Construction of measure of fractional Φ^4_3 model in full subcritical regime

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(based on joint work with M. Gubinelli and P. Rinaldi)

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Plan of the talk

1. Fractional Φ^4_3 model
2. Idea behind the construction
3. Flow equation approach
4. Coercive estimate

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Constructive Euclidean quantum field theory

Given space Ω of field configurations $\phi : \mathbb{R}^d \to \mathbb{R}$ and action $S : \Omega \to \mathbb{R}$, e.g.

$$S(\phi) = \int_{\mathbb{R}^d} \left[\phi(x)(\mathcal{Q}\phi)(x) + \lambda\phi(x)^4\right] \mathrm{d}x, \qquad \mathcal{Q} = 1 - \Delta,$$

the goal of the constructive Euclidean QFT is to make sense of the probability measure on Ω formally given by

$$\nu(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}} \exp(-S(\phi)) \prod_{x\in\mathbb{R}^d} \mathrm{d}\phi(x).$$

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Osterwalder–Schrader axioms:

1. Invariance under Euclidean transformations of $\mathbb{R}^d$.
2. Reflection positivity: Let $(\theta\phi)(x_1, \ldots, x_d) = \phi(-x_1, x_2, \ldots, x_d)$. Then

$$\int \overline{F(\theta\phi)} F(\phi)\, \nu(\mathrm{d}\phi) \geqslant 0$$

   for all functionals $F(\phi)$ that depend only on $\phi|_{\{x_1>0\}}$.
3. Regularity: exponential integrability.

Constructions of interacting models

$$S_{\varepsilon,\tau}(\phi) = \varepsilon^d \sum_{x\in\mathbb{T}^d_{\varepsilon,\tau}} \left[\phi(x)(\mathcal{Q}_\varepsilon\phi)(x) + \lambda\phi(x)^4 - r_{\varepsilon,\tau}\,\phi(x)^2\right], \quad \mathcal{Q}_\varepsilon = 1 - \Delta_\varepsilon,$$

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Existence of continuum and infinite volume limit and OS axioms for Φ^4_d model:

- $d = 2$: [60': Nelson, Glimm, Jaffe, Segal, Guerra, Rosen, Simon, ...],
- $d = 3$: [70': Glimm, Jaffe, Feldman, Park, Osterwalder, Magnen, Senéor, ...].

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Triviality of ϕ^4 model – the continuum limit does not exist or is Gaussian:

- $d = 4$: [Aizenman, Duminil-Copin (2021)],
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Several other models were constructed in $d \leqslant 3$.

Only very special models are expected to exist in $d = 4$.

Action of fractional Φ^4_3 model with cutoffs:

$$S_{\varepsilon,\tau}(\phi) = \varepsilon^3 \sum_{x\in\mathbb{T}^3_{\varepsilon,\tau}} \left[\phi(x)(\mathcal{Q}_\varepsilon\phi)(x) + \tfrac{\lambda}{2}\phi(x)^4 - r_{\varepsilon,\tau}\,\phi(x)^2\right]$$

- $\mathbb{T}^3_{\varepsilon,\tau}$ – lattice with spacings $\varepsilon \in (0,1]$ and period $\tau \in \mathbb{N}_+$,
- $\mathcal{Q}_\varepsilon = (-\Delta_\varepsilon)^{\sigma/2} + 1$,
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Measure of fractional Φ^4_3 model with cutoffs $\varepsilon \in (0,1]$, $\tau \in \mathbb{N}_+$:

$$\nu_{\varepsilon,\tau}(\mathrm{d}\phi) := \frac{1}{\mathcal{Z}} \exp(-S_{\varepsilon,\tau}(\phi)) \prod_{x\in\mathbb{T}^3_{\varepsilon,\tau}} \mathrm{d}\phi(x).$$

We are interested in the limit $\varepsilon \searrow 0, \tau \to \infty$ of $\nu_{\varepsilon,\tau}$.

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$$\nu_\tau(\mathrm{d}\phi) = Z^{-1} \exp(-V(\phi))\, \mu_\tau(\mathrm{d}\phi),$$

- μ_τ is Gaussian measure with covariance $((-\Delta)^{\sigma/2} + 1)^{-1}$,
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Subcritical regime beyond Wick renormalization $\sigma \in (3/2, 9/4]$

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Critical and supercritical regime $\sigma \in (0, 3/2]$

Continuum limit $\lim_{\varepsilon \searrow 0} \nu_{\varepsilon,\tau}$ does not exist or is Gaussian [Panis (2023+)].

Main result

Recall that the measure $\nu_{\varepsilon,\tau}$ of the fractional Φ^4_3 model depends on:

- lattice spacing $\varepsilon \in (0,1]$ and size of the torus $\tau \in \mathbb{N}_+$,
- order of fractional Laplacian σ, mass counterterm $r_{\varepsilon,\tau}$.

Theorem [D., Gubinelli, Rinaldi (2024+)]

Assumptions: $\sigma \in (3/2, 2)$.

There exists a choice of mass counterterm $(r_{\varepsilon,\tau})_{\varepsilon\in(0,1],\tau\in\mathbb{N}_+}$ such that:

(1) the continuum/infinite volume limits $\nu := \lim_{\varepsilon\searrow 0}\lim_{\tau\to\infty}\nu_{\varepsilon,\tau}$ exist (along a subsequence),

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- Similar result [Esquivel, Weber (2024+)].
- In general uniqueness of measure not expected.
- Non-triviality – every accumulation point non-Gaussian.

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Parabolic stochastic quantization

Idea: establish the bound below and use Prokhorov's theorem

$$\sup_{\varepsilon\in(0,1],\tau\in\mathbb{N}_+}\int \|\phi\|_{\mathcal{B}}\,\nu_{\varepsilon,\tau}(\mathrm{d}\phi)<\infty.$$

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Langevin dynamic in finite dimension

Measure $\nu(\mathrm{d}\phi)=\exp(-2S(\phi))\mathrm{d}\phi$ over $\mathbb{R}^n$ is invariant under dynamic

$$\mathrm{d}\phi_t=\mathrm{d}W_t-\nabla S(\phi_t)\,\mathrm{d}t.$$

Dynamical fractional Φ^4_3 model on $\mathbb{R}\times\mathbb{T}^3_{\varepsilon,\tau}$

$$\big(\partial_t+(-\Delta_\varepsilon)^{\sigma/2}+1\big)\Phi_{\varepsilon,\tau}=\xi_{\varepsilon,\tau}-\lambda\Phi^3_{\varepsilon,\tau}+r_{\varepsilon,\tau}\,\Phi_{\varepsilon,\tau}$$

- Finite-dimensional SDE in a gradient form.
- Let $\Phi_{\varepsilon,\tau}$ be the global stationary solution.
- Then $\nu_{\varepsilon,\tau}=\mathrm{Law}(\Phi_{\varepsilon,\tau}(t,\bullet))$ for all $t\in\mathbb{R}$.

Main idea of the proof of tightness

- The following bound implies tightness

$$\sup_{\varepsilon\in(0,1],\tau\in\mathbb{N}_+} \mathbb{E}\|\Phi_{\varepsilon,\tau}(t,\bullet)\|_{\mathcal{B}} < \infty.$$

- $\Phi_{\varepsilon,\tau}$ satisfies a parabolic SPDE

$$\big(\partial_t + (-\Delta_\varepsilon)^{\sigma/2} + 1\big)\Phi_{\varepsilon,\tau} = \xi_{\varepsilon,\tau} - \lambda\Phi^3_{\varepsilon,\tau} + r_{\varepsilon,\tau}\,\Phi_{\varepsilon,\tau}$$

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Strategy

- Use flow equation approach to singular SPDEs to make sense of the equation in the continuum limit.
- Apply maximum principle to derive coercive estimate implying tightness.

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Stochastic quantization equation and coarse-grained process

Dynamical fractional Φ^4_3 model

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Notation:

- $F[\varphi] := \xi - \lambda\varphi^3 + r\,\varphi$ – force,
- ξ – spacetime white noise,
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Coarse-grained process

$$\varPhi_\mu := J_\mu * \varPhi \in C^\infty, \qquad \mu \in (0,1],$$

- $\varPhi$ – solution of the dynamical fractional Φ^4_3 model,
- J_μ – smooth approximation of Dirac delta of characteristic length scale μ.

Effective force

- In the limit $\varepsilon \searrow 0$ the dynamical Φ^4_3 model becomes a singular SPDE

$$\big(\partial_t + (-\Delta)^{\sigma/2} + 1\big)\mathit{\Phi} = F[\mathit{\Phi}], \qquad F[\varphi] := \xi - \lambda\varphi^3 + r\,\varphi\,.$$

- **Idea:** Rewrite the equation as a certain equation that involves only the *coarse-grained process* $(\mathit{\Phi}_{\mu})_{\mu\in(0,1]}$.

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- **Idea:** Rewrite the equation as a certain equation that involves only the *coarse-grained process* $(\Phi_\mu)_{\mu\in(0,1]}$.

$$\big(\partial_t + (-\Delta)^{\sigma/2} + 1\big)\Phi_\mu = J_\mu * F[\Phi]$$

Effective force

A family of functionals $F_\mu[\varphi]$ depending differentiably $\mu \in [0,1]$ such that:

- the boundary condition $F_{\mu=0}[\varphi] = F[\varphi]$ holds,
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$$\big(\partial_t + (-\Delta)^{\sigma/2} + 1\big)\Phi_\mu = J_\mu * (F_\mu[\Phi_\mu] + \zeta_\mu)$$

Effective equation

$$\begin{cases} \big(\partial_t + (-\Delta)^{\sigma/2} + 1\big)\Phi_\mu = J_\mu * (F_\mu[\Phi_\mu] + \zeta_\mu) \\ \zeta_\mu = -\int_0^\mu (H_\eta[\Phi_\eta] + \mathrm{D}F_\eta[\Phi_\eta] \cdot (\partial_\eta G_\eta * \zeta_\eta))\,\mathrm{d}\eta \end{cases}$$

Notation:

- $G = (\partial_t + (-\Delta)^{\sigma/2} + 1)^{-1}$ = fractional heat kernel,
- $\partial_\eta G_\eta := \partial_\eta J_\eta * G$ – scale decomposition of the fractional heat kernel,
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Claim: System of equations for $\mu \mapsto (\Phi_\mu, \zeta_\mu)$ remains meaningful in the continuum limit $\varepsilon \searrow 0$ provided effective force $F_\mu[\varphi]$ is chosen appropriately.

Flow equation

- A natural choice for the effective force $F_{\mu}[\varphi]$ is to define it so that $H_{\mu}[\varphi] = 0$, i.e. the following **flow equation** is satisfied

$$\partial_{\mu} F_{\mu}[\varphi] + \mathrm{D} F_{\mu}[\varphi] \cdot (\partial_{\mu} G_{\mu} * F_{\mu}[\varphi]) = 0.$$

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- We choose instead $F_\mu[\varphi]$ that satisfies the flow equation up to some small error term $H_\mu[\varphi]$.

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- Multilinear ansatz for effective force

$$F_\mu[\varphi](x) = \sum_{i=0}^{i_\flat} \lambda^i \sum_{m=0}^{3i} \int F_\mu^{i,m}(x; \mathrm{d}y_1, \ldots, \mathrm{d}y_m)\ \varphi(y_1) \ldots \varphi(y_m).$$

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- We construct $F_\mu^{i,m}$ **recursively** using the above-mentioned flow equation.
- Finite collection of kernels $F_\mu^{i,m}$ plays the role of the **enhanced noise**.

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- Functionals F_μ, H_μ are expressed in terms of kernels $F_\mu^{i,m}$ (enhanced noise). To control F_μ, H_μ we prove bounds uniform in ε, τ for moments of $F_\mu^{i,m}$.

Analysis of effective equation

- Recall that we want to prove a bound for the solution of stochastic quantization equation uniform in the lattice spacing ε and lattice size τ.

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- At small scales μ the effective force does not differ much from the force, which involves a cubic nonlinearity. Consequently, $J_\mu * F_\mu[\Phi_\mu] \simeq -\lambda \Phi_\mu^3$ and coarse-grained process Φ_μ satisfies cubic fractional heat equation

$$\big(\partial_t + (-\Delta)^{\sigma/2} + 1\big)\Phi_\mu + \lambda\Phi_\mu^3 = f_\mu.$$

Plan of the talk

1. Fractional Φ^4_3 model
2. Idea behind the construction
3. Flow equation approach
4. Coercive estimate

Application of maximum principle

Lemma

If $\Psi \in C_0^2(\mathbb{R} \times \mathbb{R}^d)$ *and* $f = (\partial_t + (-\Delta)^{\sigma/2})\Psi + \Psi^3$, *then* $\|\Psi\|_{L^\infty}^3 \leqslant \|f\|_{L^\infty}$.

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Proof.

- Let $z_\star \in \mathbb{R} \times \mathbb{R}^d$ be the maximum point of Ψ.
- $(\partial_t \Psi)(z_\star) = 0$ and by positivity of kernel of $\mathrm{e}^{s\Delta}$ and Jensen's inequality $((-\Delta)^{\sigma/2}\Psi)(z_\star) = C_\sigma \int_0^\infty (\Psi(z_\star) - (\mathrm{e}^{s\Delta}\Psi)(z_\star))s^{-1-\sigma/2}\mathrm{d}s \geqslant 0.$
- Consequently, $\sup_{z \in \mathbb{R}\times\mathbb{R}^d} \Psi(z)^3 \leqslant \Psi(z_\star)^3 \leqslant f(z_\star) \leqslant \|f\|_{L^\infty}$.

To complete the proof we apply the above reasoning to $-\Psi$. □

Coercive estimate and tightness

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- Choose the terminal scale $\bar{\mu}$ random and small enough.
- Control moments of $\|\Phi\|_{\mathcal{B}}$ in terms of $|||\mu \mapsto \Phi_\mu|||_{\bar{\mu}}$ and $\bar{\mu}^{-1}$.

Conclusions and outlook

- Construction of measure of fractional Φ^4_3 model in full subcritical regime.
- Flow equation approach to singular SPDEs.
- Coercive estimate based on the maximum principle.

? Rotational invariance.

? Sine-Gordon model, Yang–Mills theory, ...