On reversible solutions of SPDEs
PIMS 4/7/2005

Aim: study scaling limits of evolving interfaces (see [Giacomin,Lebowitz,Presutti] or Funaki's St-Flour Lectures)

Equilibrium: Gibbs measure, RW model for interface: $V \in C^2(\mathbb{R})$ even

$$S_n = X_1 + \cdots + X_n, \qquad (X_i)_i \quad \text{IID}$$

$$X_i \sim \frac{1}{Z} \exp(-V(r))\, dr, \quad 0 < c_- < V'' < c_+ < \infty$$

$$\mathbb{E}[X_i] = 0, \qquad \mathbb{E}[X_i^2] = 1$$

Then the interface is $\{(n, S_n) : n \leq N+1\}$ **conditioned** on

$$\{S_1, \ldots, S_N \geq 0, \; S_0 = S_{N+1} = 0\}$$

We call μ_N^+ the law:

$$\mu_N^+(d\phi) = \frac{1}{Z_N^+} 1_{(\phi \geq 0)} \, e^{-\sum_i V(\phi_{i+1} - \phi_i)} \, d\phi$$

with $\phi_0 = \phi_{N+1} = 0$.

Under Brownian rescaling:

$$Y_t = \frac{1}{\sqrt{N}} S_{\lfloor Nt \rfloor} \implies e_t$$

where e is the normalized Brownian excursion.
Notation:

$$\nu_N := Y \circ \mu_N^+ \implies \nu$$

law of e

Natural reversible dynamics: $\phi_t \in \mathbb{R}_+^N$,

$$(\partial\phi)_i := \phi_{i+1} - \phi_i, \quad (\partial^*\phi)_i := \phi_i - \phi_{i-1}$$

$$\begin{cases} d\phi_i = \frac{1}{2}\{\partial V'(\partial^*\phi)\}_i \, dt + dw_i + dl_i \\ \phi_0(t) = \phi_{N+1}(t) = 0, \\ \phi_i \geq 0, \quad dl_i \geq 0, \quad \int_0^\infty \phi_i(t) \; dl_i(t) = 0 \end{cases}$$

$\exists!$ stationary ϕ. Funaki-Olla [SPA 01]:

$$\Phi_N(t,x) := \frac{1}{\sqrt{N}} \sum_i \phi_x(N^2 t) \, 1_{[\frac{i-1}{N}, \frac{i}{N})}(x)$$

$\Longrightarrow \Phi_N \to$ unique stationary solution of ...

Nualart and Pardoux [PTRF 92]: $\exists!(u,\eta)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dot{W} + \eta(t,x) \\ u(0,x) = u_0(x), \quad x \in [0,1] \\ u(t,0) = u(t,1) = 0, \quad t \geq 0 \\ u \geq 0,\ d\eta \geq 0,\ \int u\, d\eta = 0 \end{cases}$$

- $\dot{W}$ space-time white noise
- η is a reflecting measure on $\mathbb{R}_+ \times [0,1]$

Existence: by penalization

$$\varepsilon > 0 : \ \frac{\partial u_\varepsilon}{\partial t} = \frac{1}{2}\frac{\partial^2 u_\varepsilon}{\partial x^2} + \dot{W} + \frac{(u_\varepsilon)^-}{\varepsilon}$$

Then $u_\varepsilon \uparrow u$ uniformly as $\varepsilon \downarrow 0$.

Results on the contact set $\{(t,x) : u(t,x)\}$.

New proof of Funaki-Olla's result, based on three main properties:

1. trivial convergence of Dirichlet forms

2. uniform strong Feller property for Φ_N:

$$|P_t^N F(\Phi) - P_t^N F(\Phi')| \leq \frac{\|F\|_\infty}{t^{1/2}} \|\Phi - \Phi'\|_{L^2(0,1)}$$

$$P_t^N F(\Phi) := \mathbb{E}_\Phi[F(\Phi_N(t))]$$

3. convergence of integration by parts formulae (IbPF) for ν_N to IbPF for ν

Reflecting Brownian Motion:

$$dX = dB + dL,$$

$$X \geq 0, \quad dL \geq 0, \quad \int_0^\infty X\, dL = 0$$

associated to the Dirichlet form:

$$D(\varphi, \psi) = \frac{1}{2} \int_0^\infty \varphi'\, \psi'\, dx.$$

Informally: $L_t = \dfrac{1}{2} \displaystyle\int_0^t \delta_0(X_s)\, ds$. IbPF:

$$\int_0^\infty \varphi'\, dx = -\varphi(0) = -\delta_0(\varphi), \qquad \varphi \in C_c^1(\mathbb{R})$$

The Dirac Delta is the Revuz measure of L.

Back to interfaces: now we want the dynamics to be conservative, i.e.

$$\sum_i \phi_i(t) = \sum_i \phi_i(0) \qquad \forall t \geq 0$$

(constant droplet volume). The natural dynamics is:

$$\begin{cases} d\phi = -\frac{1}{2}\partial\partial^*\{\partial V'(\partial^*\phi)\,dt + dl_i\} + \partial dw \\ \phi_0(t) = \phi_{N+1}(t) = 0, \\ \phi_i \geq 0, \quad dl_i \geq 0, \quad \int_0^\infty \phi_i(t)\ dl_i(t) = 0 \end{cases}$$

$\exists!$ stationary ϕ after fixing the droplet volume.

$$\Phi_N(t,x) := \frac{1}{\sqrt{N}} \sum_i \phi_x(N^4 t)\, 1_{[\frac{i-1}{N},\frac{i}{N})}(x)$$

$\Phi_N \to$ unique stationary solution of ...

Stochastic Cahn-Hilliard equation: (joint with A.Debussche)

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial^2 u}{\partial x^2} + \eta(t,x)\right) + \dfrac{\partial}{\partial x}\dot{W} \\ u(t,0) = u(t,1) = 0, \\ \partial^3 u(t,0) = \partial^3 u(t,1) = 0, \\ u \geq 0, \ d\eta \geq 0, \ \int u \, d\eta = 0 \end{cases}$$

Difficulty even for existence: try penalization

$$\frac{\partial u^\epsilon}{\partial t} = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u^\epsilon}{\partial x^2} + \frac{(u^\epsilon)^-}{\epsilon}\right) + \frac{\partial}{\partial x}\dot{W}$$

no monotonicity; tightness trivial known only for stationary solutions

the 1.-2.-3. proof above gives convergence of the semigroups and Strong Feller for the limit, and this is enough to conclude

Also, the 1.-2.-3. proof gives convergence of the stationary interface.

at present, we have a problem with uniqueness for CH: we expect pathwise uniqueness (easy with different boundary conditions, difficult here because the mass of η is expected to be $+\infty$)

the Cahn-Hilliard equation is a gradient system in $H^{-1}(0,1)$: in this norm localization works very badly

Last model: at equilibrium we have pinning, i.e. the interface gets a reward $\epsilon > 0$ every time it touches the wall. the measure is $\mu^{+}_{\epsilon,N}$, the Brownian rescaling $\nu_{\epsilon,N}$

by tuning ϵ, we have a phase transition (joint with Deuschel, Giacomin):

- $\epsilon > \epsilon_c$: convergence to flat interface $\equiv 0$

- $\epsilon < \epsilon_c$: $\nu_{\epsilon,N} \Longrightarrow \nu$ (as if $\epsilon = 0$)

- $\epsilon = \epsilon_c$: $\nu_{\epsilon,N} \Longrightarrow$ law of reflecting Brownian bridge

The natural dynamics has **sticky** reflection:

$$\begin{cases} d\phi_i = 1_{(\phi_i(t)>0)} \left[\frac{1}{2}\{\partial V'(\partial^*\phi)\}_i \, dt + dw_i\right] \\ \qquad\qquad + \frac{1}{2\epsilon} 1_{(\phi_i(t)=0)} \, dt \\ \phi_0(t) = \phi_{N+1}(t) = 0, \end{cases}$$

$\exists$ stationary ϕ (unique?).

$$\Phi_N(t,x) := \frac{1}{\sqrt{N}} \sum_i \phi_x(N^2 t) \, 1_{[\frac{i-1}{N},\frac{i}{N})}(x)$$

at $\epsilon = \epsilon_c$, $\Longrightarrow \Phi_N \to$ unique stationary solution of ...

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dot{W} - : \left|\dfrac{\partial u}{\partial x}\right|^2 : \eta(t,x) \\ u(0,x) = u_0(x), \qquad x \in [0,1] \\ u(t,0) = u(t,1) = 0, \qquad t \geq 0 \\ u \geq 0,\ d\eta \geq 0,\ \int u \, d\eta = 0 \end{cases}$$

$$: \left|\frac{\partial u}{\partial x}\right|^2 := \lim_{\epsilon} \left[\left|\frac{\partial u_\epsilon}{\partial x}\right|^2 - C_\epsilon \right]$$

This renormalized term reminds of the KPZ equation, but this one has **reversible** solutions!

Done: IbPF for the reflecting Brownian motion