Normal form method for dispersive PDEs

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Normal forms for singular dynamics
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Dispersive PDEs

Nonlinear Schrödinger equation (NLS):

$$i\partial_t u = \Delta u + \mathcal{N}(u)$$

Nonlinear wave equation (NLW):

$$\partial_t^2 u = \Delta u + \mathcal{N}(u)$$

Korteweg-de Vries equation (KdV):

$$\partial_t u = \partial_x^3 u + \partial_x(u^2)$$

- in many physical situations, they are **Hamiltonian PDEs**:

$$H(u) = \int |\nabla u|^2 dx \pm \frac{1}{p+1} \int |u|^{p+1} dx$$

- Examples of **non-Hamiltonian** dispersive PDEs (?):

$$i\partial_t u = \Delta u + u^2 \quad \text{and} \quad i\partial_t u = \Delta u + \partial_x(u^2)$$

- **Two overlapping but *distinct* fields**:

Hamiltonian PDEs (in dynamical system) and **dispersive PDEs**

Normal form reductions

Goal: Apply a sequence of transforms to reduce dynamics to the "essential" part
($=$ normal form) $+$ (hopefully negligible) error

- Essential part $=$ linear dynamics
  $+$ resonant dynamics (which can not be eliminated)

General setup:

1. Separate the nonlinear part into ***(nearly) resonant*** and ***non-resonant*** parts
2. **"Eliminate"** the non-resonant part $\Longrightarrow$ introduces ***higher*** order terms
3. Repeat (or terminate the process at some finite step)

Two major normal form reductions:

- **Birkhoff normal form reductions**
  - Hamiltonian PDE technique, working on a Hamiltonian
- **Poincaré / Poincaré-Dulac normal form reductions**
  - dispersive PDE technique, working on an equation (or on an energy functional)

Birkhoff normal form reductions (in Hamiltonian PDEs):

- **work on a Hamiltonian** (for a Hamiltonian PDE) on a symplectic space
- canonical, rigid, but less flexible (needs to be Hamiltonian!!)
- **"Eliminate"**: Lie transform (by generating a new Hamiltonian flow)

Poincaré / Poincaré-Dulac normal form reductions (in dispersive PDEs):

- **work on an equation** (usually, but can be done on a Hamiltonian / energy)
- more ad hoc, less canonical, but more flexible
- **"Eliminate"**: integration by parts (in time)

superficial analogy at a philosophical level:

$$\underbrace{\text{regularity structures}}_{\text{rigid, more powerful}} \quad \text{vs} \quad \underbrace{\text{paracontrolled calculus}}_{\text{less canonical, more flexible}}$$

Birkhoff normal form reductions:

- Kuksin-Pöschel '96, Bourgain '00, Bambusi-Grébert '06, Eliasson, Craig, Wayne, $(\text{Procesi})^2$, Kappeler, Delort, Szeftel, Faou, Berti, Colliander-Kwon-Oh, Gérard-Grellier, Bernier, Robert, ...
- rational normal form: Bernier-Faou-Grébert '20

Poincaré / Poincaré-Dulac normal form reductions:

- Shatah '85, Nikolenko '86, Babin-Ilyin-Titi '11, Kwon-Oh '12, **Guo-Kwon-Oh '13**, Erdoğan-Tzirakis '13, de Suzzoni '15, Oh-Tzvetkov '17, Oh-Sosoe-Tzv. '18, Oh-Wang '18, Kishimoto, Correia, ...
- space-time resonance method: Germain-Masmoudi-Shatah '12
  - also integration by parts in spatial frequencies
- renormalization group method: Chen-Goldenfeld-Oono '94 in physics (not to be confused with those by Kupiainen, Duch, etc.)
  - "half-step" normal form (in my view)
  - Pocovnicu '13: first and second order effective dynamics for half NLW on $\mathbb{R}$ (Gérard-Grellier '12: on $\mathbb{T}$ via Birkhoff NF)

Digression: complete integrability

Q: *What is integrability?*

- Solvability via *integration* by quadratures (i.e. in an explicit manner)

Finite dimensional Hamiltonian dynamics on $\mathbb{R}^{2N}$:

$$\frac{dp}{dt} = \frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = -\frac{\partial H}{\partial p}$$

- There exist $H_1(=H), H_2, \ldots, H_N$ all in involution: $\{H_j, H_k\} = 0$

  $\Longrightarrow$ (Liouville) The system is *integrable*

- Action-angle variables (Liouville-Arnold):

$$(p,q) \overset{\text{symplect.}}{\longmapsto} (I, \varphi) \text{ such that } \frac{dI}{dt} = 0, \ \frac{d\varphi}{dt} = c(I)$$

Infinite dimensional case (= PDEs): various notions of integrability

- infinitely many conservation laws ($\Longleftarrow$ bi-Hamiltonian structure)
- Lax pair formulation
- **Action-angle coordinates**: **invariant tori** (quasi- / almost-periodic motions)

  ⋮

- **Reducibility** (to the linear equation): ***No*** Hamiltonian structure required

Kolmogorov-Arnold-Moser theory

Given an integrable Hamiltonian $H(p,q)$ ($\rightsquigarrow$ action-angle coordinates), consider a perturbed Hamiltonian $H(p,q) + \varepsilon F(p,q)$

KAM theory = study how invariant tori are deformed under a small perturbation of an integrable system

- Kuksin-Pöschel '96: first result in the PDE setting
  - **Birkhoff normal form**
  - finite-dimensional invariant tori
  - strong non-resonant assumption
  - external parameter (Cantor-like sets)
- Infinite-dimensional invariant tori: Bourgain '96, '05, Pöschel '02, ...
  - non-integrable models
- Bernier-Grébert-Robert '24: infinite-dim'l tori without an external parameter

$\Longrightarrow$ Construction of **quasi- and almost-periodic solutions**

Reducibility and normal form method

Reducibility: By a local change of the coordinates, transform

$$\partial_t x = F(x)$$

to the associated linear equation:

$$\partial_t z = dF(0)z$$

in a neighborhood of the origin (weak form of integrability)

- Nikolenko '86: Poincaré normal form method (KAM / Nash-Moser scheme)
  - non-resonant NLS on $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, assuming some Diophantine property. Also, for the heat equation
- McKean-Shatah '91: "local reducibility" (basically small data scattering)
- Chung-Guo-Kwon-Oh '17: reducibility for $i\partial_t u + \partial_x^2 u = i\partial_x(u^2)$
  - **infinite** iteration of the **Poincaré-Dulac normal form reductions**, generating higher order terms, which corresponds to the (convergent) Taylor series expansion of the Hopf-Cole transform "$w = e^{-\frac{i}{2}\int u dx}$": $i\partial_t w + \partial_x^2 w = 0$

    $\Longrightarrow$ all small solutions are *periodic in time*

Note: *No* Hamiltonian structure $\Longrightarrow$ Birkhoff normal form method is *not* applicable

Other applications:

1. **small data global well-posedness & scattering**: Shatah '85
2. almost global existence: Shatah '85, Bambusi-Delort-Grébert-Szeftel '07
3. construction of solutions: Babin-Ilyin-Titi '11, **Guo-Kwon-Oh '13** (∞-iteration)
4. **unconditional uniqueness**: Kwon-Oh '12, **Oh-Sosoe-Wang '25 (for SKdV)**
5. nonlinear smoothing & growth of Sobolev norms (also Talbot effect): Bourgain '04, Colliander-Kwon-Oh '12, Erdoğan-Tzirakis '13, Bambusi-Grébert '21, Chapouto-Killip-Vişan '24 (quasi-periodic in space)
6. long-time stability / Nekhoroshev-type stability: Bambusi-Nekhoroshev '02, Bambusi-Grébert '06, Faou-Grébert '13, Bambusi-Gérard '24
7. Arnold diffusion (divergence after a long time): Colliander-Staffilani-Keel-Takaoka-Tao '10, Carles-Faou '12 (geometric optics)
8. energy estimate (P-D NF applied to the equation satisfied by an energy):
   - higher order I-method (by adding correction terms): CKSTT '03, '08
   - quasi-invariance: Oh-Tzvetkov '17, Oh-Sosoe-Tzv. '18, Oh-Sosoe-Tzv. '21
9. **modulated dispersive PDEs** (replacing the sewing lemma): **Gubinelli-Li-Li-Oh '25**
10. **numerical scheme** (normal form integrator): **Chapouto-Forlano-Oh '25**

Small data global well-posedness & scattering

Quadratic NLW on $\mathbb{R}^d$:

$$\partial_t^2 u = \Delta u + \mathcal{N}(\partial_t u, \nabla u, \partial_t^2 u, \nabla^2 u), \qquad \mathcal{N}(v) = O(|v|^2)$$

- decay estimate: $\|u(t)\|_{L^\infty} \lesssim |t|^{-\frac{d-1}{2}}$

$$\implies \quad \|u(t)\|_Z \le \|u(0)\|_Z + \int_0^t |t|^{-\frac{d-1}{2}} \|u(t')\|_Z dt'$$

  - $d \ge 4$: $|t|^{-\frac{d-1}{2}}$ is integrable on $[1, \infty)$ $\implies$ small data global well-posedness
  - $d = 3$: $|t|^{-1}$ is log divergent $\implies$ almost global existence

Idea: Use a NF reduction to transform to a **cubic** nonlinenarity

- Poincaré normal form reduction: Shatah '85
- **non-resonant** (can remove the quadratic part completely)
  $\iff$ **null condition**: Klainerman '85 (vector field method)

Shatah '85, Klainerman '85: small data global well-posedness for $d = 3$
(almost global existence $T_\varepsilon \sim \exp(\frac{c}{\varepsilon^2})$ when $d = 2$)

W. Craig '16, "Birkhoff normal form for nonlinear wave equations" on Youtube

Craig '16: "If you have small data scattering, why care about NF?" (47:15 on Youtube)

- NF iterations $\implies$ linear equation
- scattering: $u(t) \overset{t\to\infty}{\longmapsto}$ linear solution $W(t)u_+ \overset{t\to 0}{\longmapsto}$ flow back by linear flow

Both conjugate the equation into a linear equation

$\implies$ can think of small data scattering as a NF but it is *not* explicit

Space-time resonance method: Germain-Masmoudi-Shatah '12

- Birkhoff / P-D normal form: **time resonance** (no oscillation in time)

Ex: quadratic NLS: $i\partial_t u + \Delta u = u^2$

Consider two wave packets u_0^j, $j = 1, 2$, localized at $x = 0$ and $\xi \sim \xi_j$

- group velocity $\nabla\omega(\xi_j)$ $\quad (\omega(\xi) = |\xi|^2$: dispersion relation)

  $\implies$ If $\nabla\omega(\xi_1) = \nabla\omega(\xi_2)$, two wave packets stay together (**space resonance**)

  $\iff$ $\nabla_{\xi_1}\Phi(\xi,\xi_1) = 0$ where $\Phi(\xi,\xi_1) = \omega(\xi) - \omega(\xi_1) - \omega(\xi - \xi_1)$

- Use integration by part in ξ_1 (need to work in weighted Sobolev spaces)

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Birkhoff normal form reduction (1-d NLS)

Idea: Apply a sequence of **symplectic transformations** to the nonlinear part

$$N(q) = \sum_{n_1 - n_2 + \cdots - n_{2p+2} = 0} q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2p+1}} \bar{q}_{n_{2p+2}}$$

of the original Hamiltonian into expressions involving only ***(nearly) resonant*** monomials of the form

$$q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}}$$

where $n_1 - n_2 + \cdots + n_{2r-1} - n_{2r} = 0$ and

$$|D(\bar{n})| \leq K \quad \text{with} \quad D(\bar{n}) := n_1^2 - n_2^2 + \cdots + n_{2r-1}^2 - n_{2r}^2$$

and some large $K \gg 1$ (chosen later), plus an error.

Goal: Obtain a transformed Hamiltonian $\mathcal{H}$ of the form:

$$\mathcal{H}(q) = \underbrace{H_0(q)}_{\text{original quadratic part}} + \underbrace{\mathcal{N}_0(q)}_{\text{(nearly) resonant}} + \underbrace{\mathcal{N}_r(q)}_{\text{small error}}$$

Change of coordinates: Consider a Hamiltonian flow associated to F:

$$q_t = i\frac{\partial F}{\partial \bar{q}}$$

Let $\Gamma_t = \Gamma_t(F)$ denote the flow map generated by F at time t

Lemma: Chain rule

Let Γ_t be as above. Then, for a smooth function G, we have

$$\frac{d}{dt}(G \circ \Gamma_t) = \{G, F\} \circ \Gamma_t,$$

where $\{H_1, H_2\} = i\sum_n \left[\frac{\partial H_1}{\partial q_n}\frac{\partial H_2}{\partial \bar{q}_n} - \frac{\partial H_1}{\partial \bar{q}_n}\frac{\partial H_2}{\partial q_n}\right]$ (**Poisson bracket**)

Let $\mathbf{\Gamma := \Gamma_{t=1}}$ **= Lie transform**. By Taylor expansion of $G \circ \Gamma$ centered at $t = 0$,

$$G \circ \Gamma = \sum_{k=0}^{\infty} \frac{1}{k!}\{G, F\}^{(k)},$$

where $\{G, F\}^{(k)}$ denotes the k-fold Poisson bracket of G with F, i.e.

$$\{G, F\}^{(k)} := \{\cdots\{G, \underbrace{F\}, F\}, \cdots, F\}}_{k \text{ times}}, \qquad \{G, F\}^{(0)} = G$$

General strategy: Suppose $H = \underbrace{H_0}_{\text{quadratic part}} + N$. Write N as $N = N_0 + N_1$:

$$\text{resonant part } N_0\text{: } |D(\bar{n})| \leq K$$
$$\text{non-resonant part } N_1\text{: } |D(\bar{n})| > K$$

Goal: Eliminate the non-resonant part N_1 by the Lie transform (for suitable F)

$\implies$ the transformed Hamiltonian $\boldsymbol{H' = H \circ \Gamma}$ is given by

$$\begin{aligned} H' &= H \circ \Gamma = H_0 \circ \Gamma + N_0 \circ \Gamma + N_1 \circ \Gamma \\ &= H_0 + N_0 + \boldsymbol{N_1 + \{H_0, F\}} + \{N_0, F\} + \{N_1, F\} + \text{h.o.t.} \end{aligned}$$

- Choose F satisfying the homological equation: $\boldsymbol{\{H_0, F\} = -N_1}$
  $\implies$ this removes the non-resonant part N_1
- Define resonant part N_0' and non-resonant part N_1' of H' by

$$N_0' := N_0 + \text{resonant part of } \{N_0, F\} + \{N_1, F\} + \text{h.o.t.}$$

$$N_1' := \text{non-resonant part of } \{N_0, F\} + \{N_1, F\} + \text{h.o.t.}$$

More concretely, consider (a part of) a Hamiltonian obtained at some stage:

$$N(q,\bar{q}) = \sum_{n_1 - n_2 + \cdots - n_{2r} = 0} c(\bar{n}) q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}} = \underbrace{N_0}_{\text{resonant}} + \underbrace{N_1}_{\text{non-resonant}}$$

$$H_0(q) = \sum_n n^2 |q_n|^2 \quad \Longleftarrow \quad \text{quadratic part}$$

$\Longrightarrow$ Choose $F \sim$ "$D^{-1} N_1$". More precisely,

$$F(q,\bar{q}) = \sum_{\substack{n_1 - n_2 + \cdots - n_{2r} = 0 \\ |D(\bar{n})| > K}} \frac{c(\bar{n})}{\boldsymbol{D(\bar{n})}} q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}}$$

Then, F satisfies the homological equation: $\{H_0, F\} = -N_1$

- $\{H_1, H_2\} = i \sum_n \left[\frac{\partial H_1}{\partial q_n} \frac{\partial H_2}{\partial \bar{q}_n} - \frac{\partial H_1}{\partial \bar{q}_n} \frac{\partial H_2}{\partial q_n} \right]$
- At each step, the degrees of non-resonant terms **increase at least by 2** since $\deg F \geq 4$

General case: Given $\mathcal{N}(q,\bar{q}) = \sum_{n_1-n_2+\cdots-n_{2r}=0} c(\bar{n}) q_{n_1}\bar{q}_{n_2}\cdots q_{n_{2r-1}}\bar{q}_{n_{2r}}$,

define the "size" of a Hamiltonian $\mathcal{N}$ by

$$\|\mathcal{N}\| = \sup_{*} \sum_{n} |c(\bar{n})||q^{(1)}_{n_1}||q^{(2)}_{n_2}|\cdots|q^{(2r)}_{n_{2r}}|,$$

where the supremum is taken over factors $q^{(j)}$, satisfying some condition

- ex: all factors $\lesssim 1$ in L^2, and all but two factors $\lesssim 1$ in H^1
- Algebra property: H_1 and H_2, homogeneous Hamiltonians. Then, we have
$$\|\{H_1,H_2\}\| \lesssim \|H_1\|\|H_2\|$$

Inductive hypothesis: Assume that the Hamiltonian is of the form:

$$\mathcal{H}(q) = \sum_n n^2|q_n|^2 + \underbrace{\mathcal{N}_0(q)}_{|D(\bar{n})|\le K} + \underbrace{\mathcal{N}_1(q)}_{\boldsymbol{|D(\bar{n})|>K}} + \mathcal{N}_r(q)$$

such that $\|\mathcal{N}_0\|$, $\|\mathcal{N}_1\| \lesssim 1$ and

- remainder part $\mathcal{N}_r$: $\|\mathcal{N}_r\| < K^{-C}$ for some large $C > 0$

Remark: The initial Hamiltonian satisfies this hypothesis (by Sobolev)

Given non-resonant part

$$\mathcal{N}_1 = \sum_{\substack{n_1 - n_2 + \cdots - n_{2r} = 0 \\ |D(\bar{n})| > K}} c(\bar{n}) q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}},$$

choose $F(q, \bar{q}) \sim$ "$D^{-1}\mathcal{N}_1$" such that $\{H_0, F\} = -\mathcal{N}_1$. Then, we have

$$\begin{aligned} \mathcal{H}' = \mathcal{H} \circ \Gamma &= H_0 + \mathcal{N}_0 + \mathcal{N}_1 \\ &\quad + \{H_0, F\} + \{\mathcal{N}_0, F\} + \{\mathcal{N}_1, F\} + \text{h.o.t.} + \mathcal{N}_r \circ \Gamma \\ &= H_0 + \mathcal{N}_0 + \{\mathcal{N}_0, F\} + \{\mathcal{N}_1, F\} + \text{h.o.t.} + \mathcal{N}_r \circ \Gamma \end{aligned}$$

- Since $|D(\bar{n})| > K$, we have $\|F\| \leq K^{-1} \|\mathcal{N}_1\| \lesssim K^{-1}$
- By Taylor expansion and algebra property, $\mathcal{N}_r \circ \Gamma$ is "small":

$$\|\mathcal{N}_r \circ \Gamma\| \leq \|\mathcal{N}_r\| \sum_{k=0}^{\infty} \frac{\|F\|^k}{k!} \lesssim \|\mathcal{N}_r\| \lesssim K^{-C}$$

- Higher order terms with sufficiently high degrees are also small

$\Longleftarrow$ new error part $\mathcal{N}_r'$

- Remaining terms $\mathfrak{N}$:

$$\mathfrak{N} := \sum_{k=1}^{M} \frac{1}{k!}\{\mathcal{N}_0,\, F\}^{(k)} + \overbrace{\sum_{k=1}^{M} \frac{1}{k!}\{\mathcal{N}_1,\, F\}^{(k)} + \sum_{k=2}^{M} \frac{1}{k!}\{H_0,\, F\}^{(k)}}^{\lesssim K^{-1}\|\mathcal{N}_1\|}$$

From $\|\{\mathcal{N}_0, F\}\| \lesssim K^{-1}\|\mathcal{N}_0\|\|\mathcal{N}_1\| \lesssim K^{-1}\|\mathcal{N}_1\|$

$\implies \|\{\mathcal{N}_0, F\}^{(k)}\| \lesssim K^{-k}\|\mathcal{N}_1\| \implies \|\mathfrak{N}\| \lesssim K^{-1}\|\mathcal{N}_1\|$

- With $\mathfrak{N} = \mathfrak{N}_0 + \mathfrak{N}_1$ = resonant part + non-resonant part,

$$\boxed{\mathcal{H}' = \text{new Hamiltonian } := H_0 + \mathcal{N}_0' + \mathcal{N}_1' + \mathcal{N}_r'}$$

  - new resonant part: $\mathcal{N}_0' := \mathcal{N}_0 + \mathfrak{N}_0$ with $\|\mathcal{N}_0'\| \lesssim 1$
  - new non-resonant part: $\mathcal{N}_1' := \mathfrak{N}_1$ with $\|\mathcal{N}_1'\| \lesssim \boldsymbol{K^{-1}}\|\mathcal{N}_1\|$

  $\Longleftarrow$ **gets smaller at each iteration step**

After sufficiently many iterations, we obtain

$$\mathcal{H}(q) = \sum_n n^2|q_n|^2 + \underbrace{\mathcal{N}_0(q)}_{\text{resonant}} + \mathcal{N}_r(q),$$

where $\|\mathcal{N}_0\| \lesssim 1$ and $\|\mathcal{N}_r\| \lesssim K^{-C}$

- Small data setting: **no** need to assume largeness $K \gg 1$, since the degree of non-resonant part gets higher (and hence smaller at each iteration step)
  - needs a non-resonant condition, i.e. $D(\bar{n}) \neq 0$, often coming from Diophantine assumption + external parameter

- can combine with the **dispersive** techniques (Bourgain '04, '04)

Ex 1: use the space-time estimate to obtain improved spatial estimates.

L^6-Strichartz estimate (Bourgain '93) implies

$$\max_{a \in \mathbb{Z}} \Bigg| \sum_{\substack{n_1 - n_2 + \cdots - n_6 = 0 \\ D(\bar{n}) = a}} |c(\bar{n})| |q^{(1)}_{n_1}| |q^{(2)}_{n_2}| \cdots |q^{(6)}_{n_6}| \Bigg| \lesssim n_{\max}^{0+} \prod_{j=1}^{6} \|q^{(j)}_{n_j}\|_{L^2}$$

Ex 2: (upside-down) I-method: Bourgain '04, Colliander-Kwon-Oh '12

used Birkhoff NF reductions in place of P-D NF reductions

(for getting higher order modified energies)

- Also with Bourgain's high-low method (Bourgain '98)

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- One inductively proves estimates for Hamiltonians with higher order nonlinearity, which appear in the process of the normal form reduction

Ex 2: (upside-down) I-method: Bourgain '04, Colliander-Kwon-Oh '12

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Poincaré-Dulac normal form reductions

Poincaré-Dulac Theorem: Consider a differential equation:

$$\partial_t x = Ax + F(x) = Ax + \sum_{j=a}^{\infty} f_j(x), \quad x = (x_1, x_2, \dots, x_N),$$

where $f_j(x)$ denotes nonlinear terms of degree j in x

- Under some assumption, we can introduce a sequence of changes of variables:

$$\begin{aligned} z_1 &= x + y_1, \\ z_2 &= z_1 + y_2 = x + y_1 + y_2, \\ &\vdots \\ z = z_\infty &= x + \sum_{j=1}^{\infty} y_j, \end{aligned}$$

  to reduce the system to the **canonical form**:

$$\partial_t z = Az + G(z) = Az + \sum_{j=a}^{\infty} g_j(z)$$

- Canonical form: $\partial_t z = Az + G(z) = Az + \sum_{j=a}^{\infty} g_j(z)$

  where $g_j(z) =$ *resonant* monomials of degree j in z

- After the Jth step, we have

$$\partial_t z_J = Az_J + G_J(z_J),$$

  where monomials of degree up to $J(a-1)+a-2$ in $G_J(z_J)$ are all resonant

- **Interaction representation**: $\boxed{\mathbf{x}(t) = e^{-tA}x(t),}$ etc.

$$\partial_t x = Ax + F(x) \quad \Longrightarrow \quad \partial_t \mathbf{x} = e^{-tA}F(e^{tA}\mathbf{x})$$

  Also, the resulting canonical equations become

$$\begin{cases} \partial_t \mathbf{z}_J = e^{-tA}G_J(e^{tA}\mathbf{z}_J), & \text{after the } J\text{th step} \\ \partial_t \mathbf{z} = e^{-tA}G(e^{tA}\mathbf{z}), & J = \infty \end{cases}$$

Remark: interaction representation (terminology from quantum mechanics)

- widely used in dispersive PDEs (such as the Fourier restriction norm method)
- allows us to easily capture **multilinear dispersive oscillations**

- After integrating in time, we obtain

$$\begin{cases} \mathbf{z}_J(t) = \mathbf{z}_J(0) + \displaystyle\int_0^t e^{-t'A} G_J(e^{t'A}\mathbf{z}_J(t'))dt', & \text{after the } J\text{th step} \\ \mathbf{z}(t) = \mathbf{z}(0) + \displaystyle\int_0^t e^{-t'A} G(e^{t'A}\mathbf{z}(t'))dt', & J = \infty \end{cases}$$

The main goal point of the classical Poincaré-Dulac normal form reductions is to renormalize the flow so that it is expressed in terms of resonant terms. We, however, introduce the following change of viewpoint to study dispersive PDEs

Generalized Duhamel formulation:

- After the Jth step:

$$\mathbf{x}(t) = \mathbf{x}(0) - \sum_{j=1}^{J} \big[\mathbf{y}_j(t) - \mathbf{y}_j(0) \big] + \int_0^t e^{-t'A} G_J(e^{t'A}\mathbf{z}_J(t'))dt'$$

- With $J = \infty$:

$$\mathbf{x}(t) = \mathbf{x}(0) - \sum_{j=1}^{\infty} \big[\mathbf{y}_j(t) - \mathbf{y}_j(0) \big] + \int_0^t e^{-t'A} G(e^{t'A}\mathbf{z}(t'))dt'$$

Original Duhamel formulation: $\mathbf{x}(t) = \mathbf{x}(0) + \displaystyle\int_0^t e^{-t'A} F(e^{t'A}\mathbf{x}(t'))dt'$

This change of viewpoint turned out to be useful in various settings:

- **Unconditional uniqueness** for dispersive PDEs in low regularities
  - uniqueness in the entire $C([0,T];H^s)$
  - construction of solutions *without* any auxiliary functions spaces such as Strichartz spaces or the $X^{s,b}$-spaces
  - precursor (two iterations & no mention of NF): Babin-Ilyin-Titi '11, Kwon-Oh '12
  - infinite iterations: Guo-Kwon-Oh '13, Oh-Wang '21, Kishimoto, etc.

Cubic NLS on $\mathbb{T}$: interaction representation $\mathbf{u}(t) = S(-t)u(t)$:

$$\partial_t \widehat{\mathbf{u}}_n = -i \sum_{n=n_1-n_2+n_3} e^{-i\phi(\bar{n})t} \widehat{\mathbf{u}}_{n_1} \overline{\widehat{\mathbf{u}}_{n_2}} \widehat{\mathbf{u}}_{n_3} =: -i\mathcal{N}^{(1)}(\mathbf{u})_n,$$

where $\phi(\bar{n}) := n^2 - n_1^2 + n_2^2 - n_3^2 = 2(n_2 - n_1)(n_2 - n_3)$

- Given a parameter $K = K(\|u(0)\|_{L^2}) > 0$, write

$$\mathcal{N}^{(1)}(\mathbf{u}) = \underbrace{\mathcal{N}_0^{(1)}(\mathbf{u})}_{\text{nearly resonant}} + \underbrace{\mathcal{N}_1^{(1)}(\mathbf{u})}_{\text{non-resonant}}, \text{ depending on } |\phi(\bar{n})| \le K \text{ or } > K$$

- Nearly resonant part $\mathcal{N}_0^{(1)}(\mathbf{u})$ satisfies a good estimate
- **No** estimate is available for the (highly) non-resonant part $\mathcal{N}_1^{(1)}(\mathbf{u})$

$\Longrightarrow$ Apply a NF reduction to $\mathcal{N}_1^{(1)}(\mathbf{u})$

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- **1st step of NF reductions** (= differentiation by parts)

$$\begin{aligned}
\mathcal{N}_1^{(1)}(\mathbf{u})_n &= \sum_{\substack{n=n_1-n_2+n_3 \\ |\phi(\bar{n})|>K}} e^{i\phi(\bar{n})t}\widehat{\mathbf{u}}_{n_1}\overline{\widehat{\mathbf{u}}_{n_2}}\widehat{\mathbf{u}}_{n_3} \\
&= \partial_t\bigg[\sum_{*} \frac{e^{i\phi(\bar{n})t}}{\phi(\bar{n})}\widehat{\mathbf{u}}_{n_1}\overline{\widehat{\mathbf{u}}_{n_2}}\widehat{\mathbf{u}}_{n_3}\bigg] \\
&\quad - \sum_{*} \frac{e^{i\phi(\bar{n})t}}{\phi(\bar{n})}\partial_t\big(\widehat{\mathbf{u}}_{n_1}\overline{\widehat{\mathbf{u}}_{n_2}}\widehat{\mathbf{u}}_{n_3}\big) \\
&=: \partial_t \underbrace{\mathcal{B}^{(2)}(\mathbf{u})_n}_{\text{easy}} + \underbrace{\mathcal{N}^{(2)}(\mathbf{u})_n}_{\text{quintic}}
\end{aligned}$$

- Divide the quintic term $\mathcal{N}^{(2)}(\mathbf{u})$ into

  (i) nearly resonant part $\mathcal{N}_0^{(2)}(\mathbf{u})$: bounded in H^s

  $\Longleftarrow$ modulation restriction + divisor counting argument

  (ii) non-resonant part $\mathcal{N}_1^{(2)}(\mathbf{u})$: no estimate available

  $\Longrightarrow$ 2nd step of NF reductions

- Repeat the process indefinitely

Difficulty:

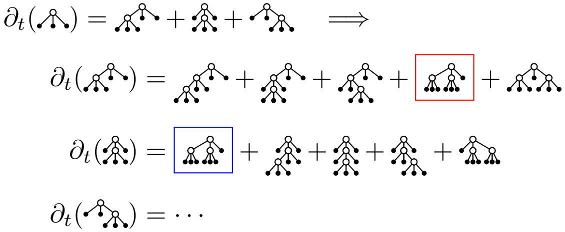
- When we apply differentiation by parts, the time derivative may fall on any of the factors $\widehat{\mathbf{u}}_{n_j}$. In general, the structure of such terms can be very complicated, depending on where the time derivative falls

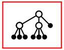
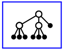
We use ***ordered trees*** *for indexing* such terms arising in the general steps of the NF reductions

- ordered trees = (ternary) trees "**with memory**"

  $\Longleftarrow$ The order in which time derivative fall matters!!

  Example:



  As ordered trees,  and  are different!!

- Indexing via ordered trees allows us to handle combinatorial complexity

After the Jth step:

$$\partial_t \mathbf{u}(t) = \partial_t \left(\sum_{j=2}^{J+1} \mathcal{B}^{(j)}(\mathbf{u}) \right) + \sum_{j=1}^{J+1} \mathcal{N}_0^{(j)}(\mathbf{u}) + \underbrace{\mathcal{N}_1^{(J+1)}(\mathbf{u})}_{\text{bad}}$$

- In order to justify the formal computations, we consider frequency truncated initial data $\mathbf{P}_{\leq N} u(0)$ and the associated *smooth* solutions (***No*** need if $u(t) \in L^3_x$)
- In general, we only have

$$\left|\mathcal{N}_1^{(J+1)}\right| \leq F(\boldsymbol{N}, \boldsymbol{J}) \quad \text{with} \lim_{\boldsymbol{N}\to\infty} F(N, J) = \infty \text{ for } \textbf{each fixed } \boldsymbol{J} \in \mathbb{N}$$

This, however, does not cause an issue since we also show

$$\lim_{\boldsymbol{J}\to\infty} F(N, J) = 0 \quad \text{for } \textbf{each } \boldsymbol{\mathit{fixed}}\ \boldsymbol{N} \in \mathbb{N}.$$

Therefore, by **first taking the limit $J \to \infty$ and then $N \to \infty$**, we conclude that the error term $\mathcal{N}_1^{(J+1)}$ vanishes in the limit (order is crucial)

Putting all together, we obtain the **normal form equation:**

$$\mathbf{u}(t) = \mathbf{u}(0) + \sum_{j=2}^{\infty} \mathcal{B}^{(j)}(\mathbf{u}) \Big|_0^t + \int_0^t \sum_{j=1}^{\infty} \mathcal{N}_0^{(j)}(\mathbf{u})(t')dt'$$

$\Longleftarrow$ $\mathcal{B}^{(j)}(\mathbf{u})$ of deg $2j-1$, $\mathcal{N}_0^{(j)}(\mathbf{u})$ of deg $2j+1$

Normal form equation: $\mathcal{B}^{(j)}(\mathbf{u})$ of deg $2j-1$, $\mathcal{N}_0^{(j)}(\mathbf{u})$ of deg $2j+1$

$$\mathbf{u}(t) = \mathbf{u}(0) + \sum_{j=2}^{\infty} \mathcal{B}^{(j)}(\mathbf{u})\Big|_0^t + \int_0^t \sum_{j=1}^{\infty} \mathcal{N}_0^{(j)}(\mathbf{u})(t')dt'$$

Moral: This infinite iteration of NF reductions allows us to **exchange analytical difficulty with algebraic / combinatorial difficulty**

- relevant analysis involves simple Cauchy-Schwarz's inequality
- can be viewed as an (*analytical*) *renormalization*
  - equivalent to the original cubic NLS for smooth solutions (in $H^{\frac{1}{6}}(\mathbb{T}) \supset L^3(\mathbb{T})$) but behaves better for rough solutions:
    - original cubic NLS: UU holds in H^s for $s \geq \frac{1}{6}$ (sharp)
    - NF equation: UU holds for $s \geq 0$ (and in $\mathcal{F}L^p$ for $p < \infty$, i.e. $\sim H^{-\frac{1}{2}+}$)

Various applications:

- nonlinear smoothing, growth of Sobolev norm, reducibility, energy estimate, ...

Note: Not to be confused with a power series expansion, where we use ordinary trees Christ '07, Oh '17, Chevyrev-Oh-Wang '22, etc.

Energy estimate: Oh-Wang '18, Oh-Sosoe-Tzvetkov '18

- Integration by parts is often useful in establishing a good energy estimate

  $\Longleftrightarrow$ NF reduction on the evolution equation $\partial_t E(\mathbf{u}) = \cdots$ satisfied by the (non-conserved) energy functional $E(\mathbf{u})$

  - higher order I-method via adding correction terms (= boundary terms in P-D NF)
  - energy estimate in the short-time Fourier restriction norm method
  - energy estimate for proving quasi-invariance

  $\vdots$

- For bookkeeping, we use "ordered bi-trees" that grow in two directions
- Defining a **modified energy** $E_\infty(u)$ **of an infinite order** is given by

$$E_\infty(u) = \|u\|_{H^s}^2 - \sum_{j=2}^{\infty} \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \mathcal{B}^{(j)}(u)(n),$$

  we obtain

$$E_\infty(u)(t) - E_\infty(u)(0) = \int_0^t \sum_{j=1}^{\infty} \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \mathcal{N}_0^{(j)}(u)(n, t')dt',$$

  where RHS satisfies good estimates

Remark: For NLW, "more classical" IBP in space (not in time) is useful

Normal form approach to stochastic dispersive PDE

Stochastic KdV on $\mathbb{T}$: $\partial_t u = \partial_x^3 u + \partial_x(u^2) + \Phi\xi$

- $\xi =$ space-time white noise
- $\Phi \in \text{HS}(L^2; H^s)$ such that "$\Phi\xi(t) \in H_x^s$"
- de Bouard-Debussche-Tsutsumi '04: local well-posedness for $s > -\frac{1}{2}$
- When $\Phi = \text{Id}$ (corresponding to the space-time white noise forcing), $s < -\frac{1}{2}$
  - Oh '09: local well-posedness
  - Oh-Quastel-Sosoe '24: global dynamics (evolution system of measures)

Unconditional uniqueness?

- uniqueness in the results above holds in (some versions of) the $X^{s,b}$-spaces
- UU is a concept of uniqueness which does not depend on *how solutions are constructed* and thus is of fundamental importance for SPDEs (ex: agreement with limits of time discretizations)

Theorem: Oh-Sosoe-Wang '25

If $\Phi \in \text{HS}(L^2; L^2)$, then SKdV is unconditionally well-posedness in $L^2(\mathbb{T})$

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If $\Phi \in \mathrm{HS}(L^2; L^2)$, then SKdV is unconditionally well-posedness in $L^2(\mathbb{T})$

- L^2 is sharp in view of the quadratic nonlinearity
- normal form approach, applicable to other stochastic dispersive PDEs
  - a new way to construct solutions to stochastic dispersive PDEs
- transform SKdV with an *additive* noise to a **normal form equation with *multiplicative* noises**

With $\boxed{\phi(\bar{n}) = n^3 - n_1^3 - n_2^3 = 3nn_1n_2,}$ consider

$$\mathcal{N}(\mathbf{u})(t,n) = in \sum_{\substack{n=n_1+n_2 \\ |\phi(\bar{n})|>K}} e^{i\phi(\bar{n})t}\widehat{\mathbf{u}}_{n_1}(t)\widehat{\mathbf{u}}_{n_2}(t)$$

Ito's lemma: $d(X_1X_2X_3) = X_2X_3dX_1 + X_1X_3dX_2 + X_1X_2dX_3$
$$+\tfrac{1}{2}X_1d\langle X_2, X_3\rangle + \tfrac{1}{2}X_2d\langle X_1, X_3\rangle + \tfrac{1}{2}X_3d\langle X_1, X_2\rangle$$

- $X_1 = \frac{e^{i\phi(\bar{n})t}}{i\phi(\bar{n})}$, $X_2 = \widehat{\mathbf{u}}_{n_1}$, and $X_3 = \widehat{\mathbf{u}}_{n_2}$ (for fixed n, n_1, and n_2)
- Only the bracket $\langle X_2, X_3\rangle$ is non-zero, but

$$\langle X_2, X_3\rangle = \mathbf{1}_{n_1+n_2=0} = 0 \text{ (since we can assume } n \neq 0)$$

Ito's lemma: $d(X_1X_2X_3) = X_2X_3dX_1 + X_1X_3dX_2 + X_1X_2dX_3$
$+\frac{1}{2}X_1d\langle X_2, X_3\rangle + \frac{1}{2}X_2d\langle X_1, X_3\rangle + \frac{1}{2}X_3d\langle X_1, X_2\rangle$

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$$\Longrightarrow \int_0^t ine^{i\phi(\bar{n})t'}\widehat{\mathbf{u}}_{n_1}(t')\widehat{\mathbf{u}}_{n_2}(t')dt$$
$$= \frac{e^{i\phi(\bar{n})t'}}{3n_1n_2}\widehat{\mathbf{u}}_{n_1}(t')\widehat{\mathbf{u}}_{n_2}(t')\Big|_0^t - 2\int_0^t \frac{e^{i\phi(\bar{n})t'}}{3n_1n_2}\widehat{\mathbf{u}}_{n_1}(t')\boldsymbol{d\widehat{\mathbf{u}}_{n_2}(t')},$$

where $$d\widehat{\mathbf{u}}_{n_2}(t) = in_2 \underbrace{\sum_{n_2=k_1+k_2} e^{3in_2k_1k_2t}\widehat{\mathbf{u}}_{k_1}(t)\widehat{\mathbf{u}}_{k_2}(t)}_{\text{continuous in time if } u(t)\in L^2_x} + \Phi(n_2)e^{itn_2^3}dB_{n_2}(t)$$

- $\int_0^t \frac{e^{i\phi(\bar{n})t'}}{3n_1n_2}\widehat{\mathbf{u}}_{n_1}(t')\Phi(n_2)e^{it'n_2^3}dB_{n_2}(t') \Longleftarrow$ **multiplicative noise!!**
- This yields UU for $s > \frac{1}{2}$
- one more NF step to get to L^2-regularity (as in the case of the deterministic KdV)
- infinite iteration is needed for SNLS (infinitely many multiplicative noise terms)

Modulated dispersive PDEs

Modulated KdV on $\mathbb{T}$: $\partial_t u + \partial_x^3 u \cdot \boldsymbol{\partial_t w} = \partial_x(u^2)$

- **modulation function** $w : \mathbb{R}_+ \to \mathbb{R}$, continuous but *not* differentiable
- **(ρ, γ)-irregularity** (Chouk-Gubinelli '15, Catellier-Gubinelli '16): Given $\rho > 0$, $0 < \gamma < 1$, and $T > 0$, w is (ρ, γ)-irregular on $[0, T]$ if we have

$$|\Phi_{t,r}^w(a)| := \left| \int_r^t e^{iaw(t')} dt' \right| \lesssim |t - r|^\gamma \langle \boldsymbol{a} \rangle^{-\boldsymbol{\rho}}, \quad \text{uniformly in } a \in \mathbb{R},\, 0 \le r \le t \le T$$

- a fractional Brownian motion of Hurst index $H \in (0, 1)$ is a.s. (ρ, γ)-irregular for any $\rho < \frac{1}{2H}$ (with some $\boldsymbol{\frac{1}{2} < \gamma < 1}$)
- a "generic" δ-Hölder continuous function is (ρ, γ)-irregular for any $\rho < \frac{1}{2\delta}$

Interaction representation: $\mathbf{u}(t) = U^w(t)^{-1} u(t)$, where $U^w(t) = e^{-\boldsymbol{w(t)}\partial_x^3}$

$$\Longrightarrow \quad \mathbf{u}(t) = u_0 + \int_0^t U^w(t')^{-1} \partial_x \big((U^w(t')\mathbf{u})^2 \big) dt'$$

Main task: Give a meaning to the Duhamel integral term

$$\mathbf{u}(t) = u_0 + \boxed{\int_0^t U^w(t')^{-1}\partial_x\big((U^w(t')\mathbf{u})^2\big)dt'}$$

Main task: Give a meaning to the Duhamel integral term

- Chouk-Gubinelli '15, C-G-Li-Li-Oh '24:

  - construct as a **nonlinear Young integral** $\mathcal{I}^{\mathbf{X}}(\mathbf{u})$ with the driver $\mathbf{X}$:

$$\mathbf{X}_{t,r}(f_1, f_2) = \int_r^t U^w(t')^{-1}\partial_x\big((U^w(t')f_1)(U^w(t')f_2)\big)dt', \quad f_1, f_2 \text{ on } \mathbb{T}$$

$$\Longrightarrow \quad \mathcal{F}_x\big(X_{t,r}(f_1,f_2)\big)(n) = in \sum_{n=n_1+n_2} \Phi^w_{t,r}(\phi(\bar{n}))\widehat{f_1}(n_1)\widehat{f_2}(n_2)$$

- $|\Phi^w_{t,r}(\phi(\bar{n}))| = \big|\int_r^t e^{i\phi(\bar{n})w(t')}dt'\big| \lesssim |t-r|^\gamma \langle n\rangle^{-\rho}\langle n_1\rangle^{-\rho}\langle n_2\rangle^{-\rho}$

  $\Longleftarrow$ yields smoothing of arbitrary degree (by taking $\rho \gg 1$)

- If $\mathbf{u} \in \mathcal{C}^{\alpha}([0,T]; H^s)$ with $\gamma + \alpha > 1$, then the **sewing lemma** (Gubinelli '04) allows us to construct $\mathcal{I}^{\mathbf{X}}(\mathbf{u}) = \mathbf{X}(\mathbf{u}_\bullet, \mathbf{u}_\bullet)$ as a nonlinear Young integral

  $\Longrightarrow$ local well-posedness of the modulated KdV in $H^s(\mathbb{T})$ for $s \geq -\rho$ (**regularization by noise**)

- Robert '24: analogous result by the Fourier restriction norm method (adapted to U^2- and V^2-spaces)

$\Longrightarrow$ uniqueness holds only conditionally in $\mathcal{C}^{\alpha}([0,T];H^s)$, $\alpha \sim \frac{1}{2}$

Goal: Implement a **normal form approach** for modulated dispersive PDEs

- $\partial_t \frac{e^{i\phi(\bar{n})t}}{i\phi(\bar{n})} = e^{i\phi(\bar{n})t}$ does *not* make sense if we replace t by $w(t)$
- With $\Phi^w_{t,r}(\phi(\bar{n})) = \int_r^t e^{i\phi(\bar{n})w(t')}dt'$, we instead use

$$\boxed{\partial_{\boldsymbol{r}} \Phi^w_{t,\boldsymbol{r}}(\phi(\bar{n})) = -e^{i\phi(\bar{n})w(\boldsymbol{r})}}$$

Then, by integration by parts, we have

$$\widehat{\mathbf{u}}(t,n) - \widehat{\mathbf{u}}(0,n) = in \int_0^t \sum_{\substack{n_1,n_2 \in \mathbb{Z}^* \\ n=n_1+n_2}} e^{i\phi(\bar{n})w(t')} \widehat{\mathbf{u}}(t',n_1)\widehat{\mathbf{u}}(t',n_2)dt'$$

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$$= \mathcal{F}_x\big(\mathbf{X}_{t,0}(\mathbf{u}(0))\big)(n) + \int_0^t \mathcal{F}_x\big(\mathcal{N}_{t,t'}(\mathbf{u}(t'))\big)(n)dt'$$

where $\mathcal{F}_x\big(\mathcal{N}_{t,t'}(\mathbf{u}(t'))\big)(n)$

$$= -2n \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^* \\ n=n_{123}}} \Phi^w_{t,t'}(\phi(n,n_{12},n_3))e^{i\phi(n_{12},n_1,n_2)w(t')}n_{12}\prod_{j=1}^{3}\widehat{\mathbf{u}}(t',n_j)$$

Normal form equation for the modulated KdV:

$$\mathbf{u}(t) = \mathbf{u}(0) + \mathbf{X}_{t,0}(\mathbf{u}(0)) + \int_0^t \mathcal{N}_{t,t'}(\mathbf{u}(t'))dt'$$

- locally well-posed in L^2 via a contraction argument in $C([0,T]; L^2)$ *without* using any auxiliary function space
- equivalent to the original modulated KdV for $\mathbf{u} \in C([0,T]; L^2)$

Theorem: Gubinelli-Li-Li-Oh '25

The modulated KdV on $\mathbb{T}$ is unconditionally well-posed in $L^2(\mathbb{T})$

- L^2 is sharp in view of the quadratic nonlinearity
- The usual NF argument requires a large parameter $K \gg 1$ to create smallness for boundary terms. In this *revised* NF argument, the boundary term is given by $\mathbf{X}_{t,0}(\mathbf{u}(0))$ which becomes small for short time intervals. Hence, **no** need for a large parameter (simplification over the usual NF argument)

Nonlinear Young integration approach via the sewing lemma:

$$\mathbf{u}(t) = \mathbf{u}(r) + \mathbf{X}_{t,r}(\mathbf{u}(r)) - (\boldsymbol{\Lambda\delta\Theta})_{t,r}, \qquad \Theta_{t,r} = \mathbf{X}_{t,r}(\mathbf{u}(r))$$

- Λ = sewing map, δ = coboundary map

Normal form equation for the modulated KdV:

$$\mathbf{u}(t) = \mathbf{u}(r) + \mathbf{X}_{t,r}(\mathbf{u}(r)) + \int_r^t \mathcal{N}_{t,t'}(\mathbf{u}(t'))dt'$$

- In the current modulated setting, *the normal form reduction with the controlled structure (i.e.* $\mathbf{u}$ *is a solution) extends the construction of the nonlinear Young integral* $\mathcal{I}^{\mathbf{X}}(\mathbf{u})$ *to the much larger class* $C([0,\tau]; H^s(\mathbb{T}))$, providing a concrete expression for $\Lambda\delta\Theta$.

Modulated cubic NLS on $\mathbb{T}$: $i\partial_t u + \partial_x^2 u \cdot \partial_t w = |u|^2 u$

Theorem: Gubinelli-Li-Li-Oh '25

The modulated cubic NLS on $\mathbb{T}$ is unconditionally well-posed in $H^{\frac{1}{6}}(\mathbb{T})$

- $H^{\frac{1}{6}}$ is sharp in view of the cubic nonlinearity
- While an infinite iteration of NF reductions is needed to prove the same result for the (unmodulated) cubic NLS, only one NF reduction suffices in the modulated setting $\Longleftarrow$ regularization by noise

Normal form integrator

The *revised* NF reduction, using

$$\partial_r \Phi^w_{t,r}(\phi(\bar{n})) = \partial_r \int_r^t e^{i\phi(\bar{n})w(t')}dt' = -e^{i\phi(\bar{n})w(r)} \quad \text{with } w(t) = t$$

yields the **normal form equation for KdV on $\mathbb{T}$:**

$$\mathbf{u}(t_{j+1}) = \mathbf{u}(t_j) + \mathbf{X}_{t_{j+1},t_j}(\mathbf{u}(t_j)) + \int_{t_j}^{t_{j+1}} \mathcal{N}_{t_{j+1},t'}(\mathbf{u}(t'))dt'$$

Numerical scheme: $\boxed{\mathbf{u}(t_{j+1}) \approx \mathbf{u}(t_j) + \mathbf{X}_{t_{j+1},t_j}(\mathbf{u}(t_j))}$

- Hofmanova-Schratz '17: exponential integrator

$$\mathbf{u}(t_{j+1}) = \mathbf{u}(t_j) + \mathbf{X}_{t_{j+1},t_j}(\mathbf{u}_\bullet) \approx \mathbf{u}(t_j) + \mathbf{X}_{t_{j+1},t_j}(\mathbf{u}(t_j))$$

$\Longrightarrow$ H^1-convergence with rate τ, assuming that a solution is in H^3

Theorem: Chapouto-Forlano-Oh '25

- Let $s \geq 0$ and $0 \leq \theta < 1$. Then, $\boldsymbol{H^s}$**-local error** $\boldsymbol{\lesssim \tau^{2-\theta}}$ for an $H^{s+2(1-\theta)}$-solution
- Let $s > \frac{1}{2}$ and $0 \leq \theta \leq \frac{1}{2}$. Then, H^s-conv. with rate $\tau^{1-\theta}$ for an $H^{s+2(1-\theta)}$-solution

- Also, H^s-convergence for $0 \leq s \leq \frac{1}{2}$ for an $H^{\frac{3}{2}+}$-solution

Modulated KdV on $\mathbb{T}$: $\partial_t u + \partial_x^3 u \cdot \partial_t w = \partial_x(u^2)$

- w is (ρ, γ)-irregular

Normal form equation for the modulated KdV on $\mathbb{T}$:

$$\mathbf{u}(t_{j+1}) = \mathbf{u}(t_j) + \mathbf{X}_{t_{j+1},t_j}(\mathbf{u}(t_j)) + \int_{t_j}^{t_{j+1}} \mathcal{N}_{t_{j+1},t'}(\mathbf{u}(t'))dt'$$

Numerical scheme: $\boxed{\mathbf{u}(t_{j+1}) \approx \mathbf{u}(t_j) + \mathbf{X}_{t_{j+1},t_j}(\mathbf{u}(t_j))}$

Theorem: Chapouto-Forlano-Oh '25

Let $s \geq 0$. Then, **H^s-local error $\lesssim \tau^{2-}$** for an H^s-solution if $\rho \gg 1$

- The modulation function w can be taken as
  - a fractional Brownian motion of Hurst index $0 < H \ll 1$
  - a "generic" δ-Hölder continuous function with $0 < \delta \ll 1$
- The scheme can be applied to the modulated cubic NLS:
  For $s \geq \frac{1}{6}$, **H^s-local error $\lesssim \tau^{2-}$** for an H^s-solution if $\rho \gg 1$