Discretisations of stochastic Allen-Cahn equations

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Singular SPDEs, invariant measures and discrete models

Joint works with Ana Djurdjevac (FU Berlin), Helena Kremp (TU Wien), Harprit Singh (University of Edinburgh)

Main example: stochastic Allen-Cahn equation

$$(\partial_t - \Delta)u = u - u^3 + \xi$$

on $[0,1] \times \mathbb{T}$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, where ξ is space-time white noise.

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Some variations:

- Replace $u - u^3$ by $f(u)$, where f is nice (e.g. globally Lipschitz)
- Replace $u - u^3$ by $P(u)$, where P is a polynomial of odd degree with negative leading order coefficient
- Replace $\mathbb{T}$ by $\mathbb{T}^d$

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Reference object, 3: The solution of the linear equation

$$(\partial_t - \Delta)\Psi = \xi$$

is $1/4 - \varepsilon$-Hölder continuous in time, $1/2 - \varepsilon$-Hölder continuous in space for any $\varepsilon > 0$.
In dimensions 2 and higher Ψ itself is a distribution.

Our focus: Discretisation, error estimates.

- Spatial scheme on scale N^{-1}
- Temporal scheme on scale M^{-1}
- Random variables sampled

Disclaimer:

- Will not comment on initial condition
- Will drop any ε-s in rates of convergence

[Gyöngy '99]: Approximation $v^{M,N}$ by

- Finite differences in space
- Finite differences in time
- Random variables from rectangular increments of W

$$\begin{aligned}\Xi_{i,j}^{N,M} &:= \xi\Big(\big[\tfrac{i}{M},\tfrac{i+1}{M}\big]\times\big[\tfrac{j}{N},\tfrac{j+1}{N}\big]\Big)\\ &= W_{\frac{i+1}{M},\frac{j+1}{N}} - W_{\frac{i+1}{M},\frac{j}{N}} - W_{\frac{i}{M},\frac{j+1}{N}} + W_{\frac{i}{M},\frac{j}{N}}.\end{aligned}$$

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The scheme:

$$\begin{aligned}M\Big(v^{N,M}\big(\frac{i+1}{M}, \frac{j}{N}\big) - v^{N,M}\big(\frac{i}{M}, \frac{j}{N}\big)\Big) &= f\Big(v^{N,M}\big(\frac{i}{M}, \frac{j}{N}\big)\Big) + MN\Xi_{i,j}^{N,M} \\ + N^2\Big(v^{N,M}\big(\frac{i}{M}, \frac{j+1}{N}\big) + v^{N,M}\big(\frac{i}{M}, \frac{j-1}{N}\big) - 2v^{N,M}\big(\frac{i}{M}, \frac{j}{N}\big)\Big)\end{aligned}$$

Theorem (Gyöngy '99)

Let $p \geq 2$. If $M^{-1} < (1/2)N^{-2}$ and f is globally Lipschitz continuous, then there exists a constant C such that for all M, N

$$\sup_{i,j} \left(\mathbb{E} \Big(u\big(\tfrac{i}{M}, \tfrac{j}{N}\big) - v^{N,M}\big(\tfrac{i}{M}, \tfrac{j}{N}\big) \Big)^p \right)^{1/p} \leq C(M^{-1/4} + N^{-1/2}).$$

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Theorem (Davie-Gaines '01)

Let $R_1, \ldots R_{NM}$ be the rectangles of the grid with meshsize M^{-1}, N^{-1}. There exists a constant $c > 0$ such that for all M, N

$$\inf_{\varphi \text{ meas.}} \left(\mathbf{E}\left(\Psi(1,0) - \varphi(\xi(R_{11}), \ldots, \xi(R_{NM}))^2\right)^{1/2} \geq c(M^{-1/4} + N^{-1/2}).$$

[Becker-Gess-Jentzen-Kloeden '23]

- Spectral Galerkin truncation in space
- Tamed exponential Euler scheme in time
- Random variables from the Wiener increments on each Fourier mode of the noise:
$$B^k_{\frac{i+1}{M}} - B^k_{\frac{i}{M}}$$

Same upper and lower bounds hold for $f(u) = u - u^3$.

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Tangential analogy: For SDEs

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t$$

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The previous discretisations "treat additive noise as multiplicative" because the stochastic integral Ψ is *approximated*.

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- *Conjectured**: [Davie-Gaines '01], [Jentzen-Kloeden '08]
- *Realistic*: $\mathcal{F}\Psi(k)$ on the (temporal) grid are still Gaussian with known covariance
- *Proved* to provide *some* improvement: from temporal rate $1/4$ to $1/2$ [Jentzen '11], [Wang '20]

Combining [Bréhier-Cui-Hong '18] and [Jentzen-Kloeden '08], consider:

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The scheme:

$$\begin{aligned} v^{M,N}(\frac{i+1}{M},\cdot) &= \Pi_N P_{M^{-1}} \Phi\Big(v^{M,N}(\frac{i}{M},\cdot)\Big) \\ &\quad + \Pi_N\Big(\Psi(\frac{i+1}{M},\cdot) - P_{M^{-1}}\Psi(\frac{i}{M},\cdot)\Big), \end{aligned}$$

where P is the heat semigroup, Π_N is the projection on the first N Fourier modes, and Φ is the solution flow of the ODE $\dot{x} = f(x)$.

Theorem (Djurdjevac-G-Kremp '24)

Let $p \geq 2$, let f have polynomially growing derivatives up to order 3 and bounded from above first derivative. Then there exists a constant C such that for all M, N

$$\Big(\mathbb{E}\sup_{i,j}\Big(u\big(\tfrac{i}{M},\tfrac{j}{N}\big) - v^{N,M}\big(\tfrac{i}{M},\tfrac{j}{N}\big)\Big)^p\Big)^{1/p} \leq C(M^{-1} + N^{-1/2}).$$

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Proposition (Djurdjevac-G-Kremp '24)

Let u be the solution with $f(x) = x$. There exists a constant $c > 0$ such that for all M, N

$$\inf_{\varphi\,\mathrm{meas.}} \left(\mathbf{E}\left(u(1,0) - \varphi(\hat{\Psi}(\frac{1}{M},0),\dots,\hat{\Psi}(1,N)\right)^2\right)^{1/2} \geq c(M^{-1} + N^{-1/2}).$$

The bottleneck is the quadrature error

$$E_M := \Big| \int_0^T P_{T-s}\big(f(\Psi_s) - f(\Psi_{k_M(s)})\big)\, ds\Big|.$$

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- Triangle inequality, regularity of Ψ: $E_M \lesssim M^{-1/4}$
- Triangle inequality, regularity of $\mathcal{F}\Psi(k)$: $E_M \lesssim M^{-1/2}$

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- Stochastic sewing [Lê '20]: if X is fBM with $H = 1/4$, then

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- second+third point: $E_M \lesssim M^{-1}$

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- *Well-motivated*: C^α for $\alpha < 0$ are the natural solution spaces in higher dimensions

  And the rate improves with lowering α: [Hairer '14] for mollifier approximations, [Ma-Zhu '21], [Ma-Wang-Yang '24] for discrete approximations in $d = 2$
- *Promising*: although Ψ is $1/4$ Hölder in time one has for all $\alpha \in (-1/2, 1/2)$, $\varepsilon > 0$

$$\|\Psi_t - \Psi_s\|_{C^\alpha(\mathbb{T})} \lesssim |t - s|^{1/4-\alpha/2-\varepsilon}.$$

- *Problematic*: in distributional spaces u^3 is not defined (and we have no renormalisation!)

Setup and assumptions:

- Scheme exactly as in [Gyöngy '99]
- Temporal scale cN^{-2}, $c < 1/8$, spatial scale N^{-1}.
- Nonlinearity $f(u) = -u^k + P(u)$, where k odd and P is polynomial of order $k-1$.

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Theorem (G-Singh '22)

Let $\theta \in (-1/2, 0]$. Then there exists an almost surely finite random variable η such that for all $n \in \mathbb{N}$

$$\sup_i \| u_{\frac{i}{N^2}} - v^N_{\frac{i}{N^2}} \|_{C^\theta(\mathbb{T}_N)} \leq \eta N^{-1/2+\theta}.$$

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Lower bound of order N^{-1} holds already for a fixed Fourier mode.

The rate one could expect from the general theory is

$$\text{true regularity} \;-\; \text{critical regularity}$$

E.g. for Φ^4_3: $-1/2 - (-1) = 1/2$
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But error and solution can be considered in different spaces:

$$\begin{aligned} \|u^3 - (v^N)^3\|_{C^{-1/2}} &= \|(u - v^N)(u^2 + (v^N)^2)\|_{C^{-1/2}} \\ &\lesssim \|u - v^N\|_{C^{-1/2}} \|u^2 + (v^N)^2\|_{C^{1/2}} \end{aligned}$$

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In higher dimensions this is done on the level of the remainders (in progress with Marco Cacace)

Thank you!