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controlled rough paths: a general Hopf-algebraic setting

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I] Introduction: Banach's fixed point method and Cauchy–Lipschitz theorem.

Let (E, d) be a complete metric space, and let $\mathcal{M}: E \longrightarrow E$. The map $\mathcal{M}$ is contracting if there exists $k < 1$ such that

$$d(\mathcal{M}(x), \mathcal{M}(y)) \leq k\, d(x, y)$$

for any $x, y \in E$. In that case, $\mathcal{M}$ admits a unique fixed point $l \in E$, obtained as $l = \lim_{k \to +\infty} \mathcal{M}^k(x)$ for any choice of $x \in E$.

Let us now consider the following initial value problem on a smooth manifold M of dimension e:

$$(1) \quad \begin{cases} \dot{Z}_t = \underbrace{\sum_{i=1}^{d} \varphi^i(Z_t)\, \dot{X}^i_t}_{:= f(t, Z_t)} \\ Z_0 \text{ given} \end{cases}$$

Here
- $X = (X^1, \ldots, X^d)$ is a path $[0, T] \longrightarrow \mathbb{R}^d$, supposed to be C^1 for the moment.
- $\varphi^1, \ldots, \varphi^d$ vector fields on M.

We shall suppose $M = \mathbb{R}^e$. Smooth vector fields are just given by $\mathcal{C}^\infty(\mathbb{R}^e, \mathbb{R}^e)$. The two integers d and e have nothing in common.

One can remark that $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is Lipschitz, namely

$$\|f(t', z') - f(t, z)\| \leq C \cdot (\|z' - z\| + |t' - t|)$$ for

some positive constant C.

Equation (1) is equivalent to the integral form

$$(1') \quad Z = \mathcal{M}_T Z$$

Here $E_T = \mathcal{C}([0,T], \mathbb{R}^e)$ endowed with the norm

$$\|\varphi\| = \sup_{t \in [0,T]} \|\varphi(t)\|_{\mathbb{R}^e} \quad \text{(it is a Banach space)}$$

and $$\boxed{\mathcal{M}_T Z(t) := Z_0 + \int_0^t f(u, Z_u)\, du.}$$

Let $E_{T,Z_0} = \{Z \in E_T,\ Z(0) = Z_0\}$.

Theorem (Cauchy–Lipschitz)

$\mathcal{M}_T : E_{T,Z_0} \to E_{T,Z_0}$ is contracting for $T < \frac{1}{C}$

Proof: Let $Z, \hat{Z} \in E_{T,Z_0}$.

$$\|\mathcal{M}_T(Z - \hat{Z})(t)\|_{\mathbb{R}^e} \leq \int_0^t \|f(u, Z_u) - f(u, \hat{Z}_u)\|\, du \leq t C \|Z - \hat{Z}\|, \text{ hence}$$

$$\|\mathcal{M}_T Z - \mathcal{M}_T \hat{Z}\| \leq T C \|Z - \hat{Z}\|.$$ ∎

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We are interested in the case when $X : [0,T] \to \mathbb{R}^d$ is not $\mathcal{C}^1$, for example X continuous γ-Hölder with $\gamma \in]0,1]$. Other situations including discontinuities (jumps) can be considered.

$$(1'') \quad z_t = z_0 + \sum_{i=1}^{d} \int_0^t \varphi^i(z_r)\, dX_r^i \quad = \text{“} \dot{X}(t)\, dt \text{''}$$

- Give a precise meaning to $(1'')$,
- Solve it (non-unique solution! Cauchy–Lipschitz fails).

Main idea: lift $(1'')$ into

$$(2) \quad \mathbf{Z}_t = \mathbf{Z}_0 + \sum_{i=1}^{d} \int_0^t \varphi^i(\mathbf{Z}_r)\, d\mathbf{X}_r^i$$

where
- $\mathbf{X}$ is a rough path over X
- The unknown $\mathbf{Z}$ lives in a Banach space $\mathcal{V}^{\mathbf{X}}_{[0,T]}$ of controlled rough paths driven by $\mathbf{X}$,
- $\varphi(\mathbf{Z})$ is defined from a Taylor-like formula involving $\varphi, D\varphi, \dots, D^N\varphi$ (N = truncation index)
- $\int_0^t \text{—}\, d\mathbf{X}_r^i$ is a rough integral operator, sharing essential properties with the Riemann integral.

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One solves (2) in $\mathcal{V}^{\mathbf{X}}_{[0,T]}$ by fixed point method again.

$\Rightarrow$ one unique solution $\mathbf{Z}$ ... but this solution depends on the choice of a rough path $\mathbf{X}$ above X ... and there are plenty of them [Friz–Victoir, Tapia–Zambotti].

II] Truncated Hopf algebras

Def: Let N be a positive integer. A graded vector space

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

is a N-truncated Hopf algebra if

- $\mathcal{H}_n = \{0\}$ for any $n \geq N+1$,
- $\dim \mathcal{H}_n < +\infty$, and $\dim \mathcal{H}_0 = 1$ (connectedness)
- $\mu : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$ graded associative
- $\Delta : \mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$ graded coassociative
- $u : \mathbb{R} \to \mathcal{H}$ and $\varepsilon : \mathcal{H} \to \mathbb{R}$ graded unit and co-unit resp.
- $\Delta\circ\mu - (\mu\otimes\mu)\tau_{23}(\Delta\otimes\Delta) : \mathcal{H}\otimes\mathcal{H} \longrightarrow \mathcal{H}\otimes\mathcal{H}$ takes values in $(\mathcal{H}\otimes\mathcal{H})_{\geq N+1} = \bigoplus_{p+q\geq N+1} \mathcal{H}_p\otimes\mathcal{H}_q$.

Proposition [ZGLM]

If $\mathcal{H}$ is a N-truncated Hopf algebra, $\mathcal{H}^*$ as well.

⚠ A N-truncated Hopf algebra is NOT a Hopf algebra!

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III] $\mathcal{H}$-rough paths

Let $T>0$, let $\Delta_T := \{(s,t) \in [0,T]^2,\ s \le t\}$.

Def. (FRIZ – H. Zhang) $\omega : \Delta_T \to [0,+\infty[$ is a control if

$$\begin{cases} \omega(t,t) = 0 & \forall t \in [0,T] \\ \omega(s,u) + \omega(u,t) \le \omega(s,t) & \text{(subadditivity)}. \end{cases}$$

ex: $\omega(s,t) = t-s$

right continuity assumption:

$$\omega(s,t+) = \omega(s,t)$$

$$:= \lim_{\varepsilon \downarrow 0} \omega(s, t+\varepsilon)$$

Let $X : [0,T] \to \mathbb{R}^d$ such that

$$\| X_t - X_s \| \le C\, \omega(s,t)^\gamma$$

with ω control and $\gamma \in]0,1]$, and let $\mathcal{H}$ be a N-truncated Hopf algebra, with $N = \lfloor \frac{1}{\gamma} \rfloor$.

Def. A $\mathcal{H}$-roughpath above X is a two-parameter family $\mathbb{X} = (\mathbb{X}_{st})_{s,t \in [0,T]}$ of truncated characters of $\mathcal{H}$ (namely, $\mathbb{X}_{st}(xy) = \mathbb{X}_{st}(x)\, \mathbb{X}_{st}(y)$ whenever $|x|+|y| \le N$) such that

- $\mathbb{X}_{su} \star \mathbb{X}_{ut} = \mathbb{X}_{st}$ (Chen's lemma),
- $|\langle \mathbb{X}_{st}, x \rangle| \le C \cdot \|x\|_{\mathcal{H}}\, \omega(s,t)^{|x|\gamma}$

- $\mathcal{H} = (\mathcal{H}^A_{Ш})_{\le N}$ T. Lyons' rough paths (weakly geometric)
- $\mathcal{H} = (\mathcal{H}^A_{BCK})_{\le N}$ M. Gubinelli's branched RPs
- $\mathcal{H} = (\mathcal{H}^A_{MKW})_{\le N}$ planarly branched RPs ([CEMM 2020])
- $\mathcal{H} = (\mathcal{H}^A_{LOT})_{\le N}$ Multi-index RPs ...

V] Controlled rough paths (M. Gubinelli 2004/2008)

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$$\mathbf{Z} = (Z^1, \dots, Z^e) : [0,T] \to \mathcal{H}^e$$

$\mathcal{H}$ N-truncated Hopf alg.

Given $\mathbb{X}$ above X, $\mathbf{Z}$ is a controlled rough path driven by $\mathbb{X}$ if, $\forall h \in \mathcal{H}^*$,

$$\| \langle h, R_{\mathbb{X}}(\mathbf{Z})_{st} \rangle \|_{\mathbb{R}^e} \leq C \|h\|_{\mathcal{H}^*} \, \omega(s,t)^{(N-|h|)\gamma}$$

with

$$\langle h, R_{\mathbb{X}}(\mathbf{Z})_{st} \rangle := \langle h, \mathbf{Z}_t \rangle - \langle \mathbb{X}_{st} \star h, \mathbf{Z}_s \rangle$$

$\mapsto$ Banach space $\mathcal{D}^{\mathbb{X}}_{[0,T]}$ with norm

$$\|\mathbf{Z}\| := \|\mathbf{Z}_0\|_{\mathcal{H}^e} + \sup_{\substack{j=0 \dots N \\ s,t \in \Delta_{[0,T]}}} \| \pi_j (R_{\mathbb{X}}(\mathbf{Z}))_{st} \|_{\mathcal{H}^e} \, \omega(s,t)^{(N-j)\gamma}$$

$\pi_j : \mathcal{H}^e \to \mathcal{H}_j^e$

Image by $\varphi \in C^\infty(\mathbb{R}^e, \mathbb{R}^e)$:

$$\langle h, \varphi(\mathbf{Z})_s \rangle := \sum_{n=0}^{N} \frac{D^n \varphi(Z_s)}{n!} \langle \Delta^{(n-1)}(h), (\mathbf{Z}_s - Z_s \mathbb{1})^{\otimes n} \rangle$$

Integral:

necessitates a family of cocycles $L_i : \mathcal{H} \to \mathcal{H}$ $\quad i = 1 \dots d$

(example: $L_i(F) = B_i^+(F) =$ for rooted forests) $\quad \boxed{\Delta L_i(x) = (I \otimes L_i)(\Delta x) + L_i(x) \otimes \mathbb{1}}$

$$\int_0^t \mathbf{Z}_r \, d\mathbb{X}^i_r := L_i(\mathbf{Z}_t) + \underbrace{\left(\int_0^t Z_r \, d\mathbb{X}^i_r \right)}_{\text{rough integral}} \mathbb{1}$$

Generalised sewing lemma (P Friz, H-Zhang 2018)

Let $\Xi = (\Xi^1, \dots, \Xi^e) : \Delta_T \to \mathbb{R}^e$ such that

$$\|\Xi_{st} - \Xi_{su} - \Xi_{ut}\| \le \sum_{j=1}^{r} \omega(s,u)^{j\delta}\, \omega(u,t)^{(r+1-j)\delta}.$$

Then $\boxed{I\Xi_{st} := \boxed{\text{RRS-}\lim_{\substack{P \text{ partition} \\ \text{of } [s,t]}}} \sum_{(u,v)\in P} \Xi_{uv}}$ exists, and

verifies $\quad I\Xi_{st} = I\Xi_{su} + I\Xi_{ut}$.

One checks that $\Xi^i_{st} := \langle \mathbf{X}_{st}, L_i(Z_s)\rangle$ verifies the condition of the generalised sewing lemma.

$\int_0^t Z_r \, d\mathbf{X}^i_r := I\Xi^i_{st}$ with Ξ^i_{st} as above.

$$\mathcal{M}_T Z(t) := Z_0 + \sum_{i=1}^{d} \int_0^t \varphi_i(Z)\, d\mathbf{X}^i_r$$

<u>Theorem</u> [Zhu, Luo, Li, Manderson 2025]

- If ω is right-continuous, $\mathcal{M}_T$ is contracting in $\mathcal{Y}^{\mathbf{X}}_{[0,T],Z_0}$ for T small enough
  $\Rightarrow$ unique solution, depending on the choice of $\mathbf{X}$ above x,
- the solution is <u>robust</u> under small changes of
  $\to$ the vector fields $\varphi^1, \dots, \varphi^d$,
  $\to$ the initial condition Z_0,
  $\to$ the driving path $\mathbf{X}$.

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