Algebraic Structures on (prime) m-Dyck paths

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Joint work with Maria Ronco

Operads, Symmetries for QFT and Singular SPDEs
Nancy
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We will talk about...

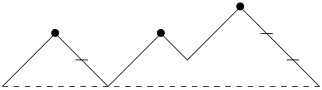
	non symmetric (Hopf) operad	$\underset{\text{Envelope}}{\overset{\text{Prim}}{\rightleftarrows}}$	operad
$m = 0$	$\mathcal{A}\mathrm{s}$		$\mathcal{L}\mathrm{ie}$
$m = 1$	$\mathcal{D}\mathrm{end}$		$\mathcal{B}\mathrm{race}$
m	$\boxed{\mathcal{D}yck^m}$		$\mathcal{B}\mathrm{race}_m$

Section 1– m-Dyck paths

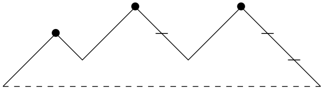
- Definitions, notations
- Grafting m-Dyck paths
- Enumerating prime m-Dyck paths
- The m-Tamari lattice

m-Dyck paths of size n

- It is a path in $(\mathbb{R}^+)^{\times 2}$, starting at $(0,0)$ and ending at $(2nm,0)$, consisting of up steps (m,m) and down steps $(1,-1)$
- It is **prime** if it only meets the x-axis at these two points.



$(2,2,3)$, not prime



$(2,3,3)$, prime

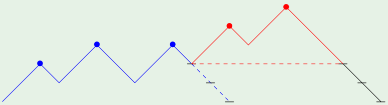
- Notation:
  - $x=(x_1,\dots,x_n)$: the list of heights of the summits of x. Note that $x_1=m$ and $x_{i+1}-x_i\leq m$.
  - Dy_n^m: the set of m-Dyck paths of size n; $d_{m,n}$ its cardinal.
  - PDy_n^m: the set of prime m-Dyck paths of size n; $p_{m,n}$ its cardinal.

Grafting m-Dyck paths

$$(x_1, \dots, x_p) \times_i (y_1, \dots, y_q) = (x_1, \dots, x_p, y_1 + i, \dots, y_q + i), \; 0 \le i \le x_p$$

Example

$$(2,3,3) \times_2 (2,3) = (2,3,3,4,5)$$



The operation $\times_0$ is associative. It corresponds to the **concatenation**.
Any $y \in \mathrm{Dy}_n^m$ writes uniquely as

$$y = y^{(1)} \times_0 \dots \times_0 y^{(r)}$$

with $y^{(j)} \in \mathrm{PDy}_{n_j}^m$ and $\sum_j n_j = n$.

Generalization: grafting m-Dyck paths along a partition

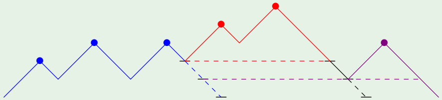
The notation $x \times_\mu y$

- $x = (x_1, \ldots, x_p) \in \mathrm{Dy}_p^m$
- $y = y^{(1)} \times_0 \ldots \times_0 y^{(r)}$ has r prime components, in Dy_q^m
- $x_p \geq \mu_1 \geq \ldots \geq \mu_r \geq 0$ an r partition **compatible with** x, denoted μ

$$x \times_\mu y = (\ldots((x \times_{\mu_1} y^{(1)}) \times_{\mu_2} y^{(2)}) \times \ldots) \times_{\mu_r} y^{(r)}$$

Example

$$(2,3,3) \times_{2 \geq 1} (2,3;2) = (2,3,3,4,5,3)$$



Enumerating prime m-Dyck paths

- Fuss-Catalan numbers: $A_s(p,r) = \frac{r}{sp+r}\binom{sp+r}{s}$
- $d_{m,n} = \frac{1}{mn+1} \cdot \binom{(m+1)n}{n} = A_{n-1}(m+1, m+1) = A_n(m+1, 1)$

Proposition (?; L-Ronco)

For $n, m \geq 1$

$$p_{m,n} = \frac{1}{n}\binom{(m+1)n-2}{n-1}$$

Idea of the proof:

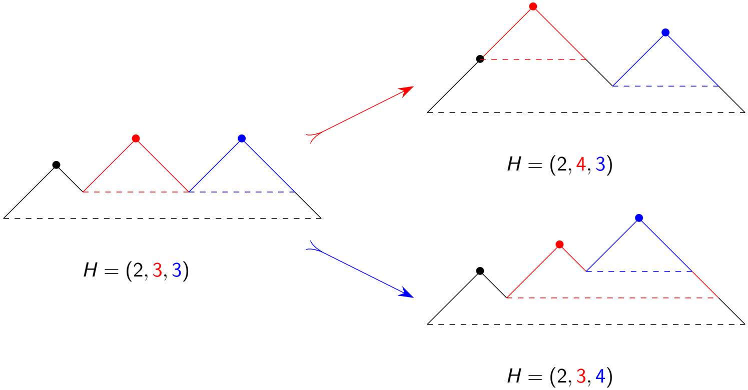
- Concatenation of prime m-Dyck paths gives $D_m(x) = \frac{P_m(x)}{1-P_m(x)}$
- use Riordan convolution formula
  $A_s(p, r+t) = \sum_{k=0}^{s} A_k(p,r)A_{s-k}(p,t)$

to get

$$p_{m,n} = A_{n-1}(m+1, m)$$

The m-Tamari lattice [Bergeron, Préville-Ratelle; Bousquet-Mélou, Fusy, Préville-Ratelle]

covering relations:



Note $(2, 3, 4) \to (2, 4, 5)$ is a covering relation

Section 2– Results by Lopez, Préville-Ratelle, Ronco, arxiv 1508.01252+ JPAA 2020

- Split of associativity, $\mathcal{D}yck^m$-algebras
- The structure of $\mathcal{D}yck^m$-algebra on the vector space spanned by m-Dyck paths
  - Labeling the down steps
  - The valuation map
  - The formula

Split of associativity

Definition: $\mathcal{D}yck^m$-algebra

It is a vector space D endowed with bilinear products $*_i$, $0 \leq i \leq m$ satisfying the relations

- $x *_i (y *_j z) = (x *_i y) *_j z$, for $0 \leq i < j \leq m$,
- $\sum_{j=0}^{i} x *_i (y *_j z) = \sum_{k=i}^{m} (x *_k y) *_i z$, for $0 \leq i \leq m$

for any elements x, y and z in D.

Facts

- $* = \sum_{i=0}^{m} *_i$ is associative
- $m = 0$, $*_0$ is associative
- $m = 1$, $*_0$ and $*_1$ satisfy the **dendriform relations**

Free $\mathcal{D}yck^m$-algebras

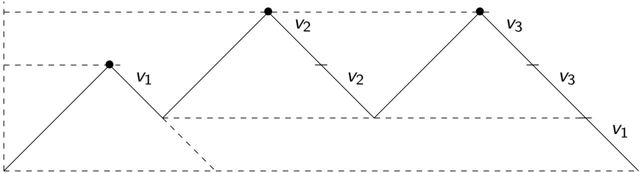
Theorem (Lopez;Préville-Ratelle;Ronco)

The free $\mathcal{D}yck^m$-algebra $\mathcal{D}^m$ generated by one element has for underlying vector space the span of the set of m-Dyck paths.
Moreover, for $x \in \mathrm{Dy}_p^m, y \in \mathrm{Dy}_q^m$, there exists an interval $J_i(x,y)$ in the m-Tamari lattice Dy_{p+q}^m such that:

$$x *_i y = \sum_{z \in J_i(x,y)} z$$

Labeling the down steps of an m-Dyck path

$(2, 3, 3) \mapsto (v_1, v_3, v_3)$



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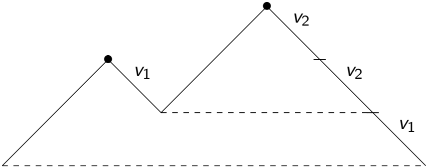
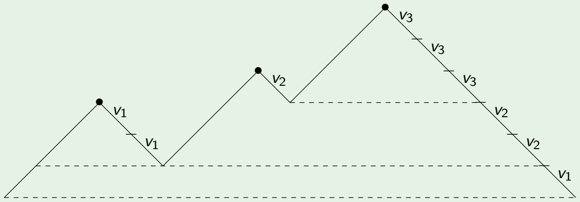


The valuation map of an m-Dyck path $x = (x_1, \dots, x_n)$

it gives the cardinal of maximum occurences at stage i

$$val(x; -) : [x_n] \to [m] = \{0, \dots, m\}$$

Example ($x = (3, 4, 6) \in \mathrm{Dy}_3^3 \mapsto (v_1, v_2, v_2, v_3, v_3, v_3)$)



$6 \mapsto 3$
$5 \mapsto 2$
$4 \mapsto 2$
$3 \mapsto 2$
$2 \mapsto 1$
$1 \mapsto 1$
$0 \mapsto 0$

Idea

$x *_i y$ will involve $x \times_j y$ with $val(x; j) = i$.
More complicated if y is not prime.

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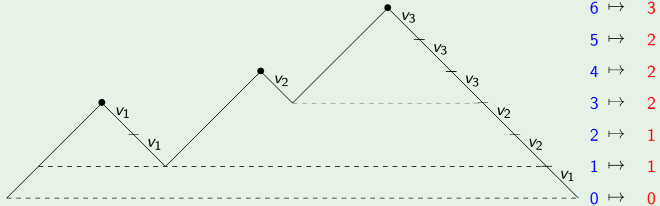


Idea

$x *_i y$ will involve $x \times_j y$ with $val(x; j) = i$.
More complicated if y is not prime.

The Formula

Theorem

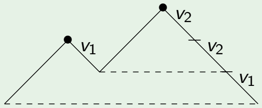
$x = (x_1, \ldots, x_p) \in \mathrm{Dy}_p^m$, $y \in \mathrm{Dy}_q^m$ with r prime components.

$$x *_i y = \sum_{\mu} x \times_\mu y$$

where $x_p \geq \mu_1 \ldots \geq \mu_r$ and $val(x; \mu_r) = i$,
is the sum of every elements in an m-Tamari interval.

Example

$x = (2, 3)$; $val : (0, 1, 2, 3) \mapsto (0, 1, 1, 2)$



To compute $(2, 3) *_1 (2, 2)$, we need the partitions of the form
$3 \geq \mu_1 \geq \mu_2$ avec $\mu_2 \in \{1, 2\}$.

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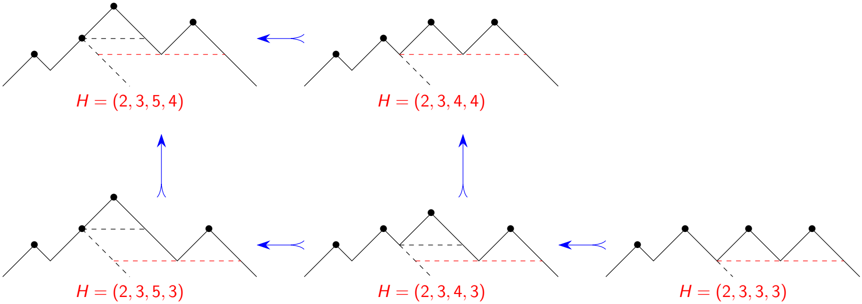
$$\begin{aligned}(2,3) *_1 (2,2) = &(2,3) \times_{3\geq 2} (2,2) + (2,3) \times_{2\geq 2} (2,2) \\ &+ (2,3) \times_{3\geq 1} (2,2) + (2,3) \times_{2\geq 1} (2,2) + (2,3) \times_{1\geq 1} (2,2)\end{aligned}$$

$$= (2,3,5,4) + (2,3,4,4) + (2,3,5,3) + (2,3,4,3) + (2,3,3,3)$$

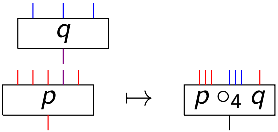


Section 3– Operad, cooperad structures on *m*-Dyck paths

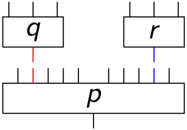
- Quick introduction to operads and co-operads
- The structure of co-operad on Dy^m.
- Reinterpreting the results by Lopez, Préville-Ratelle and Ronco

(non symmetric) operad $\mathcal{P}$

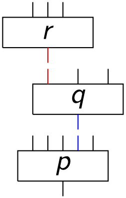
- A collection of sets (or vector spaces) $\mathcal{P}(n)$, $n \geq 1$
- Operations $\circ_i : \mathcal{P}(n) \times \mathcal{P}(n') \to \mathcal{P}(n+n'-1)$, $1 \leq i \leq n$



- A unit in $\mathcal{P}(1)$
- Relations:



parallel



sequential

Keep in mind, a non-symmetric operad consists of

graded objects, that we can either

- graft, as leaves;
- or insert, as boxes;

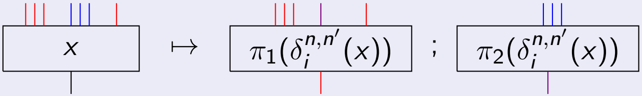
- The endomorphism operad $\mathcal{P}(n) = \mathrm{Map}(X^{\times n}, X)$
- The terminal operad $\mathcal{P}(n) = \star_n$, $\star_n \circ_i \star_{n'} = \star_{n+n'-1}$
- The operad on the symmetric group, $\mathcal{P}(n) = \Sigma_n$

$$\sigma = (2, \boxed{3}, 1) \quad \circ_3 \quad \tau = (2, 3, 1) \quad = \quad (2, 4, 5, 3, 1)$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \boxed{1} & 0 \end{bmatrix} \circ_3 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

co-operad $\mathcal{C}$

- A collection of sets (or vector spaces) $\mathcal{C}(n)$,
- Co-operations $\delta_i^{n,n'} : \mathcal{C}(n+n'-1) \to \mathcal{C}(n) \times \mathcal{C}(n'),\ 1 \leq i \leq n$



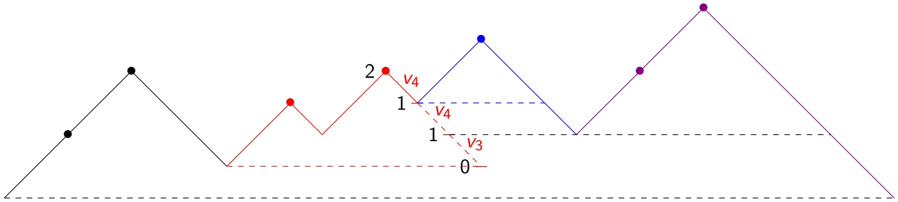
- A co-unit in $\mathcal{C}(1)$ and co-relations

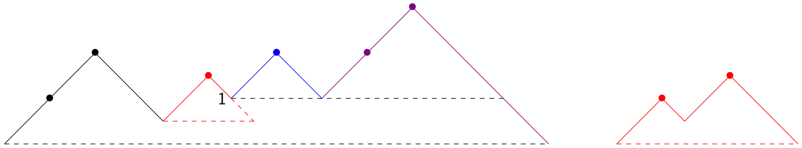
Remark

- Co-operad structure "excision of a sub-object"
- If $\mathcal{C}$ is a co-operad in Vect, then $\mathcal{P}(n) = \mathcal{C}(n)^*$ forms an operad.
- If $\mathcal{P}$ is an operad in Vect, and $\mathcal{P}(n)$ is finite dimensional $\forall n$, then $\mathcal{C}(n) = \mathcal{P}(n)^*$ forms a co-operad.

Example: $\delta_3^{6,2} : \mathrm{Dy}_7^2 \to \mathrm{Dy}_6^2 \times \mathrm{Dy}_2^2$

$x = (2, 4, 3, 4, 5, 4, 6) = ((2, 4) \times_1 (2, 3)) \times_{2+1 \geq 1+1} (2; 2, 4)$





$\pi_1(\delta_3^{6,2}(x)) = ((2, 4) \times_1 (2)) \times_{1+1 \geq 1+1} (2; 2, 4) = (2, 4, 3, 4, 4, 6)$

$\pi_2(\delta_3^{6,2}(x)) = (2, 3)$

Operadic interpretation of the results by L., P.-R. and R.

Theorem

- *For $1 \leq i \leq p$, the collection of maps*

$$\delta_i^{n,n'} : \mathrm{Dy}^m_{n+n'-1} \to \mathrm{Dy}^m_n \times \mathrm{Dy}^m_{n'}$$

endows $\{\mathcal{C}(n) := \mathrm{Dy}^m_n\}_{n\geq 1}$ with a co-operadic structure, in Sets

- *For $(x, y) \in \mathrm{Dy}^m_n \times \mathrm{Dy}^m_{n'}$ we have $(\delta_i^{n,n'})^{-1}(x, y)$ is an interval in $\mathrm{Dy}_{n+n'-1}$ for the m-Tamari order.*
- *The operad $\mathcal{C}^*$ in vector spaces is the operad $\mathcal{D}yck^m$.*

Idea of proof

There is a morphism of operads $\mathcal{D}yck^m \to \mathcal{C}^*$ sending $*_i$ to $(m, m+i)^*$.

Section 4– $\mathcal{D}yck^m$-bialgebras

Lopez, Préville-Ratelle, Ronco arxiv 1508.01252

- What is a $\mathcal{D}yck^m$-bialgebra?
- The case of *m*-Dyck paths
  - Cutting *m*-Dyck paths
  - The result

$\mathcal{D}yck^m$-bialgebra

For A, B two $\mathcal{D}yck^m$-algebras, one can build a $\mathcal{D}yck^m$-algebra on

$$A \boxtimes B = A \otimes \mathbb{K} \oplus \mathbb{K} \otimes B \oplus A \otimes B.$$

Definition: A $\mathcal{D}yck^m$-bialgebra A is

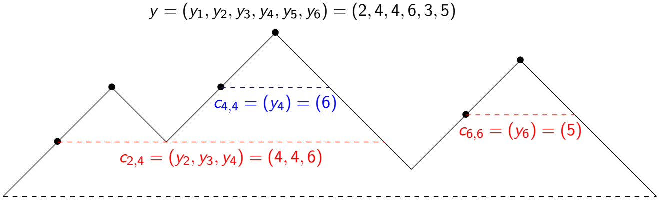
a $\mathcal{D}yck^m$-algebra,with $\Delta : A \to A \boxtimes A$ satisfying

- Δ is a morphism of $\mathcal{D}yck^m$-algebras
- $\Delta(a) = a \otimes 1 + 1 \otimes a + \bar{\Delta}(a)$ with $\bar{\Delta}(a) \in A \otimes A$.
- $\bar{\Delta}$ is coassociative.

Remark

If A is a $\mathcal{D}yck^m$-bialgebra then $A^+ = \mathbb{K} \oplus A$ is a bialgebra

Cutting m-Dyck paths

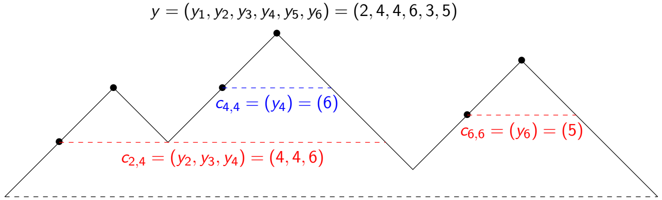


$y = (y_1, y_2, y_3, y_4, y_5, y_6) = (2, 4, 4, 6, 3, 5)$ admits the cuts:

- at vertex y_2: $c_{2,2} = (4)$ and $c_{2,4} = (4, 4, 6)$
- at vertex y_4: $c_{4,4} = (6)$
- at vertex y_6: $c_{6,6} = (5)$

An admissible cut is a collection of disjoint cuts
Example: $S_1 = \{c_{4,4}\}$, $S_2 = \{c_{2,4}, c_{6,6}\}$.

Cutting m-Dyck paths



Example: $S_1 = \{c_{4,4}\}$, $S_2 = \{c_{2,4}, c_{6,6}\}$

Notation: $y/S \in \mathrm{Dy}^m$ and $y^S \in \mathcal{D}^m$

- y/S: keep the part of y not in S:
  $y/S_1 = (y_1, y_2, y_3, \widehat{y_4}, y_5, y_6) = (2, 4, 4, 3, 5)$
  $y/S_2 = (y_1, \widehat{y_2, y_3, y_4}, y_5, \widehat{y_6}) = (y_1, y_5) = (2, 3)$.
- y^S: take the remaining parts and multiply them:
  $y^{S_1} = (2)$
  $y^{S_2} = (\tilde{y}_2, \tilde{y}_3, \tilde{y}_4) * (\tilde{y}_6) = (2, 2, 4) * (2)$

The results

Theorem

The free $\mathcal{D}yck^m$-algebra $\mathcal{D}^m$ is a $\mathcal{D}yck^m$-bialgebra, with

$$\overline{\Delta}(y) = \sum_{S \in Ad(y)} y^S \otimes y/S.$$

Consequences

$\rightarrow$ $\mathcal{D}yck^m$ is a Hopf operad
$\rightarrow$ $\mathrm{Prim}(\mathcal{D}yck^m)$ is an operad;
equivalently for B a $\mathcal{D}yck^m$-bialgebra, $\mathrm{Prim}(B)$ is a $\mathrm{Prim}(\mathcal{D}yck^m)$-algebra.

Can we say more?

Section 6– Poincaré-Birkhoff-Witt Theorem and Cartier-Milnor-Moore-Quillen Theorem

Here $\mathbb{K}$ is a field of caracteristic 0.

- Case of $\mathcal{L}\mathrm{ie}$-algebras
- Case of $\mathcal{B}\mathrm{race}$-algebras
- $\mathcal{B}\mathrm{race}^m$-algebras
- Prime m-Dyck paths and binary planar rooted trees
- A formula

The case of Lie algebras

$$\mathcal{L}\mathrm{ie}_{\mathrm{alg}} \underset{[-,-]}{\overset{U}{\rightleftarrows}} \mathcal{A}\mathrm{s}_{\mathrm{alg}} \quad (\perp)$$

PBW theorem: as a vector space $U(L) \cong S(L)$.

$$U(\mathbb{L}(V)) = T(V) = (\mathcal{C}\mathrm{om} \circ \mathcal{L}\mathrm{ie})(V)$$

Sanity check

$f_{\mathcal{L}\mathrm{ie}} = -\ln(1-x)$, $f_{\mathcal{A}\mathrm{s}} = \frac{x}{1-x}$, $f_{\mathcal{C}\mathrm{om}} = e^x - 1$
Hence generating function of $\mathcal{C}\mathrm{om} \circ \mathcal{L}\mathrm{ie}$ is

$$e^{-\ln(1-x)} - 1 = \frac{x}{1-x}$$

The case of Lie algebras

$$\mathcal{L}\mathrm{ie}_{\mathrm{alg}} \underset{[-,-]}{\overset{U}{\rightleftarrows}} \mathcal{A}\mathrm{s}_{\mathrm{alg}} \quad (\perp)$$

Cartier-Milnor-Moore-Quillen Theorem

The adjunction restricts to an adjunction

$$\mathcal{L}\mathrm{ie}_{\mathrm{alg}} \underset{\mathrm{Prim}}{\overset{U}{\rightleftarrows}} \mathcal{A}\mathrm{s}_{\mathrm{big}} \quad (\perp)$$

which is an equivalence of categories

Definition

A **brace** algebra is a vector space B equipped with operations $M_{1n} : B^{\otimes n+1} \longrightarrow B$ for $n \geq 0$, satisfying $M_{10} = \mathrm{Id}$ and

$$M_{1n}(M_{1r}(x; y_1, \dots, y_r); z_1, \dots, z_n) = \sum M_{1u}(x; z_{(1)}, M_{1,a_2}(y_1; z_{(2)}), z_{(3)}, \dots, z_{(2r-1)}, M_{1,a_{2r}}(y_k, z_{(2r)}), z_{(2r+1)}),$$

where the sum is taking over all the words (possibly empty) such that the concatenation $z_{(1)} \dots . z_{(2r+1)} = z_1 \dots z_n$.

Example

$M_{11}(M_{11}(x; y_1); z_1) = M_{11}(x, M_{11}(y_1; z_1)) + M_{12}(x; y_1, z_1) + M_{12}(x; z_1, y_1)$.
It is in particular right symmetric (pre-Lie), thus a Lie algebra.

PBW et CMMQ for brace algebras, results of Ronco and Chapoton

$$\mathrm{Br}_{alg} \underset{M}{\overset{U}{\rightleftarrows}} \mathcal{D}\mathrm{end}_{\mathrm{alg}} \quad (U \dashv M)$$

PBW

$U(B) \cong T(B)$, for B a brace algebra.
Sanity check: $T(Br(V)) \cong Dend(V)$.

CMMQ

$$\mathrm{Br}_{alg} \underset{\mathrm{Prim}}{\overset{U}{\rightleftarrows}} \mathcal{D}\mathrm{end}_{\mathrm{big}} \quad (U \dashv \mathrm{Prim})$$

is an equivalence of categories.

Theorem (L., Ronco)

- *There is an operad* $\mathcal{B}\mathrm{race}^m$ *such that any* $\mathcal{D}yck^m$*-algebra has an underlying* $\mathcal{B}\mathrm{race}^m$*-algebra structure yielding an adjunction:*

$$\mathcal{B}\mathrm{race}^m_{\mathrm{alg}} \underset{M}{\overset{U\mathcal{D}yck}{\rightleftarrows}} \mathcal{D}yck^m_{\mathrm{alg}} \quad (\perp)$$

- *For* B *a* $\mathcal{D}yck^m$*-bialgebra,* $\mathrm{Prim}(B)$ *is a sub-*$\mathcal{B}\mathrm{race}^m$*-algebra of* B*.*
- $U\mathcal{D}yck(A)$ *is a* $\mathcal{D}yck^m$*-bialgebra and* $U\mathcal{D}yck(A) \simeq T(A)$*.*
- *The adjiunction restricts to an equivalence of categories*

$$\mathcal{B}\mathrm{race}^m_{\mathrm{alg}} \underset{\mathrm{Prim}}{\overset{U\mathcal{D}yck}{\rightleftarrows}} \mathcal{D}yck^m_{\mathrm{big}} \quad (\perp)$$

Sanity check: $\mathcal{A}\mathrm{s} \circ \mathcal{B}\mathrm{race}^m \simeq \mathcal{D}yck^m$

$\mathcal{B}\mathrm{race}^m(n)$ has a basis indexed by $\mathrm{PDy}^m_n \times \Sigma_n$.

Definition

A $\mathcal{B}\mathrm{race}^m$-algebra is a brace algebra B equipped with operations $\bullet_i : B^{\otimes 2} \to B$ for $1 \leq i \leq m-1$, satisfying some relations

- Quadratic relations involving $\bullet_i$ and $\bullet_j$ for $1 \leq i < j < m$

$$(x \bullet_i y) \bullet_j z - (x \bullet_j y) \bullet_i z = x \bullet_i (y \bullet_j z) - x \bullet_j (y \bullet_i z)$$

- Relations of the form $M_{1n}(x \bullet_i y; z_1, \dots, z_n) = \dots$

Proposition

We have an explicit description of a $\mathcal{B}\mathrm{race}^m$-algebra on

$$\oplus_n \mathbb{K}[\mathrm{PDy}_n^m] \otimes V^{\otimes n}$$

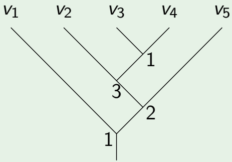
Thanks for your attention!

A bijection

between prime m-Dyck paths and some labelled rooted planar binary trees

There is a bijection $\mathrm{PDy}_n^m \to Y_n^m$, where Y_n^m is a subset of planar binary trees with n leaves and vertices labelled by elements in $\{1, \ldots, m\}$.
$s \in Y_n^m \iff \forall t \subset s, t = t^l \vee_i t^r$, the root of t^l has label $\geq i$.

Example, for $m = 3$



$\mapsto$ $(2) \times_1 [((2) \times_3 ((2) \times_1 (2))) \times_2 (2)] = (2) \times_1 (2565) = (23676)$