Multi-indice B-series

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Outline

B-series

The Ordinary Differential Equation (ODE)

$$dy = f(y_t)dt, \quad y(0) = y \in \mathbb{R}^d.$$

"An algebraic theory of integration methods" J.C.Butcher (1972) [2]

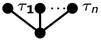
$$\begin{aligned}
y_t - y_s &= \int_s^t f(y_{t_1}) dr_1 \\
&= \int_s^t \sum_k \frac{1}{k!} f^{(k)}(y_s)(y_{r_1} - y_s)^k dr_1 \\
&= f(y_s)(t-s) + \sum_{k\in\mathbb{N}_+} \int_s^t \frac{1}{k!} f^{(k)}(y_s)(y_{r_1} - y_s)^k dr_1 \\
&= f(y_s)(t-s) \\
&+ \sum_{k\in\mathbb{N}_+} \int_s^t \frac{1}{k!} f^{(k)}(y_s) \left(\int_s^{r_1} \sum_n \frac{1}{n!} f^{(n)}(y_s)(y_{r_2} - y_s)^n dr_2 \right)^k dr_1
\end{aligned}$$

B-series

$$y_t - y_s = f(y_s)(t-s) + (f^{(1)}f)(y_s)\int_s^t \int_s^{r_1} dr_2 dr_1$$
$$+\frac{1}{2}(f^{(2)}(f,f))(y_s)\int_s^t \left(\int_s^{r_1} dr_2 \int_s^{r_1} dr_2\right) dr_1 + \dots$$

Butcher Series

$$B(a,h,f,y_0) = a(\emptyset)y_0 + \sum_{\tau\in \mathrm{T}} \frac{h^{|\tau|}a(\tau)}{S(\tau)} F_f[\tau](y_0).$$

For a tree $\tau = B_+(\tau_1, \dots, \tau_n) =$ 

, the elementary differential is

$$F_f[\tau] = f^{(n)}(F_f[\tau_1], \dots, F_f[\tau_n]).$$

The symmetry factor is

$$S(\tau) = \prod_j r_j!(S(\tau_j))^{r_j}.$$

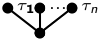
Motivation

The Ordinary Differential Equation (ODE)

$$dy = f(y_t)dt, \quad y(0) = y \in \mathbb{R}. \tag{1}$$

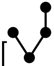
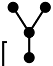
Butcher Series

$$B(a, h, f, y_0) = a(\emptyset)y_0 + \sum_{\tau \in T} \frac{h^{|\tau|}a(\tau)}{S(\tau)} F_f[\tau](y_0).$$

For a tree $\tau = B_+(\tau_1, ..., \tau_n) =$ , the elementary differential is

$$F_f[\tau] = f^{(n)} \prod_{i=1}^{n} F_f[\tau_i].$$

In 1-dimension, it is not injective from trees to elementary differentials.

$F_f[$  $] = F_f[$  $] = f^2 f^{(1)} f^{(2)}$

Multi-indices

The concept of multi-indices emerged initially within the context of studying singular stochastic partial differential equations by Otto, Sauer, Smith, and Weber [5].

Multi-indice

$$z^{\beta} := \prod_{k \in \mathbb{N}} z_k^{\beta(k)}$$

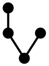
is a collection of abstract variables $(z_k)_{k \in \mathbb{N}}$.

- z_k: nodes within a rooted tree possessing k children.
- $\beta(k)$: number of nodes possessing k children within a rooted tree.
- We assume finite support for β, i.e., $|\{i \in \mathbb{N} \mid \beta(i) \neq 0\}| < \infty$.

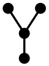
$\beta = (1, 1)$ $z_0 z_1$

$\beta = (1, 2)$ $z_0 z_1^2$

$\beta = (2, 0, 1)$ $z_0^2 z_2$



$\beta = (2, 1, 1)$ $z_0^2 z_1 z_2$



$\beta = (2, 1, 1)$ $z_0^2 z_1 z_2$

From the above examples, it should be clear that different trees can have the same multi-indice.

Populated Multi-indices

Given our aim to exclusively examine multi-indices corresponding to non-planar rooted trees, we shall focus on those fulfilling the so-called "population" condition [3].

$$[\beta] := \sum_{k\in\mathbb{N}} (1-k)\beta(k) = |\beta| - \sum_{k\in\mathbb{N}} k\beta(k) = 1.$$

$$|\beta| = \sum_{k\in\mathbb{N}} \beta(k).$$

From a tree point of view, $|\beta|$ corresponds to the number of nodes and the sum $\sum_{j\in\mathbb{N}} j\beta(j)$ corresponds to the number of edges.

 : $[(1,1)] = |(1,1)| - 0 - 1 = 1$

 : $[(1,2)] = |(1,2)| - 0 - 2 = 1.$

Link with Rooted Trees

- **Mapping multi-indices to trees**
  For any populated multi-indice $z^\beta \in \mathrm{M}_0 \setminus \{z_0\}$ and any $n \in \mathbb{N}_+$ with $\beta(n) \neq 0$, there exist populated multi-indices $z^{\beta_1}, \ldots, z^{\beta_n} \in \mathrm{M}_0$ such that $\quad z^\beta = z_n \prod_{j=1}^n z^{\beta_j}$.
  For every populated multi-indice z^β, there exists at least one tree t such that, for each $k \in \mathbb{N}$, the number of arity-k nodes in t equals $\beta(k)$.

- **Mapping trees to multi-indices**
  Consider the general form of a tree $t = B_+(t_1, \ldots, t_n)$, then we have

$$\Psi(\bullet) = z_0, \quad \Psi(t) = z_n \prod_{j=1}^{n} \Psi(t_j).$$

  One can verify that $z_n \prod_{j=1}^n \Psi(t_j)$ is populated by induction.

Multi-indices B-series

Multi-indices B-series

$$B(a,h,f,y) = a(z^0)y + \sum_{z^\beta \in \mathrm{M}_0} \frac{h^{|z^\beta|} a(z^\beta)}{S(z^\beta)} F_f[z^\beta](y),$$

- $\mathrm{M}_0 := \{z^\beta : [\beta] = 1\}$ is the set of populated multi-indices
- z^0: the empty multi-indice is $\beta(k) = 0$, for all $k \in \mathbb{N}$.
- a: a linear map from $\mathcal{M}_0$ into $\mathbb{R}$ with a finite support, where

$$\mathcal{M}_0 := \left\{ \tilde{\prod}_{j=1}^{n} z^{\beta_j} : z^{\beta_j} \in \mathrm{M},\ n \in \mathbb{N}_+ \right\}.$$

a preserves the multiplicativity of the forest product. Therefore, if $a(z^0) = 1$, a is a character of multi-indices with respect to the forest product.

Multi-indices B-series

$$B(a,h,f,y) = a(z^0)y + \sum_{z^\beta \in \mathrm{M}_0} \frac{h^{|z^\beta|} a(z^\beta)}{S(z^\beta)} F_f[z^\beta](y),$$

- $S(z^\beta)$ is the symmetry factor given by

$$S(z^\beta) := \prod_{k\in\mathbb{N}} (k!)^{\beta(k)}.$$

- $F_f[z^\beta]$: elementary differentials

$$F_f[z^\beta](y) := \prod_{k\in\mathbb{N}} \left(f^{(k)}(y)\right)^{\beta(k)},$$

Connection with local affine equivariant methods

Theorem (Munthe-Kaas and Verdier, 2016 [4])

If a smooth mapping $\varphi : \mathfrak{X}(\mathbb{R}^d) \mapsto \mathfrak{X}(\mathbb{R}^d)$ is local and affine equivariant, then its Taylor development at the zero vector field is an aromatic B-series.

In 1-dimension case, the elementary differentials of aromatic trees collapse to

$$F_f[z^\beta](y) = \prod_{k\in\mathbb{N}} \left(f^{(k)}(y)\right)^{\beta(k)}.$$

Therefore, multi-indices B-series uniquely characterize the Taylor expansion of local and affine equivariant maps.

Multi-indices B-series

Multi-indices B-series

$$B(a,h,f,y) = a(z^0)y + \sum_{z^\beta \in \mathrm{M}_0} \frac{h^{|z^\beta|} a(z^\beta)}{S(z^\beta)} F_f[z^\beta](y),$$

Proposition (Bruned,Ebrahimi-Fard,Hou 2024)

We suppose that $f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. Then the exact solution of (1) is given by a multi-indice B-series with a linear map a given by:

$$a(z^\beta) = \frac{1}{|z^\beta|} \sum_{z^\beta = z_k \prod_{i=1}^k z^{\beta_i}} \prod_{i=1}^k a(z^{\beta_i}).$$

Composition Law

The composition of two multi-indices B-series is defined as

$$B(a, h, f, \cdot) \circ B(b, h, g, y) = B(a, h, f, B(b, h, g, y))$$

Theorem (Bruned,Ebrahimi-Fard,Hou 2024)

For linear maps a and b with $b(z^0) = 1$, the composition of two multi-indices B-series satisfies

$$B(a, h, f, \cdot) \circ B(b, h, f, y) = B(b \star_2 a, h, f, y)$$

where for $z^\beta \in \mathrm{M}$

$$(b \star_2 a)(z^\beta) := < b \otimes a, \Delta_2 z^\beta > .$$

Composition Law

$$F_f[z^\alpha]\left(B(b,h,f,y)\right) = \sum_{k\in\mathbb{N}} \frac{1}{k!}\partial^k F_f[z^\alpha](y)\left(B(b,h,f,y)-y\right)^k$$

$$= \sum_{k\in\mathbb{N}} \frac{1}{k!}\partial^k F_f[z^\alpha](y)\left(\sum_{z^\beta\in \mathrm{M}_0} \frac{h^{|z^\beta|} b(z^\beta)}{S(\beta)} F_f[z^\beta](y)\right)^k$$

$$= F_f[z^\alpha](y) + \sum_{k\in\mathbb{N}_+} \sum_{z^{\beta_1},\dots,z^{\beta_k}\in \mathrm{M}_0} \frac{1}{k!}\partial^k F_f[z^\alpha](y) \prod_{j=1}^{k}\left(\frac{b(z^{\beta_j}) h^{|z^{\beta_j}|}}{S(z^{\beta_j})} F_f[z^{\beta_j}](y)\right).$$

$$F_f\left[\tilde{\prod}_{j=1}^{k} z^{\beta_j} \star_2 z^\alpha\right](y)$$

For $z^{\beta_j} \in \mathrm{M}_0$ and $\tilde{z}^{\tilde{\alpha}} \in \mathcal{M}$, define

$$\tilde{\prod}_{j=1}^{n} z^{\beta_j} \star_2 \tilde{z}^{\tilde{\alpha}} := \left(\prod_{j=1}^{n} z^{\beta_j}\right) D^n \tilde{z}^{\tilde{\alpha}}, \quad D = \sum_{k\in\mathbb{N}} z_{k+1}\partial_{z_k}$$

Proposition (morphism property of elementary differentials w.r.t. $\star_2$)

For every $z^{\beta_j} \in \mathrm{M}_0$ and $z^{\alpha} \in \mathrm{M}_0$, one has

$$F_f\left[\tilde{\prod}_{j=1}^{n} z^{\beta_j} \star_2 z^{\alpha}\right] = \left(\prod_{j=1}^{n} F_f[z^{\beta_j}]\right) \partial^n F_f[z^{\alpha}].$$

- proof inspired by "Composition and substitution of Regularity Structures B-series (Bruned, 2023)"[1].
- Taylor expansion to elementary differentials around y
- Use the morphism property
- Δ_2 is the dual of $\star_2$

Substitution Law

Firstly, the substitution of two multi-indices B-series is defined as:

$$B(a, h, f, \cdot) \circ_s B(b, h, g, y) := B(a, h, B(b, h, g, y), y).$$

Since the substitution is replacing f by $B(b, h, g, y)$, by applying the definition of elementary differentials, one has

$$B(a, h, f, \cdot) \circ_s B(b, h, g, y) = a(z^0)y + \sum_{z^\beta \in \mathrm{M}_0} \frac{a(z^\beta) h^{|z^\beta|}}{S(z^\beta)} \hat{F}_g[z^\beta](y),$$

where

$$\hat{F}_g[z^\beta](y) = \prod_{k \in \mathbb{N}} \left(\partial^k B(b, h, g, y) \right)^{\beta(k)}.$$

Substitution Law

$$F_{g,f}\left[z^{\beta} \blacktriangleright z^{\alpha}\right] = \sum_{k\in\mathbb{N}} \alpha(k) F_f[z^{\alpha}] \frac{\partial^k F_g[z^{\beta}]}{f^{(k)}} = \sum_{k\in\mathbb{N}} \alpha(k) F_f[z^{\alpha}] \frac{F_g[D^k z^{\beta}]}{F_f[z_k]}$$

Definition

The insertion of $z^{\beta} \in \mathrm{M}_0$ into $z^{\alpha} \in \mathrm{M}_0$ is defined to be

$$z^{\beta} \blacktriangleright z^{\alpha} := \sum_{k\in\mathbb{N}} \left(D^k z^{\beta}\right) \left(\partial_{z_k} z^{\alpha}\right).$$

The simultaneous insertion of $z^{\beta_j} \in \mathrm{M}_0$ into $z^{\alpha} \in \mathrm{M}_0$ is

$$\tilde{\prod}_{j=1}^{n} z^{\beta_j} \star_1 z^{\alpha} := \sum_{k_1,\ldots,k_n\in\mathbb{N}} \left(\prod_{j=1}^{n} D^{k_j} z^{\beta_j}\right) \left[\left(\prod_{j=1}^{n} \partial_{z_{k_j}}\right) z^{\alpha}\right].$$

Substitution Law

Theorem (Bruned, Ebrahimi-Fard, Hou 2024)

If $b(z^0) = 0$ and $(b \star_1 a)(z^0)$ is set to be $a(z^0)$, one has

$$B(a, h, f, y) \circ_s B(b, h, g, y) = B(b \star_1 a, h, g, y)$$

- proof inspired by "Composition and substitution of Regularity Structures B-series (Bruned, 2023)"[1].
- By the duality, $RHS = \sum_{z^\beta \in \mathrm{M}_0} \frac{a(z^\beta) h^{|z^\beta|}}{S(z^\beta)} F_g \left[M_b(z^\beta) \right](y)$,

  where $M_b(z^\beta) := \sum_{\tilde{z}^{\tilde{\alpha}} \in \mathcal{M}} \frac{b(\tilde{z}^{\tilde{\alpha}}) h^{|\tilde{z}^{\tilde{\alpha}}|}}{S(\tilde{z}^{\tilde{\alpha}})} \tilde{z}^{\tilde{\alpha}} \star_1 z^\beta$
- Then, the proof boils down to show that

$$F_g \left[M_b(z^\beta) \right] = \hat{F}_g[z^\beta] \quad \text{for any } z^\beta \in \mathrm{M}_0.$$

- Any $z^\beta \in \mathrm{M}_0$ can be expressed as $\tilde{\prod}_{j=1}^{n} z^{\beta_j} \star_2 z_0$.
- $M_b \left(\tilde{z}^{\tilde{\beta}} \star_2 z^\beta \right) = M_b \left(\tilde{z}^{\tilde{\beta}} \right) \star_2 M_b \left(z^\beta \right)$.

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