Renormalisation of singular SPDEs on Riemannian manifolds

Harprit Singh

May 28, 2024

A class of singular SPDEs

A class of singular SPDEs

Let M Riemannian manifold,

A class of singular SPDEs

Let M Riemannian manifold, $E \to M$ a vector bundle with a connection and metric.

A class of singular SPDEs

Let M Riemannian manifold, $E \to M$ a vector bundle with a connection and metric. Consider sub-critical equations of the form

$$(\partial_t + \mathfrak{L})u = F(u, \nabla u, ..., \nabla^k u, \xi) \ ,$$

A class of singular SPDEs

Let M Riemannian manifold,$E \to M$ a vector bundle with a connection and metric.Consider sub-critical equations of the form

$$(\partial_t + \mathfrak{L})u = F(u, \nabla u, ..., \nabla^k u, \xi) \ ,$$

where

- u (generalised) section of a vector bundle $E \to M$

A class of singular SPDEs

Let M Riemannian manifold, $E \to M$ a vector bundle with a connection and metric. Consider sub-critical equations of the form

$$(\partial_t + \mathfrak{L})u = F(u, \nabla u, ..., \nabla^k u, \xi) \ ,$$

where

- u (generalised) section of a vector bundle $E \to M$
- $\mathfrak{L}$ is an elliptic operator of order $> k$ on E,

A class of singular SPDEs

Let M Riemannian manifold, $E \to M$ a vector bundle with a connection and metric. Consider sub-critical equations of the form

$$(\partial_t + \mathfrak{L})u = F(u, \nabla u, ..., \nabla^k u, \xi) \ ,$$

where

- u (generalised) section of a vector bundle $E \to M$
- $\mathfrak{L}$ is an elliptic operator of order $> k$ on E,
- ξ is an irregular bundle valued (stochastic) noise.

A class of singular SPDEs

Let M Riemannian manifold, $E \to M$ a vector bundle with a connection and metric. Consider sub-critical equations of the form

$$(\partial_t + \mathfrak{L})u = F(u, \nabla u, ..., \nabla^k u, \xi) \ ,$$

where

- u (generalised) section of a vector bundle $E \to M$
- $\mathfrak{L}$ is an elliptic operator of order $> k$ on E,
- ξ is an irregular bundle valued (stochastic) noise.

Solution theories:

A class of singular SPDEs

Let M Riemannian manifold, $E \to M$ a vector bundle with a connection and metric. Consider sub-critical equations of the form

$$(\partial_t + \mathfrak{L})u = F(u, \nabla u, ..., \nabla^k u, \xi) \ ,$$

where

- u (generalised) section of a vector bundle $E \to M$
- $\mathfrak{L}$ is an elliptic operator of order $> k$ on E,
- ξ is an irregular bundle valued (stochastic) noise.

Solution theories:

- Para-controlled Calculus [GIP15],

A class of singular SPDEs

Let M Riemannian manifold, $E \to M$ a vector bundle with a connection and metric. Consider sub-critical equations of the form

$$(\partial_t + \mathfrak{L})u = F(u, \nabla u, ..., \nabla^k u, \xi) \ ,$$

where

- u (generalised) section of a vector bundle $E \to M$
- $\mathfrak{L}$ is an elliptic operator of order $> k$ on E,
- ξ is an irregular bundle valued (stochastic) noise.

Solution theories:

- Para-controlled Calculus [GIP15],(geometric setting [BB16]).

A class of singular SPDEs

Let M Riemannian manifold,$E \to M$ a vector bundle with a connection and metric.Consider sub-critical equations of the form

$$(\partial_t + \mathfrak{L})u = F(u, \nabla u, ..., \nabla^k u, \xi) \ ,$$

where

- u (generalised) section of a vector bundle $E \to M$
- $\mathfrak{L}$ is an elliptic operator of order $> k$ on E,
- ξ is an irregular bundle valued (stochastic) noise.

Solution theories:

- Para-controlled Calculus [GIP15],(geometric setting [BB16]).
- Regularity Structures [Hai14],

A class of singular SPDEs

Let M Riemannian manifold, $E \to M$ a vector bundle with a connection and metric. Consider sub-critical equations of the form

$$(\partial_t + \mathfrak{L})u = F(u, \nabla u, ..., \nabla^k u, \xi) \ ,$$

where

- u (generalised) section of a vector bundle $E \to M$
- $\mathfrak{L}$ is an elliptic operator of order $> k$ on E,
- ξ is an irregular bundle valued (stochastic) noise.

Solution theories:

- Para-controlled Calculus [GIP15],(geometric setting [BB16]).
- Regularity Structures [Hai14],(geometric setting [DDD18], [HS23]).

A class of singular SPDEs

Let M Riemannian manifold, $E \to M$ a vector bundle with a connection and metric. Consider sub-critical equations of the form

$$(\partial_t + \mathfrak{L})u = F(u, \nabla u, ..., \nabla^k u, \xi) \ ,$$

where

- u (generalised) section of a vector bundle $E \to M$
- $\mathfrak{L}$ is an elliptic operator of order $> k$ on E,
- ξ is an irregular bundle valued (stochastic) noise.

Solution theories:

- Para-controlled Calculus [GIP15],(geometric setting [BB16]).
- Regularity Structures [Hai14],(geometric setting [DDD18], [HS23]).
- Renormalisation group flow [Kup14, Duc21].

A class of singular SPDEs

Let M Riemannian manifold, $E \to M$ a vector bundle with a connection and metric. Consider sub-critical equations of the form

$$(\partial_t + \mathfrak{L})u = F(u, \nabla u, ..., \nabla^k u, \xi) \ ,$$

where

- u (generalised) section of a vector bundle $E \to M$
- $\mathfrak{L}$ is an elliptic operator of order $> k$ on E,
- ξ is an irregular bundle valued (stochastic) noise.

Solution theories:

- Para-controlled Calculus [GIP15],(geometric setting [BB16]).
- Regularity Structures [Hai14],(geometric setting [DDD18], [HS23]).
- Renormalisation group flow [Kup14, Duc21].
- Multi-indices [OSSW21].

Some important examples

Some important examples

Φ^4_d-equation

Some important examples

Φ^4_d-equation

Let $E \to M$ vector bundle with metric and connection.

Some important examples

Φ^4_d-equation

Let $E \to M$ vector bundle with metric and connection.

$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \xi \ ,$$

Some important examples

Φ^4_d-equation

Let $E \to M$ vector bundle with metric and connection.

$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \xi \ ,$$

for ξ an E valued space time white noise.

Some important examples

Φ^4_d-equation

Let $E \to M$ vector bundle with metric and connection.

$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \xi \ ,$$

for ξ an E valued space time white noise.

Stochastic Yang–Mills heat flow

Some important examples

Φ^4_d-equation

Let $E \to M$ vector bundle with metric and connection.

$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \xi \ ,$$

for ξ an E valued space time white noise.

Stochastic Yang–Mills heat flow

Let $P \to M$ principle G-bundle,

Some important examples

Φ^4_d-equation

Let $E \to M$ vector bundle with metric and connection.

$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \xi \ ,$$

for ξ an E valued space time white noise.

Stochastic Yang–Mills heat flow

Let $P \to M$ principle G-bundle, $|\cdot|_{\mathfrak{g}}$ an Ad-invariant scalar product on $\mathfrak{g}$.

Some important examples

Φ^4_d-equation

Let $E \to M$ vector bundle with metric and connection.

$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \xi \ ,$$

for ξ an E valued space time white noise.

Stochastic Yang–Mills heat flow

Let $P \to M$ principle G-bundle, $|\cdot|_{\mathfrak{g}}$ an Ad-invariant scalar product on $\mathfrak{g}$. Consider principal connections $\omega = \omega^{ref} + A$

Some important examples

Φ^4_d-equation

Let $E \to M$ vector bundle with metric and connection.

$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \xi \ ,$$

for ξ an E valued space time white noise.

Stochastic Yang–Mills heat flow

Let $P \to M$ principle G-bundle, $|\cdot|_{\mathfrak{g}}$ an Ad-invariant scalar product on $\mathfrak{g}$.
Consider principal connections $\omega = \omega^{ref} + A$

$$\partial_t A = -(d^\omega)^* F_\omega - d^\omega (d^\omega)^*(A) + \xi \ ,$$

Some important examples

Φ^4_d-equation

Let $E \to M$ vector bundle with metric and connection.

$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \xi \ ,$$

for ξ an E valued space time white noise.

Stochastic Yang–Mills heat flow

Let $P \to M$ principle G-bundle, $|\cdot|_{\mathfrak{g}}$ an Ad-invariant scalar product on $\mathfrak{g}$. Consider principal connections $\omega = \omega^{ref} + A$

$$\partial_t A = -(d^\omega)^* F_\omega - d^\omega (d^\omega)^*(A) + \xi \ ,$$

for ξ a $\Omega^1(M, \mathrm{Ad}(\mathfrak{g}))$-valued space time white noise.

Some important examples

Φ^4_d-equation

Let $E \to M$ vector bundle with metric and connection.

$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \xi \ ,$$

for ξ an E valued space time white noise.

Stochastic Yang–Mills heat flow

Let $P \to M$ principle G-bundle, $|\cdot|_{\mathfrak{g}}$ an Ad-invariant scalar product on $\mathfrak{g}$. Consider principal connections $\omega = \omega^{ref} + A$

$$\partial_t A = -(d^\omega)^* F_\omega - d^\omega (d^\omega)^*(A) + \xi \ ,$$

for ξ a $\Omega^1(M, \mathrm{Ad}(\mathfrak{g}))$-valued space time white noise.

Content

1. Meta Theorem of subcritical SPDEs
2. Regularity Structures on Manifolds and Vector Bundles
3. Construction of $\{\mathfrak{Dif}_T\}_{T\in\mathfrak{T}_-}$ on Manifolds
4. Some open (algebraic) avenues

Section 1

Meta Theorem of subcritical SPDEs

Setup

Meta equation

Consider a subcritical equation

$$\partial_t u + \mathcal{L}u = F(u, \nabla u, ..., \nabla^n u) + \xi \tag{1}$$

where

Setup

Meta equation

Consider a subcritical equation

$$\partial_t u + \mathcal{L}u = F(u, \nabla u, ..., \nabla^n u) + \xi \tag{1}$$

where

- $\mathcal{L}$ is an elliptic operator on $\mathbb{T}^d$ of order $> n$

Setup

Meta equation

Consider a subcritical equation

$$\partial_t u + \mathcal{L}u = F(u, \nabla u, ..., \nabla^n u) + \xi \tag{1}$$

where

- $\mathcal{L}$ is an elliptic operator on $\mathbb{T}^d$ of order $> n$
- $\partial_t + \mathcal{L}$ is space-time translation invariant

Setup

Meta equation

Consider a subcritical equation

$$\partial_t u + \mathcal{L}u = F(u, \nabla u, ..., \nabla^n u) + \xi \tag{1}$$

where

- $\mathcal{L}$ is an elliptic operator on $\mathbb{T}^d$ of order $> n$
- $\partial_t + \mathcal{L}$ is space-time translation invariant
- $\xi \in \mathcal{D}'(\mathbb{R} \times \mathbb{T}^d)$ space time white noise.

Setup

Meta equation

Consider a subcritical equation

$$\partial_t u + \mathcal{L} u = F(u, \nabla u, ..., \nabla^n u) + \xi \tag{1}$$

where

- $\mathcal{L}$ is an elliptic operator on $\mathbb{T}^d$ of order $> n$
- $\partial_t + \mathcal{L}$ is space-time translation invariant
- $\xi \in \mathcal{D}'(\mathbb{R} \times \mathbb{T}^d)$ space time white noise.

Problem:

Setup

Meta equation

Consider a subcritical equation

$$\partial_t u + \mathcal{L}u = F(u, \nabla u, ..., \nabla^n u) + \xi \qquad (1)$$

where

- $\mathcal{L}$ is an elliptic operator on $\mathbb{T}^d$ of order $> n$
- $\partial_t + \mathcal{L}$ is space-time translation invariant
- $\xi \in \mathcal{D}'(\mathbb{R} \times \mathbb{T}^d)$ space time white noise.

Problem:

- Consider $(\partial_t + \mathcal{L})v = \xi$

Setup

Meta equation

Consider a subcritical equation

$$\partial_t u + \mathcal{L} u = F(u, \nabla u, ..., \nabla^n u) + \xi \tag{1}$$

where

- $\mathcal{L}$ is an elliptic operator on $\mathbb{T}^d$ of order $> n$
- $\partial_t + \mathcal{L}$ is space-time translation invariant
- $\xi \in \mathcal{D}'(\mathbb{R} \times \mathbb{T}^d)$ space time white noise.

Problem:

- Consider $(\partial_t + \mathcal{L})v = \xi$
- Schauder estimates may not provide enough regularity for the non-linearity $F(v, \nabla v, ..., \nabla^n v)$ to be well defined!

Naive hope

Naive hope

Let ρ_ϵ smooth mollifiers

Naive hope

Let ρ_ϵ smooth mollifiers such that and $\xi_\epsilon := \rho_\epsilon \star \xi$

Naive hope

Let ρ_ϵ smooth mollifiers such that and $\xi_\epsilon := \rho_\epsilon \star \xi \to \xi$ as $\epsilon \to 0$.

Naive hope

Let ρ_ϵ smooth mollifiers such that and $\xi_\epsilon := \rho_\epsilon \star \xi \to \xi$ as $\epsilon \to 0$.Consider solutions u_ϵ

Naive hope

Let ρ_ϵ smooth mollifiers such that and $\xi_\epsilon := \rho_\epsilon \star \xi \to \xi$ as $\epsilon \to 0$.Consider solutions u_ϵ of

$$\partial_t u_\epsilon + \mathcal{L} u_\epsilon = F(u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \xi_\epsilon. \tag{2}$$

Naive hope

Let ρ_ϵ smooth mollifiers such that and $\xi_\epsilon := \rho_\epsilon \star \xi \to \xi$ as $\epsilon \to 0$.Consider solutions u_ϵ of

$$\partial_t u_\epsilon + \mathcal{L} u_\epsilon = F(u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \xi_\epsilon. \tag{2}$$

Generically, u_ϵ does not converge as $\epsilon \to 0$, one needs *renormalisation*.

The theory of regularity structures provides the following type of result:

The theory of regularity structures provides the following type of result:

Metatheorem

The theory of regularity structures provides the following type of result:

Metatheorem

Let G, $\mathcal{L}$, ξ as well as ρ_ϵ and ξ_ϵ be as above.

The theory of regularity structures provides the following type of result:

Metatheorem

Let G, $\mathcal{L}$, ξ as well as ρ_ϵ and ξ_ϵ be as above. Then, there exists a finite index set $\mathfrak{T}_-$

The theory of regularity structures provides the following type of result:

Metatheorem

Let G, $\mathcal{L}$, ξ *as well as* ρ_ϵ *and* ξ_ϵ *be as above. Then, there exists a finite index set* $\mathfrak{T}_-$ *and non-linearities* $\Upsilon^G[\tau]$

The theory of regularity structures provides the following type of result:

Metatheorem

Let G, $\mathcal{L}$, ξ as well as ρ_ϵ and ξ_ϵ be as above. Then, there exists a finite index set $\mathfrak{T}_-$ and non-linearities $\Upsilon^G[\tau]$ depending only on G, $\mathcal{L}$, ξ

The theory of regularity structures provides the following type of result:

Metatheorem

Let G, $\mathcal{L}$, ξ as well as ρ_ϵ and ξ_ϵ be as above. Then, there exists a finite index set $\mathfrak{T}_-$ and non-linearities $\Upsilon^G[\tau]$ depending only on G, $\mathcal{L}$, ξ and as well as constants $c_\epsilon^\tau \in \mathbb{R}$ depending additionally on ρ_ϵ

The theory of regularity structures provides the following type of result:

Metatheorem

Let G, $\mathcal{L}$, ξ as well as ρ_ϵ and ξ_ϵ be as above. Then, there exists a finite index set $\mathfrak{T}_-$ and non-linearities $\Upsilon^G[\tau]$ depending only on G, $\mathcal{L}$, ξ and as well as constants $c_\epsilon^\tau \in \mathbb{R}$ depending additionally on ρ_ϵ such for u_ϵ

The theory of regularity structures provides the following type of result:

Metatheorem

Let G, $\mathcal{L}$, ξ as well as ρ_ϵ and ξ_ϵ be as above. Then, there exists a finite index set $\mathfrak{T}_-$ and non-linearities $\Upsilon^G[\tau]$ depending only on G, $\mathcal{L}$, ξ and as well as constants $c_\epsilon^\tau \in \mathbb{R}$ depending additionally on ρ_ϵ such for u_ϵ satisfying

$$\partial_t + \mathcal{L}u_\epsilon = F(u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \sum_{\tau \in \mathfrak{T}_-} c_\epsilon^\tau \Upsilon^F[\tau](u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \xi_\epsilon \ ,$$

The theory of regularity structures provides the following type of result:

Metatheorem

Let G, $\mathcal{L}$, ξ as well as ρ_ϵ and ξ_ϵ be as above. Then, there exists a finite index set $\mathfrak{T}_-$ and non-linearities $\Upsilon^G[\tau]$ depending only on G, $\mathcal{L}$, ξ and as well as constants $c_\epsilon^\tau \in \mathbb{R}$ depending additionally on ρ_ϵ such for u_ϵ satisfying

$$\partial_t + \mathcal{L}u_\epsilon = F(u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \sum_{\tau \in \mathfrak{T}_-} c_\epsilon^\tau \Upsilon^F[\tau](u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \xi_\epsilon \ ,$$

there exists $u \in \mathcal{D}'((0,T) \times \mathbb{R}^d)$

The theory of regularity structures provides the following type of result:

Metatheorem

Let G, $\mathcal{L}$, ξ as well as ρ_ϵ and ξ_ϵ be as above. Then, there exists a finite index set $\mathfrak{T}_-$ and non-linearities $\Upsilon^G[\tau]$ depending only on G, $\mathcal{L}$, ξ and as well as constants $c_\epsilon^\tau \in \mathbb{R}$ depending additionally on ρ_ϵ such for u_ϵ satisfying

$$\partial_t + \mathcal{L}u_\epsilon = F(u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \sum_{\tau \in \mathfrak{T}_-} c_\epsilon^\tau \Upsilon^F[\tau](u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \xi_\epsilon \ ,$$

there exists $u \in \mathcal{D}'((0, T) \times \mathbb{R}^d)$ independent of the choice of $\{\rho_\epsilon\}_{\epsilon \in (0,1)}$

The theory of regularity structures provides the following type of result:

Metatheorem

Let G, $\mathcal{L}$, ξ as well as ρ_ϵ and ξ_ϵ be as above. Then, there exists a finite index set $\mathfrak{T}_-$ and non-linearities $\Upsilon^G[\tau]$ depending only on G, $\mathcal{L}$, ξ and as well as constants $c_\epsilon^\tau \in \mathbb{R}$ depending additionally on ρ_ϵ such for u_ϵ satisfying

$$\partial_t + \mathcal{L}u_\epsilon = F(u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \sum_{\tau \in \mathfrak{T}_-} c_\epsilon^\tau \Upsilon^F[\tau](u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \xi_\epsilon \ ,$$

there exists $u \in \mathcal{D}'((0,T) \times \mathbb{R}^d)$ independent of the choice of $\{\rho_\epsilon\}_{\epsilon\in(0,1)}$ such that $u_\epsilon \to u$ in probability as $\epsilon \to 0$.

The theory of regularity structures provides the following type of result:

Metatheorem

Let G, $\mathcal{L}$, ξ as well as ρ_ϵ and ξ_ϵ be as above. Then, there exists a finite index set $\mathfrak{T}_-$ and non-linearities $\Upsilon^G[\tau]$ depending only on G, $\mathcal{L}$, ξ and as well as constants $c_\epsilon^\tau \in \mathbb{R}$ depending additionally on ρ_ϵ such for u_ϵ satisfying

$$\partial_t + \mathcal{L}u_\epsilon = F(u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \sum_{\tau \in \mathfrak{T}_-} c_\epsilon^\tau \Upsilon^F[\tau](u_\epsilon, \nabla u_\epsilon, ..., \nabla^n u_\epsilon) + \xi_\epsilon \ ,$$

there exists $u \in \mathcal{D}'((0,T) \times \mathbb{R}^d)$ independent of the choice of $\{\rho_\epsilon\}_{\epsilon \in (0,1)}$ such that $u_\epsilon \to u$ in probability as $\epsilon \to 0$.

Note that this metatheorem purposefully kept several aspects vague, see [BCCH20, Theorem 2.22].

Roadmap of the proof

Roadmap of the proof

Some vocabulary

Roadmap of the proof

Some vocabulary

- A *structure space* is a vector space/bundle T.

Roadmap of the proof

Some vocabulary

- A *structure space* is a vector space/bundle T.
  Elements similar to abstract Taylor polynomials.

Roadmap of the proof

Some vocabulary

- A *structure space* is a vector space/bundle T.
  Elements similar to abstract Taylor polynomials.
- A *model* Z gives analytic "meaning" to elements of T,

Roadmap of the proof

Some vocabulary

- A *structure space* is a vector space/bundle T.
  Elements similar to abstract Taylor polynomials.
- A *model* Z gives analytic "meaning" to elements of T, similarly to the map $P(X) \mapsto p(x)$ abstract polynomial to polynomial function.

Roadmap of the proof

Some vocabulary

- A *structure space* is a vector space/bundle T.
  Elements similar to abstract Taylor polynomials.
- A *model* Z gives analytic "meaning" to elements of T, similarly to the map $P(X) \mapsto p(x)$ abstract polynomial to polynomial function.
- Let $\mathcal{M}$ the *space of models*.

Roadmap of the proof

Some vocabulary

- A *structure space* is a vector space/bundle T.
  Elements similar to abstract Taylor polynomials.
- A *model* Z gives analytic "meaning" to elements of T, similarly to the map $P(X) \mapsto p(x)$ abstract polynomial to polynomial function.
- Let $\mathcal{M}$ the *space of models*.
- Given a model $Z \in \mathcal{M}$ and $\gamma > 0$ we denote by $\mathcal{D}^\gamma$ the *space of modelled distributions* (maps $(t,x) \mapsto f(x) \in T$).

Roadmap of the proof

Some vocabulary

- A *structure space* is a vector space/bundle T.
  Elements similar to abstract Taylor polynomials.
- A *model* Z gives analytic "meaning" to elements of T, similarly to the map $P(X) \mapsto p(x)$ abstract polynomial to polynomial function.
- Let $\mathcal{M}$ the *space of models*.
- Given a model $Z \in \mathcal{M}$ and $\gamma > 0$ we denote by $\mathcal{D}^\gamma$ the *space of modelled distributions* (maps $(t, x) \mapsto f(x) \in T$).
- There exists a *reconstruction operator*

$$\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{D}' \ .$$

Diagram

$$
\begin{array}{ccccc}
G & & Z(\zeta) & & \mathfrak{u}^{G,Z(\zeta)} \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
\mathrm{Eq} & \times & \mathscr{M} & \xrightarrow{\quad\mathcal{S}_A\quad} & \mathcal{D}^{\gamma} \\
 & & \Psi \Big\uparrow & & \Big\downarrow \mathcal{R} \\
\mathrm{Eq} & \times & \mathcal{C} & \xrightarrow{\quad\mathcal{S}_C\quad} & \mathcal{D}' \\
\rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} \\
G & & \zeta & & u^{G,\zeta} = \mathcal{R}\mathfrak{u}^{G,Z(\zeta)}
\end{array}
$$

Diagram

$$
\begin{array}{ccccc}
G & & Z(\zeta) & & \mathfrak{u}^{G,Z(\zeta)} \\
\cap & & \cap & & \cap \\
\mathrm{Eq} & \times & \mathscr{M} & \xrightarrow{\quad \mathcal{S}_A \quad} & \mathcal{D}^{\gamma} \\
 & & \Psi \uparrow & & \downarrow \mathcal{R} \\
\mathrm{Eq} & \times & \mathcal{C} & \xrightarrow{\quad \mathcal{S}_C \quad} & \mathcal{D}' \\
\cup & & \cup & & \cup \\
G & & \zeta & & u^{G,\zeta} = \mathcal{R}\mathfrak{u}^{G,Z(\zeta)}
\end{array}
$$

This factors the classical solution map $\mathcal{S}_C$.

Diagram

$$
\begin{array}{ccccc}
G & & Z(\zeta) & & \mathfrak{u}^{G,Z(\zeta)} \\
\rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} \\
\mathrm{Eq} & \times & \mathscr{M} & \xrightarrow{\quad \mathcal{S}_A \quad} & \mathcal{D}^\gamma \\
 & & \Psi \Big\uparrow & & \Big\downarrow \mathcal{R} \\
\mathrm{Eq} & \times & \mathcal{C} & \xrightarrow{\quad \mathcal{S}_C \quad} & \mathcal{D}' \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
G & & \zeta & & u^{G,\zeta} = \mathcal{R}\mathfrak{u}^{G,Z(\zeta)}
\end{array}
$$

This factors the classical solution map $\mathcal{S}_C$. The maps $\mathcal{S}_A$ and $\mathcal{R}$ are continuous,

Diagram

$$
\begin{array}{ccccc}
G & & Z(\zeta) & & \mathfrak{u}^{G,Z(\zeta)} \\
\Cap & & \Cap & & \Cap \\
\mathrm{Eq} & \times & \mathscr{M} & \xrightarrow{\ \mathcal{S}_A\ } & \mathcal{D}^{\gamma} \\
 & & \Psi \uparrow & & \downarrow \mathcal{R} \\
\mathrm{Eq} & \times & \mathcal{C} & \xrightarrow{\ \mathcal{S}_C\ } & \mathcal{D}' \\
\Cup & & \Cup & & \Cup \\
G & & \zeta & & u^{G,\zeta} = \mathcal{R}\mathfrak{u}^{G,Z(\zeta)}
\end{array}
$$

This factors the classical solution map $\mathcal{S}_C$. The maps $\mathcal{S}_A$ and $\mathcal{R}$ are continuous,but Ψ is not.

Diagram

$$\begin{array}{ccccc} G & & Z(\zeta) & & \mathfrak{u}^{G,Z(\zeta)} \\ \Cap & & \Cap & & \Cap \\ \mathrm{Eq} & \times & \mathscr{M} & \xrightarrow{\quad \mathcal{S}_A \quad} & \mathcal{D}^\gamma \\ & & \Psi \uparrow & & \downarrow \mathcal{R} \\ \mathrm{Eq} & \times & \mathcal{C} & \xrightarrow{\quad \mathcal{S}_C \quad} & \mathcal{D}' \\ \Cup & & \Cup & & \Cup \\ G & & \zeta & & u^{G,\zeta} = \mathcal{R}\mathfrak{u}^{G,Z(\zeta)} \end{array}$$

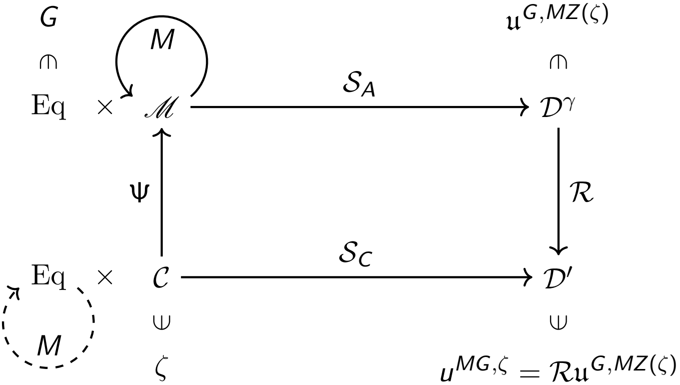
This factors the classical solution map $\mathcal{S}_C$. The maps $\mathcal{S}_A$ and $\mathcal{R}$ are continuous,but Ψ is not. In general, as $\xi_\epsilon \to \xi$ the models $\Psi(\xi_\epsilon) = Z(\xi_\epsilon)$ do *not* converge.

There is a renormalisation group $\mathfrak{G}_-$

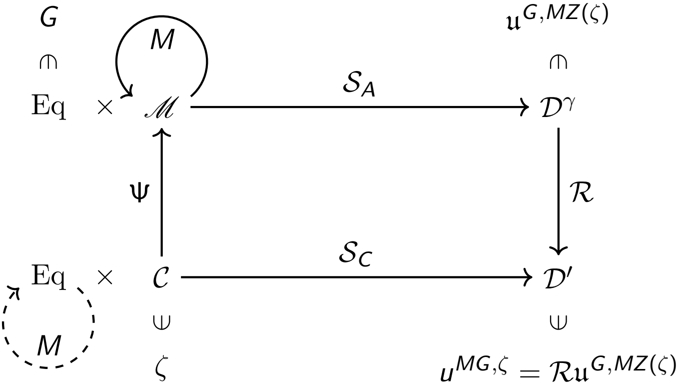
There is a renormalisation group $\mathfrak{G}_-$ acting on $\mathscr{M}$

There is a renormalisation group $\mathfrak{G}_-$ acting on $\mathscr{M}$ and "space of right hand sides" Eq

There is a renormalisation group $\mathfrak{G}_-$ acting on $\mathscr{M}$ and "space of right hand sides" Eq such that the following diagram commutes

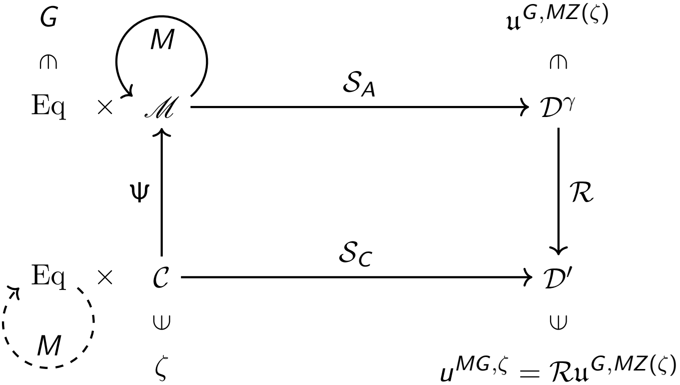


There is a renormalisation group $\mathfrak{G}_-$ acting on $\mathscr{M}$ and "space of right hand sides" Eq such that the following diagram commutes



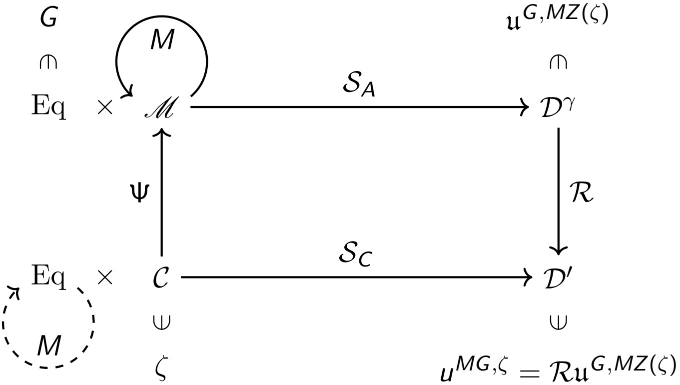
Choosing $M_\epsilon \in \mathfrak{G}_-$

There is a renormalisation group $\mathfrak{G}_-$ acting on $\mathscr{M}$ and "space of right hand sides" Eq such that the following diagram commutes



Choosing $M_\epsilon \in \mathfrak{G}_-$ such that $\xi_\epsilon \mapsto M_\epsilon \Psi(\xi_\epsilon)$ is continuous,

There is a renormalisation group $\mathfrak{G}_-$ acting on $\mathscr{M}$ and "space of right hand sides" Eq such that the following diagram commutes



Choosing $M_\epsilon \in \mathfrak{G}_-$ such that $\xi_\epsilon \mapsto M_\epsilon \Psi(\xi_\epsilon)$ is continuous, concludes the sketch.

The corpus of Regularity Structures

The corpus of Regularity Structures

The robust and general theory can be subdivided as follows:

1. General analytic solution theory.
   "the analytic step", original breakthrough [Hai14].

The corpus of Regularity Structures

The robust and general theory can be subdivided as follows:

1. General analytic solution theory.
   "the analytic step", original breakthrough [Hai14].
2. Construction of the regularity structure, a positive and a negative renormalisation Group.
   "constructive step", [BHZ19].

The corpus of Regularity Structures

The robust and general theory can be subdivided as follows:

1. General analytic solution theory.
   "the analytic step", original breakthrough [Hai14].
2. Construction of the regularity structure, a positive and a negative renormalisation Group.
   "constructive step", [BHZ19].
3. Action of the renormalisation group on the equation.
   "algebraic step", [BCCH20], [BB21].

The corpus of Regularity Structures

The robust and general theory can be subdivided as follows:

1. General analytic solution theory.
   "the analytic step", original breakthrough [Hai14].
2. Construction of the regularity structure, a positive and a negative renormalisation Group.
   "constructive step", [BHZ19].
3. Action of the renormalisation group on the equation.
   "algebraic step", [BCCH20], [BB21].
4. Convergence of renormalised models.
   "the stochastic step", [CH16], [HS23].

The corpus of Regularity Structures

The robust and general theory can be subdivided as follows:

1. General analytic solution theory.
   "the analytic step", original breakthrough [Hai14].
2. Construction of the regularity structure, a positive and a negative renormalisation Group.
   "constructive step", [BHZ19].
3. Action of the renormalisation group on the equation.
   "algebraic step", [BCCH20], [BB21].
4. Convergence of renormalised models.
   "the stochastic step", [CH16], [HS23].

Aim of this talk: Present a generalisation to Manifolds and Vector bundles.

The corpus of Regularity Structures

The robust and general theory can be subdivided as follows:

1. General analytic solution theory.
   "the analytic step", original breakthrough [Hai14].
2. Construction of the regularity structure, a positive and a negative renormalisation Group.
   "constructive step", [BHZ19].
3. Action of the renormalisation group on the equation.
   "algebraic step", [BCCH20], [BB21].
4. Convergence of renormalised models.
   "the stochastic step", [CH16], [HS23].

Aim of this talk: Present a generalisation to Manifolds and Vector bundles. First three steps in full generality.

Section 2

Regularity Structures on Manifolds and Vector Bundles

Regularity Structures on Manifolds and Vector Bundles

Setting the talk

Let M a Riemannian manifold, $E \to M$ and $F^i \to M$ are vector bundles as above.

Regularity Structures on Manifolds and Vector Bundles

Setting the talk

Let M a Riemannian manifold, $E \to M$ and $F^i \to M$ are vector bundles as above. We study subcritical SPDEs of the form

$$\partial_t u + \mathcal{L}u = \sum_{i=0}^{m} G_i(u, \nabla u, ..., \nabla^n u)\xi_i \ ,$$

Regularity Structures on Manifolds and Vector Bundles

Setting the talk

Let M a Riemannian manifold, $E \to M$ and $F^i \to M$ are vector bundles as above. We study subcritical SPDEs of the form

$$\partial_t u + \mathcal{L}u = \sum_{i=0}^{m} G_i(u, \nabla u, ..., \nabla^n u)\xi_i \ ,$$

- u is a (generalized) section of the vector bundle $E \to M$,

Regularity Structures on Manifolds and Vector Bundles

Setting the talk

Let M a Riemannian manifold, $E \to M$ and $F^i \to M$ are vector bundles as above. We study subcritical SPDEs of the form

$$\partial_t u + \mathcal{L} u = \sum_{i=0}^{m} G_i(u, \nabla u, ..., \nabla^n u)\xi_i \ ,$$

- u is a (generalized) section of the vector bundle $E \to M$,
- $\mathcal{L}$ is an elliptic operator on E of degree $> n$,

Regularity Structures on Manifolds and Vector Bundles

Setting the talk

Let M a Riemannian manifold, $E \to M$ and $F^i \to M$ are vector bundles as above. We study subcritical SPDEs of the form

$$\partial_t u + \mathcal{L} u = \sum_{i=0}^{m} G_i(u, \nabla u, ..., \nabla^n u)\xi_i \ ,$$

- u is a (generalized) section of the vector bundle $E \to M$,
- $\mathcal{L}$ is an elliptic operator on E of degree $> n$,
- ξ_i are F^i valued random distributions (e.g. F^i-valued white noise),

Regularity Structures on Manifolds and Vector Bundles

Setting the talk

Let M a Riemannian manifold, $E \to M$ and $F^i \to M$ are vector bundles as above. We study subcritical SPDEs of the form

$$\partial_t u + \mathcal{L} u = \sum_{i=0}^{m} G_i(u, \nabla u, ..., \nabla^n u)\xi_i \ ,$$

- u is a (generalized) section of the vector bundle $E \to M$,
- $\mathcal{L}$ is an elliptic operator on E of degree $> n$,
- ξ_i are F^i valued random distributions (e.g. F^i-valued white noise),
- $G_i : E \times \left(TM^* \otimes E\right) \times ... \times \left((TM^*)^{\otimes n} \otimes E\right) \to L(F^i, E)$ are local functions.

Regularity Structures on Manifolds and Vector Bundles

Setting the talk

Let M a Riemannian manifold, $E \to M$ and $F^i \to M$ are vector bundles as above. We study subcritical SPDEs of the form

$$\partial_t u + \mathcal{L}u = \sum_{i=0}^{m} G_i(u, \nabla u, ..., \nabla^n u)\xi_i \ ,$$

- u is a (generalized) section of the vector bundle $E \to M$,
- $\mathcal{L}$ is an elliptic operator on E of degree $> n$,
- ξ_i are F^i valued random distributions (e.g. F^i-valued white noise),
- $G_i : E \times \left(TM^* \otimes E\right) \times ... \times \left((TM^*)^{\otimes n} \otimes E\right) \to L(F^i, E)$ are local functions.

Solution theory on Manifold and Vector bundles

Main results of [HS23]

Solution theory on Manifold and Vector bundles

Main results of [HS23]

Generalises the theory of RS:

Solution theory on Manifold and Vector bundles

Main results of [HS23]

Generalises the theory of RS:Automates full solution theory with only the following steps on equation by equation basis:

Solution theory on Manifold and Vector bundles

Main results of [HS23]

Generalises the theory of RS:Automates full solution theory with only the following steps on equation by equation basis:

- Identification of canonical (finite dimensional!) renormalisation sub-group.

Solution theory on Manifold and Vector bundles

Main results of [HS23]

Generalises the theory of RS:Automates full solution theory with only the following steps on equation by equation basis:

- Identification of canonical (finite dimensional!) renormalisation sub-group.
- Convergence of models.

Solution theory on Manifold and Vector bundles

Main results of [HS23]

Generalises the theory of RS:Automates full solution theory with only the following steps on equation by equation basis:

- Identification of canonical (finite dimensional!) renormalisation sub-group.
- Convergence of models.

As an application construct solutions to

Solution theory on Manifold and Vector bundles

Main results of [HS23]

Generalises the theory of RS:Automates full solution theory with only the following steps on equation by equation basis:

- Identification of canonical (finite dimensional!) renormalisation sub-group.
- Convergence of models.

As an application construct solutions to

- $\phi^4_3(E)$:

Solution theory on Manifold and Vector bundles

Main results of [HS23]

Generalises the theory of RS:Automates full solution theory with only the following steps on equation by equation basis:

- Identification of canonical (finite dimensional!) renormalisation sub-group.
- Convergence of models.

As an application construct solutions to

- $\phi^4_3(E)$:

$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \infty \cdot u + \xi$$

Solution theory on Manifold and Vector bundles

Main results of [HS23]

Generalises the theory of RS:Automates full solution theory with only the following steps on equation by equation basis:

- Identification of canonical (finite dimensional!) renormalisation sub-group.
- Convergence of models.

As an application construct solutions to

- $\phi^4_3(E)$:

$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \infty \cdot u + \xi$$

- g-PAM:

Solution theory on Manifold and Vector bundles

Main results of [HS23]

Generalises the theory of RS:Automates full solution theory with only the following steps on equation by equation basis:

- Identification of canonical (finite dimensional!) renormalisation sub-group.
- Convergence of models.

As an application construct solutions to

- $\phi^4_3(E)$:
$$(\partial_t + \triangle^E)u = -|u|^2 \cdot u + \infty \cdot u + \xi$$
- g-PAM:
$$(\partial_t + \triangle)u = f(u)(\nabla u, \nabla u) - \infty \cdot \mathrm{tr}\, f(u) g^2(u) + g(u)\left(\xi - \infty \cdot g'(u)\xi\right)$$

We also locally solve ϕ_4^3:

We also locally solve ϕ^3_4:

$$(\partial_t + \triangle)u = u^2 - \infty - \infty \cdot s(x) + \xi \ ,$$

We also locally solve ϕ^3_4:

$$(\partial_t + \triangle)u = u^2 - \infty - \infty \cdot s(x) + \xi \ ,$$

where $s : M \to \mathbb{R}$ scalar curvature.

We also locally solve ϕ^3_4:

$$(\partial_t + \triangle)u = u^2 - \infty - \infty \cdot s(x) + \xi \ ,$$

where $s : M \to \mathbb{R}$ scalar curvature.

Observation for generalised Laplacians and white noise

We also locally solve ϕ^3_4:

$$(\partial_t + \triangle)u = u^2 - \infty - \infty \cdot s(x) + \xi \ ,$$

where $s : M \to \mathbb{R}$ scalar curvature.

Observation for generalised Laplacians and white noise

If the fastest divergence is slower than $\frac{1}{\epsilon^2}$, then renomalisation constants are sufficient.

We also locally solve ϕ^3_4:

$$(\partial_t + \triangle)u = u^2 - \infty - \infty \cdot s(x) + \xi \ ,$$

where $s : M \to \mathbb{R}$ scalar curvature.

Observation for generalised Laplacians and white noise

If the fastest divergence is slower than $\frac{1}{\epsilon^2}$, then renomalisation constants are sufficient.

Previous works:

We also locally solve ϕ^3_4:

$$(\partial_t + \triangle)u = u^2 - \infty - \infty \cdot s(x) + \xi \ ,$$

where $s : M \to \mathbb{R}$ scalar curvature.

Observation for generalised Laplacians and white noise

If the fastest divergence is slower than $\frac{1}{\epsilon^2}$, then renomalisation constants are sufficient.

Previous works:

- Previous work using RS:

We also locally solve ϕ^3_4:

$$(\partial_t + \triangle)u = u^2 - \infty - \infty \cdot s(x) + \xi \ ,$$

where $s : M \to \mathbb{R}$ scalar curvature.

Observation for generalised Laplacians and white noise

If the fastest divergence is slower than $\frac{1}{\epsilon^2}$, then renomalisation constants are sufficient.

Previous works:

- Previous work using RS:
  2d-PAM [DDD18], Paycha et al. (unpublished).

We also locally solve ϕ^3_4:

$$(\partial_t + \triangle)u = u^2 - \infty - \infty \cdot s(x) + \xi \ ,$$

where $s : M \to \mathbb{R}$ scalar curvature.

Observation for generalised Laplacians and white noise

If the fastest divergence is slower than $\frac{1}{\epsilon^2}$, then renomalisation constants are sufficient.

Previous works:

- Previous work using RS:
  2d-PAM [DDD18], Paycha et al. (unpublished).
- Using Paracontrolled Calculus:

We also locally solve ϕ^3_4:

$$(\partial_t + \triangle)u = u^2 - \infty - \infty \cdot s(x) + \xi \ ,$$

where $s : M \to \mathbb{R}$ scalar curvature.

Observation for generalised Laplacians and white noise

If the fastest divergence is slower than $\frac{1}{\epsilon^2}$, then renomalisation constants are sufficient.

Previous works:

- Previous work using RS:
  2d-PAM [DDD18], Paycha et al. (unpublished).
- Using Paracontrolled Calculus:
  PAM: [BB16], [Mou21].

We also locally solve ϕ^3_4:

$$(\partial_t + \triangle)u = u^2 - \infty - \infty \cdot s(x) + \xi \ ,$$

where $s : M \to \mathbb{R}$ scalar curvature.

Observation for generalised Laplacians and white noise

If the fastest divergence is slower than $\frac{1}{\epsilon^2}$, then renomalisation constants are sufficient.

Previous works:

- Previous work using RS:
  2d-PAM [DDD18], Paycha et al. (unpublished).
- Using Paracontrolled Calculus:
  PAM: [BB16], [Mou21]. ϕ^4_3: [BDFT23] based on [JP22].

Quick summary: Analytic step

Quick summary: Analytic step

Some important steps are:

Quick summary: Analytic step

Some important steps are:

- Bigrading $T = \bigoplus_{\alpha,\delta} T_{\alpha,\delta}$, approximate identity $\Pi_p\tau_p \sim \Pi_q\Gamma_{q,p}\tau_p$.

Quick summary: Analytic step

Some important steps are:

- Bigrading $T = \bigoplus_{\alpha,\delta} T_{\alpha,\delta}$, approximate identity $\Pi_p \tau_p \sim \Pi_q \Gamma_{q,p} \tau_p$.
- Polynomials get replaced by Jets.

Quick summary: Analytic step

Some important steps are:

- Bigrading $T = \bigoplus_{\alpha,\delta} T_{\alpha,\delta}$, approximate identity $\Pi_p \tau_p \sim \Pi_q \Gamma_{q,p} \tau_p$.
- Polynomials get replaced by Jets.
- Schauder estimates are more involved:

Quick summary: Analytic step

Some important steps are:

- Bigrading $T = \bigoplus_{\alpha,\delta} T_{\alpha,\delta}$, approximate identity $\Pi_p \tau_p \sim \Pi_q \Gamma_{q,p} \tau_p$.
- Polynomials get replaced by Jets.
- Schauder estimates are more involved:The lift of K has still the form

$$\mathcal{K}_\gamma f(p) = If(p) + Jf(p) + (\mathcal{N}_\gamma f)(p) \ .$$

Quick summary: Analytic step

Some important steps are:

- Bigrading $T = \bigoplus_{\alpha,\delta} T_{\alpha,\delta}$, approximate identity $\Pi_p \tau_p \sim \Pi_q \Gamma_{q,p} \tau_p$.
- Polynomials get replaced by Jets.
- Schauder estimates are more involved:The lift of K has still the form

$$\mathcal{K}_\gamma f(p) = If(p) + Jf(p) + (\mathcal{N}_\gamma f)(p) \ .$$

But new term needs $E_q \tau_p$needs to be added,

Quick summary: Analytic step

Some important steps are:

- Bigrading $T = \bigoplus_{\alpha,\delta} T_{\alpha,\delta}$, approximate identity $\Pi_p \tau_p \sim \Pi_q \Gamma_{q,p} \tau_p$.
- Polynomials get replaced by Jets.
- Schauder estimates are more involved:The lift of K has still the form

$$\mathcal{K}_\gamma f(p) = If(p) + Jf(p) + (\mathcal{N}_\gamma f)(p) \ .$$

But new term needs $E_q \tau_p$needs to be added,to get the crucial estimate

$$|\Gamma_{q,p}(I_p + J_p)\tau_p - (I_q + J_q)\Gamma_{q,p}\tau_p - E_q\tau_p|_k \lesssim |\tau| d(p,q)^{\delta - k} \ .$$

Quick summary: Analytic step

Some important steps are:

- Bigrading $T = \bigoplus_{\alpha,\delta} T_{\alpha,\delta}$, approximate identity $\Pi_p \tau_p \sim \Pi_q \Gamma_{q,p} \tau_p$.
- Polynomials get replaced by Jets.
- Schauder estimates are more involved:The lift of K has still the form

$$\mathcal{K}_\gamma f(p) = If(p) + Jf(p) + (\mathcal{N}_\gamma f)(p) \ .$$

But new term needs $E_q \tau_p$needs to be added,to get the crucial estimate

$$|\Gamma_{q,p}(I_p + J_p)\tau_p - (I_q + J_q)\Gamma_{q,p}\tau_p - E_q \tau_p|_k \lesssim |\tau| d(p,q)^{\delta - k} \ .$$

- Local non-linearity $G : E \to F$ corresponds to section of $\pi_E^* F$.

Quick summary: Constructive & algebraic step

Quick summary: Constructive & algebraic step

Some important steps are:

Quick summary: Constructive & algebraic step

Some important steps are:

- Construction of the regularity structure from rule.

Quick summary: Constructive & algebraic step

Some important steps are:

- Construction of the regularity structure from rule.
  - Efficient indexing of vector bundles.

Quick summary: Constructive & algebraic step

Some important steps are:

- Construction of the regularity structure from rule.
  - Efficient indexing of vector bundles.
  - Use symmetric sets to "attach" jets to trees.

Quick summary: Constructive & algebraic step

Some important steps are:

- Construction of the regularity structure from rule.
  - Efficient indexing of vector bundles.
  - Use symmetric sets to “attach” jets to trees.
- Coloring operations replace coproducts.

Quick summary: Constructive & algebraic step

Some important steps are:

- Construction of the regularity structure from rule.
  - Efficient indexing of vector bundles.
  - Use symmetric sets to "attach" jets to trees.
- Coloring operations replace coproducts.
- Large renormalisation group $\mathfrak{G}_-$ built from $\{\mathfrak{Dif}_T\}_{T\in\mathfrak{T}_-}$, spaces of differential operators associated to negative trees in $\mathfrak{T}_-$.

Quick summary: Constructive & algebraic step

Some important steps are:

- Construction of the regularity structure from rule.
  - Efficient indexing of vector bundles.
  - Use symmetric sets to “attach” jets to trees.
- Coloring operations replace coproducts.
- Large renormalisation group $\mathfrak{G}_-$ built from $\{\mathfrak{Dif}_T\}_{T\in\mathfrak{T}_-}$, spaces of differential operators associated to negative trees in $\mathfrak{T}_-$.
- Effect of $g \in \mathfrak{G}_-$ on equation $\partial u + \mathcal{L}u = F(u,\xi)$:

Quick summary: Constructive & algebraic step

Some important steps are:

- Construction of the regularity structure from rule.
  - Efficient indexing of vector bundles.
  - Use symmetric sets to "attach" jets to trees.
- Coloring operations replace coproducts.
- Large renormalisation group $\mathfrak{G}_-$ built from $\{\mathfrak{Dif}_T\}_{T\in\mathfrak{T}_-}$, spaces of differential operators associated to negative trees in $\mathfrak{T}_-$.
- Effect of $g \in \mathfrak{G}_-$ on equation $\partial u + \mathcal{L}u = F(u,\xi)$:

$$\partial_t u^g + \mathcal{L}u^g = F(u^g,\xi) + \sum_{T\in\mathfrak{T}_-} \langle (D_T F)(u^g), g(T)\rangle \ .$$

Quick summary: Constructive & algebraic step

Some important steps are:

- Construction of the regularity structure from rule.
  - Efficient indexing of vector bundles.
  - Use symmetric sets to "attach" jets to trees.
- Coloring operations replace coproducts.
- Large renormalisation group $\mathfrak{G}_-$ built from $\{\mathfrak{Dif}_T\}_{T\in\mathfrak{T}_-}$, spaces of differential operators associated to negative trees in $\mathfrak{T}_-$.
- Effect of $g \in \mathfrak{G}_-$ on equation $\partial u + \mathcal{L}u = F(u,\xi)$:

$$\partial_t u^g + \mathcal{L}u^g = F(u^g,\xi) + \sum_{T\in\mathfrak{T}_-} \langle (D_T F)(u^g), g(T)\rangle \ .$$

All these steps are done in *full generality*.

Section 3

Construction of $\{\mathfrak{Dif}_T\}_{T\in\mathfrak{T}\_}$ on Manifolds

Symmetric sets and Vector bundle assignments

Symmetric sets and Vector bundle assignments

Let $\mathfrak{S}$ a set of types.

Symmetric sets and Vector bundle assignments

Let $\mathfrak{S}$ a set of types. Let $\mathrm{Iso}(T^1, T^2)$ the set of all type preserving bijections $T^1 \to T^2$.

Symmetric sets and Vector bundle assignments

Let $\mathfrak{S}$ a set of types. Let $\mathrm{Iso}(T^1, T^2)$ the set of all type preserving bijections $T^1 \to T^2$.

Definition: Symmetric sets [CCHS22]

Symmetric sets and Vector bundle assignments

Let $\mathfrak{S}$ a set of types. Let $\mathrm{Iso}(T^1, T^2)$ the set of all type preserving bijections $T^1 \to T^2$.

Definition: Symmetric sets [CCHS22]

A symmetric set $\mathfrak{s}$ consists of an index set $A_{\mathfrak{s}}$

Symmetric sets and Vector bundle assignments

Let $\mathfrak{G}$ a set of types. Let $\mathrm{Iso}(T^1, T^2)$ the set of all type preserving bijections $T^1 \to T^2$.

Definition: Symmetric sets [CCHS22]

A symmetric set $\mathfrak{s}$ consists of an index set $A_\mathfrak{s}$ and a triple

$$\mathfrak{s} = \left(\{T_\mathfrak{s}^a\}_{a\in A_s},\ \{\mathfrak{t}_\mathfrak{s}^a\}_{a\in A_\mathfrak{s}},\ \{\Gamma_\mathfrak{s}^{a,b}\}_{a,b\in A_\mathfrak{s}}\right)\ ,$$

Symmetric sets and Vector bundle assignments

Let $\mathfrak{S}$ a set of types. Let $\mathrm{Iso}(T^1, T^2)$ the set of all type preserving bijections $T^1 \to T^2$.

Definition: Symmetric sets [CCHS22]

A symmetric set $\mathfrak{s}$ consists of an index set $A_\mathfrak{s}$ and a triple

$$\mathfrak{s} = \left(\{T_\mathfrak{s}^a\}_{a\in A_s}, \ \{\mathfrak{t}_\mathfrak{s}^a\}_{a\in A_\mathfrak{s}}, \ \{\Gamma_\mathfrak{s}^{a,b}\}_{a,b\in A_\mathfrak{s}}\right) ,$$

where $(T_\mathfrak{s}^a, t_\mathfrak{s}^a)$ are finite typed sets

Symmetric sets and Vector bundle assignments

Let $\mathfrak{S}$ a set of types. Let $\mathrm{Iso}(T^1, T^2)$ the set of all type preserving bijections $T^1 \to T^2$.

Definition: Symmetric sets [CCHS22]

A symmetric set $\mathfrak{s}$ consists of an index set $A_\mathfrak{s}$ and a triple

$$\mathfrak{s} = \left(\{T^a_\mathfrak{s}\}_{a\in A_s},\ \{\mathfrak{t}^a_\mathfrak{s}\}_{a\in A_\mathfrak{s}},\ \{\Gamma^{a,b}_\mathfrak{s}\}_{a,b\in A_\mathfrak{s}}\right)\ ,$$

where $(T^a_\mathfrak{s}, t^a_\mathfrak{s})$ are finite typed sets and $\Gamma^{a,b}_\mathfrak{s} \subset \mathrm{Iso}(T^b_\mathfrak{s}, T^a_\mathfrak{s})$ a non-empty set

Symmetric sets and Vector bundle assignments

Let $\mathfrak{S}$ a set of types. Let $\mathrm{Iso}(T^1, T^2)$ the set of all type preserving bijections $T^1 \to T^2$.

Definition: Symmetric sets [CCHS22]

A symmetric set $\mathfrak{s}$ consists of an index set $A_\mathfrak{s}$ and a triple

$$\mathfrak{s} = \left(\{T^a_\mathfrak{s}\}_{a\in A_s}, \{\mathfrak{t}^a_\mathfrak{s}\}_{a\in A_\mathfrak{s}}, \{\Gamma^{a,b}_\mathfrak{s}\}_{a,b\in A_\mathfrak{s}}\right) ,$$

where $(T^a_\mathfrak{s}, t^a_\mathfrak{s})$ are finite typed sets and $\Gamma^{a,b}_\mathfrak{s} \subset \mathrm{Iso}(T^b_\mathfrak{s}, T^a_\mathfrak{s})$ a non-empty set satisfying for $a, b, c \in A_\mathfrak{s}$

$$\gamma \in \Gamma^{a,b}_\mathfrak{s} \quad \Rightarrow \quad \gamma^{-1} \in \Gamma^{b,a}_\mathfrak{s} ,$$

$$\gamma \in \Gamma^{a,b}_\mathfrak{s} , \quad \bar\gamma \in \Gamma^{b,c}_\mathfrak{s} \quad \Rightarrow \quad \gamma \circ \bar\gamma \in \Gamma^{a,c}_\mathfrak{s} .$$

Symmetric sets and Vector bundle assignments

Let $\mathfrak{S}$ a set of types. Let $\mathrm{Iso}(T^1, T^2)$ the set of all type preserving bijections $T^1 \to T^2$.

Definition: Symmetric sets [CCHS22]

A symmetric set $\mathfrak{s}$ consists of an index set $A_\mathfrak{s}$ and a triple

$$\mathfrak{s} = \left(\{T^a_\mathfrak{s}\}_{a\in A_s},\ \{\mathfrak{t}^a_\mathfrak{s}\}_{a\in A_\mathfrak{s}},\ \{\Gamma^{a,b}_\mathfrak{s}\}_{a,b\in A_\mathfrak{s}}\right)\ ,$$

where $(T^a_\mathfrak{s}, t^a_\mathfrak{s})$ are finite typed sets and $\Gamma^{a,b}_\mathfrak{s} \subset \mathrm{Iso}(T^b_\mathfrak{s}, T^a_\mathfrak{s})$ a non-empty set satisfying for $a, b, c \in A_\mathfrak{s}$

$$\begin{aligned} \gamma \in \Gamma^{a,b}_\mathfrak{s} \quad &\Rightarrow \quad \gamma^{-1} \in \Gamma^{b,a}_\mathfrak{s}\ , \\ \gamma \in \Gamma^{a,b}_\mathfrak{s}\ , \quad \bar\gamma \in \Gamma^{b,c}_\mathfrak{s} \quad &\Rightarrow \quad \gamma \circ \bar\gamma \in \Gamma^{a,c}_\mathfrak{s}\ . \end{aligned}$$

(Connected groupoid in the category of typed sets.)

Let $W = (W^{\mathfrak{t}})_{\mathfrak{t}\in\mathfrak{S}}$ be vector bundle assignment.

Let $W = (W^{\mathfrak{t}})_{\mathfrak{t}\in\mathfrak{S}}$ be vector bundle assignment.

1. For T a typed set, let

$$W^{\otimes T} := \bigotimes_{x\in T} W^{\mathfrak{t}(x)} \,.$$

Let $W = (W^{\mathfrak{t}})_{\mathfrak{t}\in\mathfrak{S}}$ be vector bundle assignment.

1. For T a typed set, let

$$W^{\otimes T} := \bigotimes_{x\in T} W^{\mathfrak{t}(x)} .$$

2. Any $\psi \in \mathrm{Iso}(T, \bar{T})$

Let $W = (W^{\mathfrak{t}})_{\mathfrak{t}\in\mathfrak{S}}$ be vector bundle assignment.

1. For T a typed set, let
$$W^{\otimes T} := \bigotimes_{x\in T} W^{\mathfrak{t}(x)} .$$

2. Any $\psi \in \mathrm{Iso}(T, \bar{T})$ gives a map $W^{\otimes T} \to W^{\otimes \bar{T}}$

Let $W = (W^{\mathfrak{t}})_{\mathfrak{t}\in\mathfrak{S}}$ be vector bundle assignment.

1. For T a typed set, let

$$W^{\otimes T} := \bigotimes_{x\in T} W^{\mathfrak{t}(x)} .$$

2. Any $\psi \in \mathrm{Iso}(T,\bar{T})$ gives a map $W^{\otimes T} \to W^{\otimes \bar{T}}$ characterised by

$$W_p^{\otimes T} \ni w_p = \otimes_{x\in T} w_p^x \mapsto \psi \cdot w_p = \otimes_{y\in\bar{T}} w_p^{\psi^{-1}(y)} .$$

Let $W = (W^{\mathfrak{t}})_{\mathfrak{t}\in\mathfrak{S}}$ be vector bundle assignment.

1. For T a typed set, let

$$W^{\otimes T} := \bigotimes_{x\in T} W^{\mathfrak{t}(x)} .$$

2. Any $\psi \in \mathrm{Iso}(T, \bar{T})$ gives a map $W^{\otimes T} \to W^{\otimes \bar{T}}$ characterised by

$$W_p^{\otimes T} \ni w_p = \otimes_{x\in T} w_p^x \mapsto \psi \cdot w_p = \otimes_{y\in \bar{T}} w_p^{\psi^{-1}(y)} .$$

3. Define

$$W^{\otimes \mathfrak{s}} = \Big\{ w \in \prod_{a\in A_{\mathfrak{s}}} W^{\otimes T^a_{\mathfrak{s}}} \ : \ w^{(a)} = \gamma_{a,b} \cdot w^{(b)} \qquad \forall a, b \in A_{\mathfrak{s}} \ , \ \forall \gamma_{a,b} \in \Gamma^{a,b}_{\mathfrak{s}} \Big\} .$$

Let $W = (W^{\mathfrak{t}})_{\mathfrak{t}\in\mathfrak{G}}$ be vector bundle assignment.

1. For T a typed set, let
$$W^{\otimes T} := \bigotimes_{x\in T} W^{\mathfrak{t}(x)} .$$

2. Any $\psi \in \mathrm{Iso}(T,\bar{T})$ gives a map $W^{\otimes T} \to W^{\otimes \bar{T}}$characterised by
$$W_p^{\otimes T} \ni w_p = \otimes_{x\in T} w_p^x \mapsto \psi \cdot w_p = \otimes_{y\in\bar{T}} w_p^{\psi^{-1}(y)} .$$

3. Define
$$W^{\otimes \mathfrak{s}} = \Big\{ w \in \prod_{a\in A_{\mathfrak{s}}} W^{\otimes T^a_{\mathfrak{s}}} \,:\, w^{(a)} = \gamma_{a,b}\cdot w^{(b)} \quad \forall a,b\in A_{\mathfrak{s}} \,,\ \forall \gamma_{a,b}\in \Gamma^{a,b}_{\mathfrak{s}} \Big\} .$$

Trees for regularity structures

Consider T with edge types $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_0 \cup \mathcal{E}_-$.

Trees for regularity structures

Consider T with edge types $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_0 \cup \mathcal{E}_-$.

- $\mathcal{E}_+$ will encode kernels

Trees for regularity structures

Consider T with edge types $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_0 \cup \mathcal{E}_-$.

- $\mathcal{E}_+$ will encode kernels
- $\mathcal{E}_0$ place holder for jets

Trees for regularity structures

Consider T with edge types $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_0 \cup \mathcal{E}_-$.

- $\mathcal{E}_+$ will encode kernels
- $\mathcal{E}_0$ place holder for jets
- $\mathcal{E}_-$ will encode noises.

Trees for regularity structures

Consider T with edge types $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_0 \cup \mathcal{E}_-$.

- $\mathcal{E}_+$ will encode kernels
- $\mathcal{E}_0$ place holder for jets
- $\mathcal{E}_-$ will encode noises.

A subcritical rule R defines $\mathfrak{T}$.

Trees for regularity structures

Consider T with edge types $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_0 \cup \mathcal{E}_-$.

- $\mathcal{E}_+$ will encode kernels
- $\mathcal{E}_0$ place holder for jets
- $\mathcal{E}_-$ will encode noises.

A subcritical rule R defines $\mathfrak{T}$. For a fixed tree T we define

Trees for regularity structures

Consider T with edge types $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_0 \cup \mathcal{E}_-$.

- $\mathcal{E}_+$ will encode kernels
- $\mathcal{E}_0$ place holder for jets
- $\mathcal{E}_-$ will encode noises.

A subcritical rule R defines $\mathfrak{T}$. For a fixed tree T we define

- $E_T^- := \{e \in E_T \,|\mathfrak{e}(e) \in \mathcal{E}_-\}$,

Trees for regularity structures

Consider T with edge types $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_0 \cup \mathcal{E}_-$.

- $\mathcal{E}_+$ will encode kernels
- $\mathcal{E}_0$ place holder for jets
- $\mathcal{E}_-$ will encode noises.

A subcritical rule R defines $\mathfrak{T}$. For a fixed tree T we define

- $E_T^- := \{e \in E_T \,|\mathfrak{e}(e) \in \mathcal{E}_-\}$,
- $\mathfrak{T}_T^- := \{T_E \subset T \mid E \subset E_T^-\}$,

Trees for regularity structures

Consider T with edge types $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_0 \cup \mathcal{E}_-$.

- $\mathcal{E}_+$ will encode kernels
- $\mathcal{E}_0$ place holder for jets
- $\mathcal{E}_-$ will encode noises.

A subcritical rule R defines $\mathfrak{T}$. For a fixed tree T we define

- $E_T^- := \{e \in E_T \ |\mathfrak{e}(e) \in \mathcal{E}_-\}$,
- $\mathfrak{T}_T^- := \{T_E \subset T \mid E \subset E_T^-\}$,

Let

$$\mathfrak{T}_- := \bigcup_{T\in\mathfrak{T}} \mathfrak{T}_T^-/\sim \ .$$

Trees for regularity structures

Consider T with edge types $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_0 \cup \mathcal{E}_-$.

- $\mathcal{E}_+$ will encode kernels
- $\mathcal{E}_0$ place holder for jets
- $\mathcal{E}_-$ will encode noises.

A subcritical rule R defines $\mathfrak{T}$. For a fixed tree T we define

- $E_T^- := \{e \in E_T \,|\mathfrak{e}(e) \in \mathcal{E}_-\}$,
- $\mathfrak{T}_T^- := \{T_E \subset T \mid E \subset E_T^-\}$,

Let

$$\mathfrak{T}_- := \bigcup_{T\in\mathfrak{T}} \mathfrak{T}_T^-/\sim \, .$$

One renormalises equations by associating to each $T \in \mathfrak{T}_-$ a multi-linear differential operator $\mathfrak{d}_T \in \mathfrak{Dif}_T$.

Multi-linear differential operators associated to negative trees

Multi-linear differential operators associated to negative trees

Let S be a finite set.

Multi-linear differential operators associated to negative trees

Let S be a finite set. Let $\{W^s\}_{s\in S}$ and W be vector bundles over M.

Multi-linear differential operators associated to negative trees

Let S be a finite set. Let $\{W^s\}_{s\in S}$ and W be vector bundles over M. A map

$$\mathcal{A} : \prod_{s\in S} \mathcal{C}^\infty(W^s) \to \mathcal{C}^\infty(W)$$

Multi-linear differential operators associated to negative trees

Let S be a finite set. Let $\{W^s\}_{s\in S}$ and W be vector bundles over M. A map

$$\mathcal{A} : \prod_{s\in S} \mathcal{C}^\infty(W^s) \to \mathcal{C}^\infty(W)$$

is called *multi-linear differential operator* of order k,

Multi-linear differential operators associated to negative trees

Let S be a finite set. Let $\{W^s\}_{s\in S}$ and W be vector bundles over M. A map

$$\mathcal{A} : \prod_{s\in S} \mathcal{C}^\infty(W^s) \to \mathcal{C}^\infty(W)$$

is called *multi-linear differential operator* of order k, if it factors through the k-jet bundle via a multi-linear bundle morphism,

Multi-linear differential operators associated to negative trees

Let S be a finite set. Let $\{W^s\}_{s\in S}$ and W be vector bundles over M. A map

$$\mathcal{A} : \prod_{s\in S} \mathcal{C}^\infty(W^s) \to \mathcal{C}^\infty(W)$$

is called *multi-linear differential operator* of order k, if it factors through the k-jet bundle via a multi-linear bundle morphism, i.e. there exists $T_{\mathcal{A}} \in L(\otimes_{s\in S} J^k W^s, W)$,

Multi-linear differential operators associated to negative trees

Let S be a finite set. Let $\{W^s\}_{s\in S}$ and W be vector bundles over M. A map

$$\mathcal{A} : \prod_{s\in S} \mathcal{C}^\infty(W^s) \to \mathcal{C}^\infty(W)$$

is called *multi-linear differential operator* of order k, if it factors through the k-jet bundle via a multi-linear bundle morphism, i.e. there exists $T_{\mathcal{A}} \in L(\otimes_{s\in S} J^k W^s, W)$, such that the following diagram commutes

Multi-linear differential operators associated to negative trees

Let S be a finite set. Let $\{W^s\}_{s\in S}$ and W be vector bundles over M. A map

$$\mathcal{A} : \prod_{s\in S} \mathcal{C}^\infty(W^s) \to \mathcal{C}^\infty(W)$$

is called *multi-linear differential operator* of order k, if it factors through the k-jet bundle via a multi-linear bundle morphism, i.e. there exists $T_\mathcal{A} \in L(\otimes_{s\in S} J^k W^s, W)$, such that the following diagram commutes

$$\begin{array}{ccc} \prod_{s\in S} \mathcal{C}^\infty(W^s) & \xrightarrow{\mathcal{A}} & \mathcal{C}^\infty(W) \\ \Big\downarrow{\scriptstyle j^k_\cdot \times \dots \times j^k_\cdot} & \nearrow{\scriptstyle T_\mathcal{A}} & \\ \mathcal{C}^\infty(\prod_{s\in S} J^k W^s) & & \end{array} .$$

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$.

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant*

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant* if for all $f_s \in \mathcal{C}(W^s)$ one has

$$\mathcal{A}(\prod_{s\in S} f_s) = \mathcal{A}(\prod_{s\in S} f_{\gamma(s)})$$

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant* if for all $f_s \in \mathcal{C}(W^s)$ one has

$$\mathcal{A}(\prod_{s\in S} f_s) = \mathcal{A}(\prod_{s\in S} f_{\gamma(s)})$$

(i.e. $T_{\mathcal{A}}$ is γ-symmetric.)

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant* if for all $f_s \in \mathcal{C}(W^s)$ one has

$$\mathcal{A}(\prod_{s\in S} f_s) = \mathcal{A}(\prod_{s\in S} f_{\gamma(s)})$$

(i.e. $T_{\mathcal{A}}$ is γ-symmetric.)
For $\sigma \in \mathfrak{L}$, $\alpha \in \mathbb{R}$ and a symmetric set $\mathfrak{s} = (S, i, \Gamma)$

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant* if for all $f_s \in \mathcal{C}(W^s)$ one has

$$\mathcal{A}(\prod_{s\in S} f_s) = \mathcal{A}(\prod_{s\in S} f_{\gamma(s)})$$

(i.e. $T_{\mathcal{A}}$ is γ-symmetric.)
For $\sigma \in \mathfrak{L}$, $\alpha \in \mathbb{R}$ and a symmetric set $\mathfrak{s} = (S, i, \Gamma)$ where the type set is given by $\mathfrak{L}$

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant* if for all $f_s \in \mathcal{C}(W^s)$ one has

$$\mathcal{A}(\prod_{s\in S} f_s) = \mathcal{A}(\prod_{s\in S} f_{\gamma(s)})$$

(i.e. $T_{\mathcal{A}}$ is γ-symmetric.)
For $\sigma \in \mathfrak{L}$, $\alpha \in \mathbb{R}$ and a symmetric set $\mathfrak{s} = (S, i, \Gamma)$ where the type set is given by $\mathfrak{L}$ and the index set $A_{\mathfrak{s}}$ consists only of one element.

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant* if for all $f_s \in \mathcal{C}(W^s)$ one has

$$\mathcal{A}(\prod_{s\in S} f_s) = \mathcal{A}(\prod_{s\in S} f_{\gamma(s)})$$

(i.e. $T_{\mathcal{A}}$ is γ-symmetric.)
For $\sigma \in \mathfrak{L}$, $\alpha \in \mathbb{R}$ and a symmetric set $\mathfrak{s} = (S, i, \Gamma)$ where the type set is given by $\mathfrak{L}$ and the index set $A_{\mathfrak{s}}$ consists only of one element. We define

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant* if for all $f_s \in \mathcal{C}(W^s)$ one has

$$\mathcal{A}(\prod_{s\in S} f_s) = \mathcal{A}(\prod_{s\in S} f_{\gamma(s)})$$

(i.e. $T_{\mathcal{A}}$ is γ-symmetric.)
For $\sigma \in \mathfrak{L}$, $\alpha \in \mathbb{R}$ and a symmetric set $\mathfrak{s} = (S, i, \Gamma)$ where the type set is given by $\mathfrak{L}$ and the index set $A_{\mathfrak{s}}$ consists only of one element. We define

$$\mathfrak{Dif}_\alpha(\mathfrak{s}, \sigma)$$

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant* if for all $f_s \in \mathcal{C}(W^s)$ one has

$$\mathcal{A}(\prod_{s\in S} f_s) = \mathcal{A}(\prod_{s\in S} f_{\gamma(s)})$$

(i.e. $T_{\mathcal{A}}$ is γ-symmetric.)
For $\sigma \in \mathfrak{L}$, $\alpha \in \mathbb{R}$ and a symmetric set $\mathfrak{s} = (S, i, \Gamma)$ where the type set is given by $\mathfrak{L}$ and the index set $A_{\mathfrak{s}}$ consists only of one element. We define

$$\mathfrak{Dif}_\alpha(\mathfrak{s}, \sigma) \tag{3}$$

as the set of all multilinear differential operators which are

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant* if for all $f_s \in \mathcal{C}(W^s)$ one has

$$\mathcal{A}(\prod_{s\in S} f_s) = \mathcal{A}(\prod_{s\in S} f_{\gamma(s)})$$

(i.e. $T_{\mathcal{A}}$ is γ-symmetric.)
For $\sigma \in \mathfrak{L}$, $\alpha \in \mathbb{R}$ and a symmetric set $\mathfrak{s} = (S, i, \Gamma)$ where the type set is given by $\mathfrak{L}$ and the index set $A_{\mathfrak{s}}$ consists only of one element. We define

$$\mathfrak{Dif}_\alpha(\mathfrak{s}, \sigma) \tag{3}$$

as the set of all multilinear differential operators which are

- of order $\max\{n \in \mathbb{N} \cup \{-\infty\} \ : n \leq -\alpha\}$

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant* if for all $f_s \in \mathcal{C}(W^s)$ one has

$$\mathcal{A}(\prod_{s\in S} f_s) = \mathcal{A}(\prod_{s\in S} f_{\gamma(s)})$$

(i.e. $T_{\mathcal{A}}$ is γ-symmetric.)
For $\sigma \in \mathfrak{L}$, $\alpha \in \mathbb{R}$ and a symmetric set $\mathfrak{s} = (S, i, \Gamma)$ where the type set is given by $\mathfrak{L}$ and the index set $A_{\mathfrak{s}}$ consists only of one element. We define

$$\mathfrak{Dif}_\alpha(\mathfrak{s}, \sigma) \tag{3}$$

as the set of all multilinear differential operators which are

- of order $\max\{n \in \mathbb{N} \cup \{-\infty\} \ : n \leq -\alpha\}$
- $\prod_{s\in S} \mathcal{C}^\infty(V^{i(s)})$ to $\mathcal{C}^\infty(V^\sigma)$

Let γ be a permutation of the set S such that $W^s = W^{\gamma(s)}$ for each $s \in S$. The operator $\mathcal{A}$ is called γ-*invariant* if for all $f_s \in \mathcal{C}(W^s)$ one has

$$\mathcal{A}(\prod_{s\in S} f_s) = \mathcal{A}(\prod_{s\in S} f_{\gamma(s)})$$

(i.e. $T_{\mathcal{A}}$ is γ-symmetric.)
For $\sigma \in \mathfrak{L}$, $\alpha \in \mathbb{R}$ and a symmetric set $\mathfrak{s} = (S, i, \Gamma)$ where the type set is given by $\mathfrak{L}$ and the index set $A_{\mathfrak{s}}$ consists only of one element. We define

$$\mathfrak{Dif}_\alpha(\mathfrak{s}, \sigma) \tag{3}$$

as the set of all multilinear differential operators which are

- of order $\max\{n \in \mathbb{N} \cup \{-\infty\} \ : n \leq -\alpha\}$
- $\prod_{s\in S} \mathcal{C}^\infty(V^{i(s)})$ to $\mathcal{C}^\infty(V^\sigma)$
- γ invariant for all $\gamma \in \Gamma$.

Definition of $\mathfrak{Dif}_T$

Definition of $\mathfrak{Dif}_T$

For a tree $T \in \mathfrak{T}_-$,

Definition of $\mathfrak{Dif}_T$

For a tree $T \in \mathfrak{T}_-$,we set

$$\mathfrak{Dif}_T := \mathfrak{Dif}_{|T|}\big(\mathfrak{s}_T, \mathrm{ind}(\rho_T)\big)$$

Definition of $\mathfrak{Dif}_T$

For a tree $T \in \mathfrak{T}_-$,we set

$$\mathfrak{Dif}_T := \mathfrak{Dif}_{|T|}(\mathfrak{s}_T, \mathrm{ind}(\rho_T)) \tag{4}$$

where $\mathfrak{s}_T = (S_T, i_T, \Gamma_T)$ is given by

Definition of $\mathfrak{Dif}_T$

For a tree $T \in \mathfrak{T}_-$,we set

$$\mathfrak{Dif}_T := \mathfrak{Dif}_{|T|}\big(\mathfrak{s}_T, \mathrm{ind}(\rho_T)\big) \tag{4}$$

where $\mathfrak{s}_T = (S_T, i_T, \Gamma_T)$ is given by

- $S_T = N_T \setminus (\{\rho_T\} \cup L_T)$,

Definition of $\mathfrak{Dif}_T$

For a tree $T \in \mathfrak{T}_-$,we set

$$\mathfrak{Dif}_T := \mathfrak{Dif}_{|T|}\big(\mathfrak{s}_T, \mathrm{ind}(\rho_T)\big) \tag{4}$$

where $\mathfrak{s}_T = (S_T, i_T, \Gamma_T)$ is given by

- $S_T = N_T \setminus (\{\rho_T\} \cup L_T)$,
- $i_T = \mathrm{dif}_T$,

Definition of $\mathfrak{Dif}_T$

For a tree $T \in \mathfrak{T}_-$,we set

$$\mathfrak{Dif}_T := \mathfrak{Dif}_{|T|}\big(\mathfrak{s}_T, \mathrm{ind}(\rho_T)\big) \tag{4}$$

where $\mathfrak{s}_T = (S_T, i_T, \Gamma_T)$ is given by

- $S_T = N_T \setminus (\{\rho_T\} \cup L_T)$,
- $i_T = \mathrm{dif}_T$,
- Γ_T consists of all tree symmetries restricted to S_T.

Some open (algebraic) avenues

Some open (algebraic) avenues

It would be nice to better understand the algebraic structure behind the following settings:

Some open (algebraic) avenues

It would be nice to better understand the algebraic structure behind the following settings:

- Regularity structures on Manifolds and Vector bundles [HS23].

Some open (algebraic) avenues

It would be nice to better understand the algebraic structure behind the following settings:

- Regularity structures on Manifolds and Vector bundles [HS23].
- Identify (local) divergences for non-translation invariant SPDEs [HS23, Sin23].

Some open (algebraic) avenues

It would be nice to better understand the algebraic structure behind the following settings:

- Regularity structures on Manifolds and Vector bundles [HS23].
- Identify (local) divergences for non-translation invariant SPDEs [HS23, Sin23].
- Regularity structures for ultra-parabolic SPDEs (via homogeneous Lie groups) [MS23].

Some open (algebraic) avenues

It would be nice to better understand the algebraic structure behind the following settings:

- Regularity structures on Manifolds and Vector bundles [HS23].
- Identify (local) divergences for non-translation invariant SPDEs [HS23, Sin23].
- Regularity structures for ultra-parabolic SPDEs (via homogeneous Lie groups) [MS23].

It would also be interesting to make sense of wedge products and Hodge star in Rough Geometric Integration [CCHS19],[CS24].

Thank you for your attention!