta u^\tau + \frac{\lambda}{\sqrt{\log \tau}}\, u^\tau \xi^\tau$$

- $\nearrow$ • phase transition [Caravenna-Sun-Zygouras '19]
- $\rightsquigarrow$ • $\lambda < \hat\lambda = \sqrt{2\pi}$ Gaussian [Caravenna-Sun-Zygouras, Cosco-Zeitouni, ...]
- $\wr$ • $\lambda = \hat\lambda$ SHF [Caravenna-Sun-Zygouras '23, Tsai '24]

$$\partial_t h^\tau = \tfrac{1}{2}\Delta h^\tau + \frac{\lambda}{\sqrt{\log \tau}} \langle h^\tau, Q h^\tau \rangle + \xi^\tau \quad \rightsquigarrow \quad \lambda < \hat\lambda \quad \text{Gaussian}$$

[Caravenna-Sun-Zygouras, Gu, Chatterjee-Dunlap, ...]

Preliminary Remarks & Tools

* η_τ & η_{eff} are stationary Markov, $\eta_\varepsilon^{(t)} \sim \mu_\varepsilon$ & $\eta_{eff}^{(t)} \sim \mu$
* P^τ & P^{eff} their semigroups
  $\mathcal{L}^\tau$ & $\mathcal{L}^{eff}$ their generators
  } acting on $L^2(\mu)$

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  } acting on $L^2(\mu)$

Goal : $\forall \varphi \in L^2(\mu)$,

$$\left(P_t^\tau \varphi\right)_t \xrightarrow[\tau \uparrow \infty]{L^2(\mu)} \left(P_t^{eff} \varphi\right)_t$$

By the classical Trotter-Kato theorem, it suffices

$\forall \varphi \in L^2(\mu)$,

$$\left(1 - \mathcal{L}^\tau\right)^{-1} \varphi \xrightarrow[\tau \uparrow \infty]{L^2(\mu)} \left(1 - \mathcal{L}^{eff}\right)^{-1} \varphi$$

State Space & Generator

Space of observables $L^2(\mathbb{P})$, $\mathbb{P} \sim$ space white noise

- $L^2(\mathbb{P}) = \bigoplus_{n\geq 0} H_n$, $H_n \perp H_m$, $m \neq n$.
- $H_n = \mathrm{Span}\langle :\hat{\eta}(k_1)\cdots\hat{\eta}(k_n): \,|\, k_1,\dots,k_n \in \mathbb{R}^2 \rangle$, $H_n \simeq L^2_{sym}((\mathbb{R}^2)^n) = \Gamma L^2_n$
- $F \in L^2(\mathbb{P})$, $F \longleftrightarrow (f_n)$, $F = \sum_{n\geq 0} \left(\int \hat{f}_n(k_1,\dots,k_n) : \eta(dk_1)\cdots\eta(dk_n): \right)$

Generators $\mathcal{L}^\tau = \underbrace{S^\tau}_{\text{symmetric}} + \underbrace{A^\tau}_{\text{antisymmetric}}$, $\mathcal{L}^{eff}$

Symmetric Operators : $S^\tau = \nu_2 \mathcal{L}_0^{e_1} + \mathcal{L}_0^{e_2}$, $\mathcal{L}^{eff} = c_{eff}\lambda^{4/3}\mathcal{L}_0^{e_1} + \mathcal{L}_0^{e_2}$

$\mathcal{L}_0^{e_1}, \mathcal{L}_0^{e_2} : \Gamma L^2_n \longrightarrow \Gamma L^2_n$

diagonal in chaos

$\mathcal{L}_0^{e_1} = \frac{1}{2}\partial_1^2$, $\mathcal{L}_0^{e_2} = \frac{1}{2}\partial_2^2$

diagonal in Fourier

State Space & Generator

Antisymmetric Operator

$$\mathcal{A}^{\tau} : \Gamma L^2_n \longrightarrow \Gamma L^2_{n-1} \oplus \Gamma L^2_{n+1}$$

non-diagonal in chaos

$$\mathcal{A}^{\tau} = \underbrace{\mathcal{A}^{\tau}_{+}}_{\text{creation}} + \underbrace{\mathcal{A}^{\tau}_{-}}_{\text{annihilation}}$$

$$\mathcal{A}^{\tau}_{+} : \Gamma L^2_n \longrightarrow \Gamma L^2_{n+1} \qquad \& \qquad \mathcal{A}^{\tau}_{-} = \left(-\mathcal{A}^{\tau}_{+}\right)^*$$

$$f \in \Gamma L^2_n, \quad \mathcal{A}^{\tau}_{+} f(k_1, k_2, \dots, k_{n+1}) \propto \lambda \nu_{\tau}^{3/4} \, e_1 \cdot (k_1 + k_2) \, \mathcal{K}^{\tau}_{k_1, k_2} \, f(k_1 + k_2, k_3, \dots, k_{n+1})$$

[NEEDS SYMMETRIZATION!!]

The singularity of $\mathcal{A}^{\tau}_{+}$

- $|k_1|, |k_2| \gg |k_1 + k_2|, |k_{3:n+1}| \rightsquigarrow$ Singularity in Fourier
- $|k_1|, |k_2| \lesssim |k_1 + k_2|, |k_{3:n+1}| \rightsquigarrow$ Singularity in Chaos

The Resolvent Equation

→ Understand $(1 - \mathcal{L}^\tau) v^\tau = \varphi \in \Gamma L^2_1$

✓ Assume $\exists\ \mathcal{H}^\tau$ non-negative & chaos-diagonal s.t.

$$\mathcal{H}^\tau = - A^\tau_- (1 - \mathcal{S}^\tau - \mathcal{H}^\tau)^{-1} A^\tau_+$$

✓ Then v^τ satisfies

$$v^\tau = \underbrace{(1 - \mathcal{S}^\tau - \mathcal{H}^\tau)^{-1} A^\tau_+ v^\tau}_{\in \bigoplus_{n \geq 2} \Gamma L^2_n} + \underbrace{(1 - \mathcal{S}^\tau - \mathcal{H}^\tau)^{-1} \varphi}_{\in \Gamma L^2_1}$$

Resolvent Convergence follows if

(1) $(v^\tau_n)_{n \geq 2} \longrightarrow 0$ in ΓL^2

(2) $(1 - \mathcal{S}^\tau - \mathcal{H}^\tau)^{-1} \varphi \longrightarrow (1 - \mathcal{L}^{\text{eff}})^{-1} \varphi$ in ΓL^2_1

Challenges & (Novel) Ideas

Challenges:

(A) Singularity of $\mathcal{A}_+^\tau$ in Fourier

(B) Singularity of $\mathcal{A}_+^\tau$ in Chaos

(C) Identification of $\mathcal{U}^\tau$

(Novel) Ideas:

1. Split $\mathcal{A}_+^\tau = \mathcal{A}_+^\# + \mathcal{A}_+^\flat$

$\mathcal{A}_+^\#$ singular in Fourier, regular in Chaos

$\mathcal{A}_+^\flat$ regular in Fourier, singular in chaos

2. The #-Replacement Lemma $\exists\ G^\varepsilon : \mathbb{R} \to \mathbb{R}$ locally bdd s.t.

1) $G^\tau(x) \longrightarrow \mathcal{C}_{\text{eff}} \lambda^{4/3}$

2) $\forall \varphi,\ \left[-\mathcal{A}_-^\# \left(1 - S^\tau - G^\tau(-\mathcal{L}_0)\mathcal{L}_0^{e_1}\right)^{-1} \mathcal{A}_+^\# - G^\tau(-\mathcal{L}_0)\mathcal{L}_0^{e_1} \right] \varphi \to 0$

3. Observe that for $\varphi \in \Pi L^2_n$

$$\| (1 - S^\tau)^{-1/2} \mathcal{A}_+^\flat \varphi \| \lesssim \sqrt{n}\, \| (-\nu_\tau \mathcal{L}_0^{e_1})^{1/2} \varphi \|$$

The Approsimate Resolvent

(I) Define

$$S^{\#} = (1 - S^{\tau} - G^{\tau}(-\mathcal{L}_0)\mathcal{L}_0^{e_1})^{-1} A_+^{\#} S^{\#} + (1 - S^{\tau} - G^{\tau}(-\mathcal{L}_0)\mathcal{L}_0^{e_1})^{-1} \varphi$$

(II) Prove ① $(S_n^{\#})_{n \geq 2} \longrightarrow 0$ in ΓL^2

② $(1 - S^{\tau} - G^{\tau}(-\mathcal{L}_0)\mathcal{L}_0^{e_1})^{-1} \varphi \longrightarrow (1 - \mathcal{L}^{eff})^{-1} \varphi$

(III) Show that $\| (-\nu_{\tau} \mathcal{L}_0^{e_1})^{1/2} S^{\#} \| \downarrow 0$ in τ & n.

Then,

(III) $\Rightarrow \| A^b S^{\#} \|_{\Gamma H_{\tau}^{-1}} \downarrow 0 \Rightarrow \| S^{\#} - \nu^{\tau} \| \downarrow 0$ (*)

and

(*) & (I) $\Rightarrow \nu^{\tau} \longrightarrow (1 - \mathcal{L}^{eff})^{-1} \varphi$ in ΓL^2

Conclusions

* Identified the Large-scale behaviour of a CRITICAL SPDE beyond pathwise approaches.

* Introduced Novel tools for out-of-equilibrium, irreversible statistical mechanics stationary models.

(SELF-)INTERACTING

SPDEs

$$\partial_t h = \frac{1}{2}\Delta h + \lambda\left[(\partial_1 h)^2 - (\partial_2 h)^2\right] + \xi \qquad \text{(AKPZ)}$$

⇝ Weak Coupling J/W Erhard,
⇝ Strong Coupling

$$\partial_t v = \frac{1}{2}\Delta v - \lambda\, v\cdot\nabla v + \nabla p - \nabla^{\perp}\xi\,, \quad \nabla\cdot v = 0 \qquad \text{(SNS)}$$

⇝ Weak Coupling [C.-Kiedrowski]

DIFFUSIONS

$$dX_t = \lambda\, \mathrm{curl}\, \zeta(X_t)\,dt + dB_t \qquad \text{(}\Delta\text{CGFF)}$$

⇝ Strong Coupling [Toth-Valko '11, C.-Haunschmid-Tonimelli '21,
• [Chatterjee-Kerfe-Otto-Wang '23, Otto-Wagner '24, -]
• [Armstrong-Bou Rabee-Kuusi '24]

$$dX_t = -\lambda^2 \int_0^t \nabla V(X_t - X_s)\,ds\,dt + dB_t \qquad \text{(SRBP)}$$

⇝ Weak Coupling [C.-Giles '24]

IPS

Asymmetric Simple Exclusion Process
Lattice Gas Models

Thank You!