Transport of Gaussian measures under the flow of semilinear (S)PDEs: quasi-invariance and singularity

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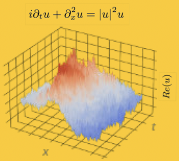
Based on joint works with J. Forlano (Monash University), J. Coe (University of Edinburgh), M. Hairer (EPFL).

Shameless advertisment

HIM trimester, Bonn, 10 May – 20 August 2027:



Trimester Program

“Nonlinear evolution equations with noise”

May 10 - August 20, 2027

Organizers:
Bjoern Bringmann (Princeton), Herbert Koch (Bonn),
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Transport of measures under the flow of (S)PDEs

Consider a PDE, e.g.

$$i\partial_t u - \Delta u + |u|^2 u = 0,$$

or a SPDE, e.g.

$$\partial_t u - \Delta u + u \cdot \nabla u = \nabla p + \xi,$$

and suppose that the initial data u_0 satisfies

$$\mathrm{Law}(u_0) = \mu \sim \exp\Big(- E(u) \Big)$$

for some quantity such that the RHS makes sense, e.g. $E(u) = \langle Au, u \rangle$.

What can we say about $\mathrm{Law}(u(t))$?

Can we use $\mathrm{Law}(u(t))$ to deduce information about the flow?

ODEs and invariant measures

Formally, consider the ODE on $\mathbb{R}^d$

$$\dot{u} = b(u),$$

and assume $\operatorname{div}(b) = 0$. If $\operatorname{Law}(u(t)) = \mu_t$, then

$$\partial_t \mu_t = -\operatorname{div}(b\mu_t) = -\nabla b \cdot \mu_t.$$

Large class of invariant measures: if $E(u)$ is an invariant quantity,

$$\nabla b \cdot \exp(-E(u)) = \partial_t \exp(-E(u)) = 0,$$

so

$$\mu = \exp(-E(u))du \text{ is invariant.}$$

Bourgain's invariant measure argument

Bourgain '94: Consider quintic NLS, posed on $\mathbb{T}$:

$$iu_t - \Delta u + |u|^4 u = 0$$

This is Hamiltonian in $u, \bar{u}$, with Hamiltonian

$$H(u, \bar{u}) = \frac{1}{2} \int |\nabla u|^2 + \frac{1}{6} \int |u|^6.$$

Therefore, the following measure is conserved:

$$d\rho = \text{``}\exp\Big(- H(u, \bar{u}) - M(u)\Big) du d\bar{u}\text{''}.$$

Rigorously,

$$d\rho = \exp\Big(- \frac{1}{6} \int |u|^6 \Big) d\mu,$$

with μ Gaussian with inverse covariance $1 - \Delta$.
The typical u has regularity $u \in H^{\frac{1}{2}-\varepsilon} \setminus H^{\frac{1}{2}}$ $\rightsquigarrow$ global existence does not have a deterministic theory.

Bourgain's invariant measure argument

By the local well posedness theory, if

$$\|u\|_{H^\sigma} \le M, \quad |\delta| \le M^{-\beta}$$

then

$$\|u(t_0+\delta)\|_{H^\sigma} \lesssim M.$$

Therefore if $\mathrm{Law}(u_0) \sim \rho$,

$$\rho(\sup_{0\le T} \|u(t)\|_{H^\sigma} \gg M) \le \sum_{k=0}^{T/M^\beta} \rho(\|u(kT/M^\beta)\|_{H^\sigma} > M)$$

$$\text{Invariance} \Rightarrow = \sum_{k=0}^{T/M^\alpha} \rho(\|u_0\|_{H^\sigma} > M)$$

$$\text{LargeDeviationEstimate} \Rightarrow \lesssim \frac{T}{M^\alpha} \exp(-cM^2) = o(1).$$

Moreover, for ρ-a.e. initial data, $\|u(t)\|_{H^\sigma} \lesssim \log(2+t)^{\frac{1}{2}}$.

Invariance for dispersive PDEs

Many results about invariance for dispersive PDEs.

- Bourgain '96: cubic NLS on $\mathbb{T}^2$,
- Deng, Tzvetkov, Visciglia '14-'15: Benjamin-Ono equation,
- Nahmod, Oh, Rey-Bellet, Staffilani '12: derivative NLS,
- Oh, Killip, Visan, Chapouto, Kishimoto '09,'19: KdV and gKdV,
- Burq, Tzvetkov, Bourgain, Bulut '06, '14: radial NLS on the unit ball,
- Gubinelli, Koch, Oh, T., Robert, Tzvetkov '21: cubic stochastic wave equation on $\mathbb{T}^2$ and $\mathcal{M}^2$,
- Sun, Tzvetkov, Wang, Liang '20, '23: fractional NLS,
- Oh, Robert, Sosoe '20: sine-Gordon equation,
- Deng, Nahmod, Yue '19-'22: NLS on $\mathbb{T}^2$ and and Hartree NLS on $\mathbb{T}^3$,
- Bringmann '20: Hartree NLW on $\mathbb{T}^3$,
- Bringmann, Deng, Nahmod, Yue '22: cubic wave equation on $\mathbb{T}^3$,
- Dinh, Rougerie, '22: NLS with trapping potential,

and many more.

Quasi-invariance and Bourgain's argument

Natural question: what happens when the initial data does not correspond to an *invariant* measure?

Remark: In Bourgain's argument, we used invariance only in the step

$$\rho(\|u(t_0)\|_{H^\sigma} > M) = \rho(\|u(0)\|_{H^\sigma} > M).$$

However, we just need $\lesssim$.

Definition

We say that a flow $\Phi_t(u_0) = u(t)$ is *quasi-invariant* with respect to the measure μ if

$$\mathrm{Law}(u(t)) \ll \mu \text{ when } \mathrm{Law}(u_0) = \mu,$$

or equivalently,

$$(\Phi_t)_\# \mu \ll \mu.$$

Density

Suppose that $\mu \sim \exp\big(-E(u)\big)$, and

$$\mu_t := (\Phi_t)_{\#}\mu = f_t \mu.$$

What can we say about f_t?
Formally, f_t solves the transport equation

$$\partial_t f_t = -b \cdot \nabla f_t - \mathcal{Q} f_t, \quad \mathcal{Q} := \frac{d}{dt} E(u(t))|_{t=0}$$

Solving this equation, we obtain

$$f_t = \exp\Big(\int_0^t \mathcal{Q}(\Phi_{-t'}(u_0))dt'\Big).$$

Can we use this to show quasi-invariance?

Quasi-invariance

Meta-Theorem:
Cruzeiro '83, Ambrosio-Figalli '06, Tzvetkov '15, Planchon-Tzvetkov-Visciglia '20

Let

$$\mu = \frac{1}{Z} \exp\Big(- E(u) \Big) du$$

be such that

$$\mathcal{Q} \in \exp(L)(\mu).$$

Then the measure μ is quasi invariant. Moreover, if the equation is globally well-posed, it is enough to have

$$\mathcal{Q} \in \exp(L)_{\text{loc}}(\mu).$$

Proof: Gronwall argument in $\exp(L)$.

Space-time estimates for the density

Let $\mu_s \sim \exp\Big(-\frac{1}{2}\|u\|^2_{H^s}\Big)$, and consider the equations on $\mathbb{T}$:

$$\text{(3-NLS)} \qquad i\partial_t u - i\partial_x^3 u + |u|^2 u = 0$$

$$\text{(FNLS)} \qquad i\partial_t u + (-\partial_x^2)^\alpha u + (|u|^2 - 2\int_{\mathbb{T}} |u|^2)u = 0.$$

When $s < s_c(\alpha)$, $|\mathcal{Q}(u)| = +\infty$ for $\mu_s -$ a.e. u.

Theorems:

We have that

$$\log f_t(u) = \int_0^t \mathcal{Q}(\Phi_{-t'}(u))dt$$

is well-defined for so μ_s-a.e. u. Moreover, the flow is quasi-invariant with respect to μ_s

- Debussche - Tsutsumi: 3-NLS, for $s > \frac{1}{2}$,
- Forlano - T.: FNLS for $s > s_*(\alpha)$, with $s_*(\alpha) < \frac{1}{2}$. This is better than deterministic well-posedeness for $1 < \alpha < \frac{1}{20}(17 + 3\sqrt{21}) \approx 1.537$.

Lagrangian approach

Recall the formula for the density

$$f_t = \exp\Big(\int_0^t \mathcal{Q}(\Phi_{-t'}(u_0))dt'\Big).$$

When u is distributed according to μ_s, $\mathcal{Q}(u)$ is *ill-defined.*
However, if $S(t)u_0$ is the solution of the *linear* equation,

$$\mathbb{E}\Big|\int_0^t \mathcal{Q}(S(-t')(u_0))dt'\Big|^2 < \infty,$$

so it is *well defined.*

Conjecture

The measure μ_s is quasi-invariant if and only if

$$\mathbb{E}\Big|\int_0^t \mathcal{Q}(S(-t')(u_0))dt'\Big|^2 < \infty.$$

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$$\mathbb{E}\Big|\int_0^t \mathcal{Q}(S(-t')(u_0))dt'\Big|^2 < \infty. \quad \rightsquigarrow \text{Normal form boundary term}$$

"Discrete Gronwall" argument for the density

Let $t, s > 0$. For any functional F, we have that

$$\begin{aligned}\int F(u_0) f_{t+s}(u_0) d\mu(u_0) &= \int F(\Phi_{t+s}(u_0)) d\mu(u_0) \\ &= \int F(\Phi_s(u_0)) f_t(u_0) d\mu(u_0) \\ &= \int F(u_0) f_t(\Phi_{-s}(u_0)) f_s(u_0) d\mu(u_0).\end{aligned}$$

Therefore, for fixed $\tau > 0$,

$$f_{(k+1)\tau}(u_0) = f_{k\tau} \circ \Phi_{-\tau}(u_0) \times f_\tau(u_0).$$

By Hölder and a recursive argument, for $T \gg \tau$,

$$\|f_T\|_{L^p} \leq \|f_\tau\|_{L^{p\frac{T}{\tau}}}^{\frac{T}{\tau}}.$$

We obtain the estimates by

- Choosing τ to be a stopping time,
- Local-well-posedness theory in $[0, \tau]$.

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We obtain the estimates by

- Choosing τ to be a stopping time,
- Local-well-posedness theory in $[0, \tau] \rightsquigarrow$ nonlinear flow $\approx$ linear flow.

SPDEs

Prototypical SPDE: for an appropriate noise ξ, consider on $\mathbb{T}^d$

$$(\text{SQE}) \qquad \partial_t u = \Delta u + u^3 + \xi.$$

Suppose $\exists!$ invariant measure ρ_ξ. What can we say about ρ_ξ?

Let μ_ξ be the invariant measure for

$$\partial_t u = \Delta u + \xi.$$

Natural guess: $\rho_\xi \ll \mu_\xi$.

Strong Feller property $\Rightarrow$

$$\rho_\xi \ll \mu_\xi \Leftrightarrow \mu_\xi \text{ is quasi-invariant.}$$

SPDEs

Consider the SDE

$$du = b(u)dt + \sigma dW_t.$$

Then the evolution ρ_t of an initial measure ρ_0 satisfies the Fokker-Plank equation

$$\partial_t \rho_t = -\operatorname{div}(b\rho_t) + \frac{1}{2}\operatorname{tr}(D^2(\sigma\sigma^T \rho_t)).$$

Parabolic equation, but no semi-explicit solution.
Stochastic technique: by Girsanov, it is enough to show quasi-invariance for

$$du = b(u)dt + \sigma h(t)dt + \sigma dW_t,$$

where h is "any" adapted process in $L^2(\mathbb{R}^d) \rightsquigarrow$ control theory problem.
Mattingly - Suidan '04: If $u = \underbrace{\text{linear solution}}_{\in C^{\alpha-\epsilon}} + \underbrace{v(t)}_{\in H^{\alpha+\frac{d}{2}}} \Rightarrow$ quasi-invariance.
Hairer - Kusuoka - Nagoji '24: In the case of (SQE), this is sharp.

Wave equation

Let $\alpha > 0$. Consider on $\mathbb{T}^2$

$$u_{tt} + u_t - \Delta u + u^3 = (-\Delta)^{-\alpha}\xi,$$

where ξ is a space-time *white noise*. Let μ_α be the invariant measure for the linear equation.

- Oh - Tzvetkov '20: μ_α is quasi-invariant for the PDE (without damping).
- T. - Forlano '24: $\exists!$ invariant measure ρ_α.

Problem:

$$u \in C^{\alpha-\epsilon}, u - \text{linear solution} \notin C^{\alpha+1}.$$

Theorem: Forlano - T. '24

The measure μ_α is quasi-invariant.

Careful! In the wave case $\not\Rightarrow \rho_\alpha \ll \mu_\alpha$.

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“Gronwall” argument for the density for SPDEs

Let $t, s > 0$. For any functional F, we have that

$$\begin{aligned}\int F(u_0) f_{t+\tau}(u_0) d\mu(u_0) &= \int \mathbb{E}[F(\Phi_{t+\tau}(u_0,\xi))] d\mu(u_0) \\ &= \int \mathbb{E}[F(\Phi_t(u_0,\xi))] f_\tau(u_0) d\mu(u_0) \\ &\leq \Big(\int \mathbb{E}[F(\Phi_t(u_0,\xi))]^{q'} d\mu(u_0) \Big)^{\frac{1}{q'}} \|f_\tau\|_{L^q} \\ &\leq \Big(\int F(u_0)^{q'} f_t(u_0) d\mu(u_0) \Big)^{\frac{1}{q'}} \|f_\tau\|_{L^q}\end{aligned}$$

By a recursive argument, for $T \gg \tau$,

$$\|f_T\|_{L^p} \leq \|f_\tau\|_{L^{p\frac{T}{\tau}}}^{\frac{T}{\tau}} \Rightarrow \|f_T\|_{L^p}^p \leq \int \exp\Big(pT \, \frac{d}{dt} \log(f_t)\Big|_{t=0}\Big) d\mu$$

We obtain the result by estimating the exponential on the RHS.

Stochastic Navier Stokes

Consider the stochastic Navier-Stokes equation on $\mathbb{T}^2$

$$\partial_t \omega + \Delta \omega = -\Delta^{-1} \nabla^\perp \omega \cdot \nabla \omega + (-\Delta)^{-\frac{\alpha-1}{2}} \xi.$$

Flandoli–Maslowski '95: for $\alpha > 0$, $\exists!$ invariant measure ρ_α.
Is the invariant measure absolutely continuous with respect to the Gaussian?

Theorem: Coe - Hairer - T. '25+

Let μ_α be the invariant measure for the linear equation. Then

$$\rho_\alpha \ll \mu_\alpha.$$

Idea:

$$\omega(t) = z(t) + v(t),$$
$$v(t) \in C^\alpha,$$
$$(\partial_t + \Delta) z(t) = -\Delta^{-1} \nabla^\perp \omega \varolessthan \nabla z + (-\Delta)^{-\frac{\alpha-1}{2}} \xi.$$

Use Girsanov to remove $v(t)$, Gronwall for $\mathrm{Law}(z(t))$.

Is quasi-invariance "universal"?

Conjecture

The measure μ_s is quasi-invariant if and only if

$$\mathbb{E}\Big|\int_0^t \mathcal{Q}(S(-t')(u_0))dt'\Big|^2 < \infty.$$

Consider Szegő equation on $\mathbb{T}$, for $\Pi_{>0}$ the projection on positive frequencies

$$iu_t = \Pi_{>0}(|u|^2 u),$$

Green $\Leftrightarrow$ $s > 1$ or $s = \frac{1}{2}$.

Theorem Coe-T., '24

Consider the Gaussian measure $d\mu_s(u) \sim \exp\Big(-\frac{1}{2}\|u\|^2_{H^s}\Big)du$. Then the flow $\Phi_t^{\text{Szegő}}$ of the Szegő equation satisfies:

- If $s > 1$, the measure μ_s is quasi-invariant with respect to the flow $\Phi_t^{\text{Szegő}}$,
- If $\frac{1}{2} < s < 1$ (and $s \neq \frac{3}{4}$), the evolved measure $(\Phi_t^{\text{Szegő}})_{\#}\mu_s$ is singular with respect to μ_s for a.e. t,
- If $s = \frac{1}{2}$, the measure μ_s is invariant (Burq-Thomann-Tzvetkov '18).

Heuristic for singularity

Recall that formally, $(\Phi_t)_{\#}\mu_s = f_t\mu_s$, with

$$f_t(u_0) = \exp\left(\|u_0\|_{H^s}^2 - \|\Phi_{-t}(u_0)\|_{H^s}^2\right).$$

When $(\Phi_t)_{\#}\mu_s \perp \mu_s$, we expect

$$f_t = 0 \ \ \mu_s - \text{a.e.} \Leftrightarrow \log(f_t) = -\infty \ \ \mu_s - \text{a.e.} \Leftrightarrow \|u_0\|_{H^s}^2 \lll \|\Phi_{-t}(u_0)\|_{H^s}^2$$

Similarly, we also expect

$$\log(f_t) = \infty \ \ (\Phi_t)_{\#}\mu_s - \text{a.e.} \Leftrightarrow f_t \circ \Phi_t = \infty \ \ \mu_s - \text{a.e.} \Leftrightarrow \|u_0\|_{H^s}^2 \lll \|\Phi_t(u_0)\|_{H^s}^2.$$

Therefore, we conjecture

$$(\Phi_t)_{\#}\mu_s \perp \mu_s \Rightarrow \|\Phi_t(u_0)\|_{H^s}^2 \text{ has a minimum in 0 for } \mu_s - \text{a.e.} u_0.$$

An abstract singularity result

Theorem Coe-T., '24

Let $g(\cdot,\cdot)$ be a measurable function with $g(x,y) > 0 \Rightarrow g(y,x) < 0$. Suppose that for μ_s-a.e. u_0, and for every $|t| \ll_{u_0} 1$,

$$g(\Phi_t(u_0), u_0) > 0.$$

Then there exists a countable set $\mathscr{N} \subseteq \mathbb{R}$ such that for every $t \in \mathbb{R} \setminus \mathscr{N}$, we have

$$(\Phi_t)_{\#}\mu_s \perp \mu_s.$$

From the previous slide, we guess for $s < 1$:

$$g(\Phi_t(u_0), u_0) = \lim_{N\to\infty} \|P_N \Phi_t(u_0)\|_{H^s}^2 - \|P_N u_0\|_{H^s}^2 \stackrel{?}{=} \infty$$

for μ_s-a.e. u_0.

Singularity

Want to show

$$\lim_{N\to\infty} \|P_N\Phi_t(u_0)\|_{H^s}^2 - \|P_N u_0\|_{H^s}^2 = \infty$$

Issue: for $s < 1$,

$$\frac{d^2}{dt^2}\|P_N\Phi_t(u_0)\|_{H^s}^2\Big|_{t=0} \sim \underbrace{N^{2-2s}}_{\to\infty}(4s-3)I_s,$$

with $I_s > 0$. For $\frac{1}{2} < s < \frac{3}{4}$, we actually have

$$\lim_{N\to\infty} \|P_N\Phi_t(u_0)\|_{H^s}^2 - \|P_N u_0\|_{H^s}^2 = -\infty.$$

$\Rightarrow$ Singularity, but incorrect intuition!