Cryptographic proofs in Approxis

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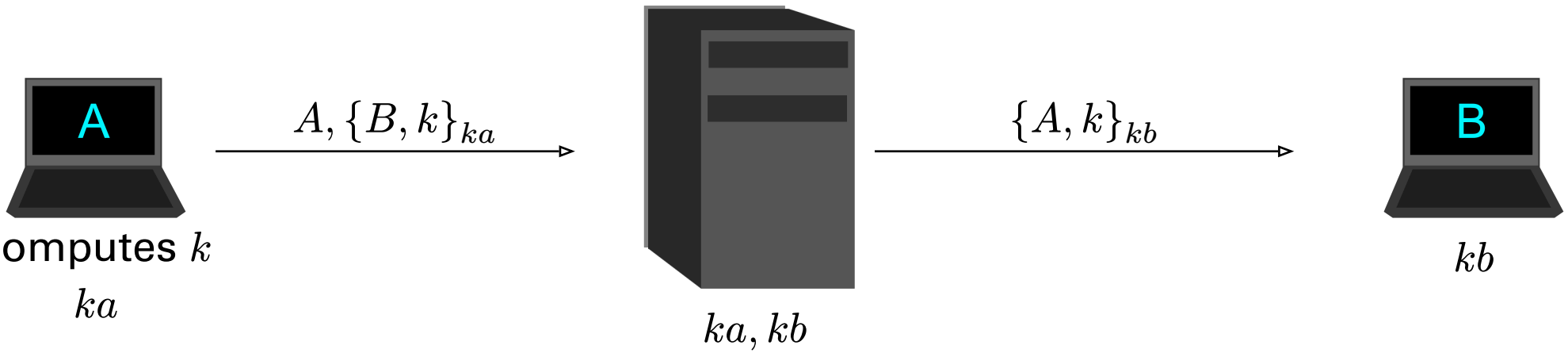
Wide-Mouthed Frog (WMF) protocol



Separation logic

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- Hoare triple $\{l \mapsto x\} l := 42 \{l \mapsto 42\}$.

Iris examples (https://iris-project.org)

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… Hoare triple ⇝ unary property

Cryptography: indistinguishability between 2 “sides”

Contextual equivalence

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$C_k := \text{if } \square = k \text{ then } () \text{ else loop}$

... Hard to prove e.g. by induction on the context

$\rightsquigarrow$ we prove logical refinements: $e \underset{\log}{\ll} e' : T$

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$\text{rand}\ 3 \underset{\log}{\ll} \text{rand}\ 1 + 2 * \text{rand}\ 1$

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$↯\ \frac{1}{N+1} \vdash \text{rand } N \underset{\log}{\ll} \text{let } x = \text{rand } N \text{ in if } x = 0 \text{ then } -1 \text{ else } x$

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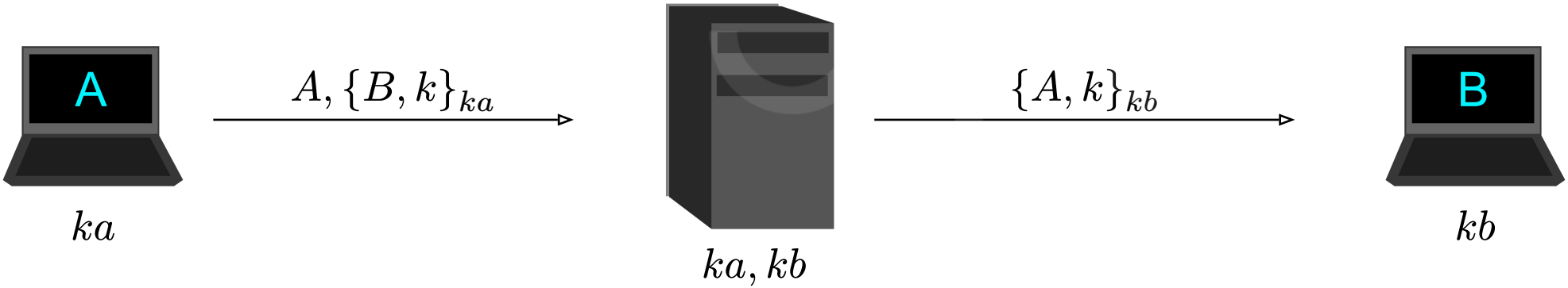
Computational security: properties up to a negligible error

What is a secure protocol ?



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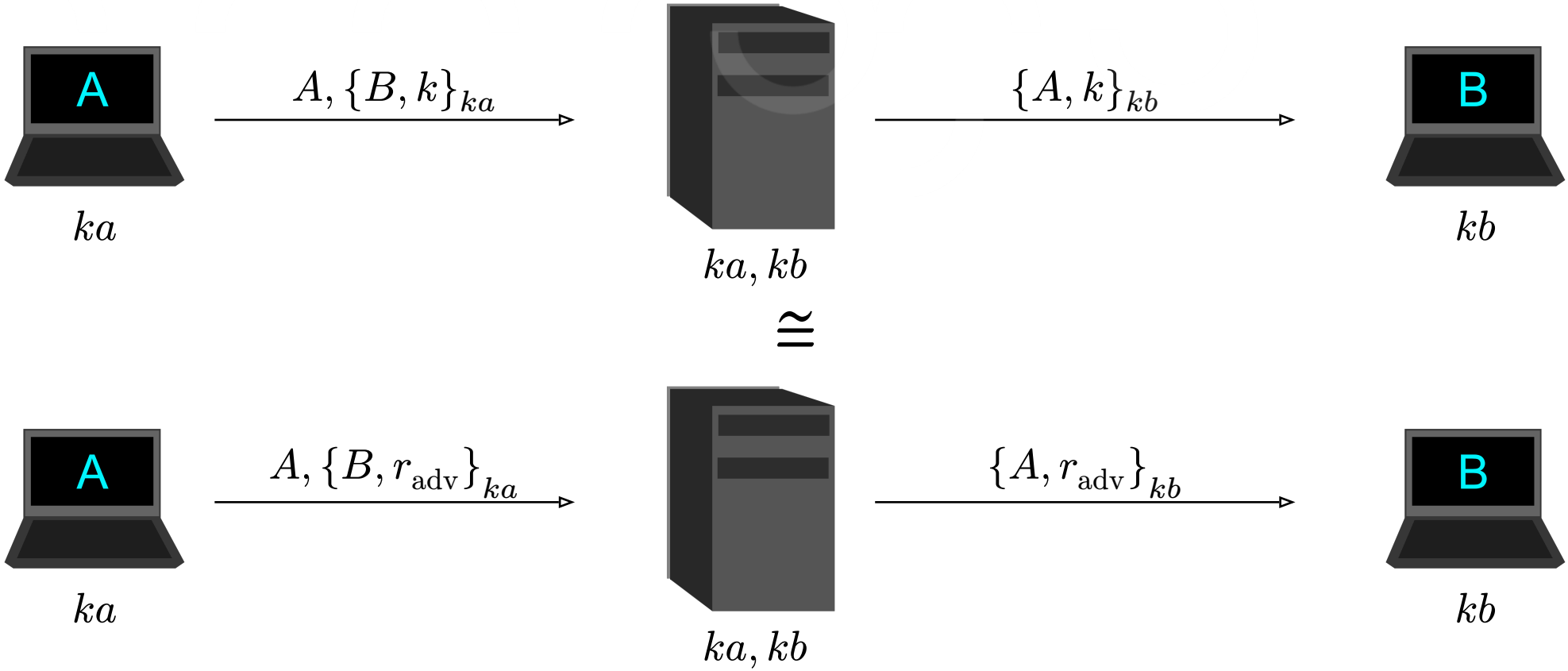

A protocol as a program

Eavesdropping attacker

Active attacker

A protocol as a program

Eavesdropping attacker

WMF eav

$ka, kb :=_{\$} \text{keygen}$

$k :=_{\$} \{0,1\}^{\eta}$

// First message

let msg1 $= (A, \text{senc}(ka, (B, k)))$ in

// Second message

let mdec $= \text{sdec}(\text{snd}\,\text{m}, ka)$ in

let sender $=$ fst m in

$(\text{msg1}, \text{senc}(kb, (\text{sender}, \text{snd}\,\text{msg})))$

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Assumption on the encryption scheme

$\mathcal{L}^{\Sigma}_{\text{real}}$	$\mathcal{L}^{\Sigma}_{\text{rand}}$
$k :=_{\$} \Sigma.\text{keygen}$	ENCRYPT(m):
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IND-CPA$ game for symmetric encryption

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IND-CPA$ game for symmetric encryption

$(\text{senc}, \text{sdec}, \text{keygen})$ satisfies IND-CPA$ if $\mathcal{L}^{\Sigma}_{\text{real}}$ and $\mathcal{L}^{\Sigma}_{\text{rand}}$ are indistinguishable

Proof

WMF

$ka, kb :=_{\$} \text{keygen}$

FirstMessageA():

$k :=_{\$} \{0,1\}^{\eta}$

send $(A, \text{senc}(ka, (B, k)))$

SecondMessageS(m):

msg $:=$ sdec(snd m, ka)

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$r :=_{\$} \{0,1\}^{2\eta}, ka, kb :=_{\$}$ keygen

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send (A, r)

SecondMessageS(m):

msg := sdec(snd m, ka)

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Proof

WMF

$r_{\text{adv}} :=_{\$} \{0,1\}^{\eta}, ka, kb :=_{\$}$ keygen

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```
Lemma wmf_eav_delay_true__wmf_eav_true :
  ⊢ REL init_scheme (wmf_eav_delay #true #true) <<
      init_scheme (wmf_eav #true):
(lrel_rand * ((lrel_id * lrel_output) * (lrel_output * ()))).
```

- Eavesdropping attacker: proof complete ✓
- Active attacker: proof incomplete ✗ require more intricate games

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- consider only polynomial contexts (contextual equivalence: too strong).

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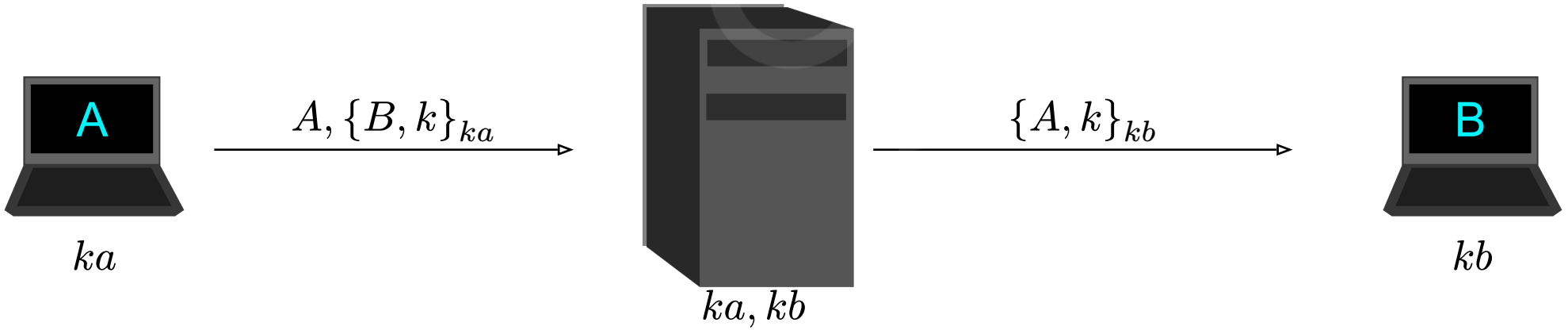
- other examples of cryptographic proof: attack on a protocol, KEMDEM, formalized abstract encryption schemes, security of One-Time Pad.

⇨ Pull request to clutch repository

Weakened WMF



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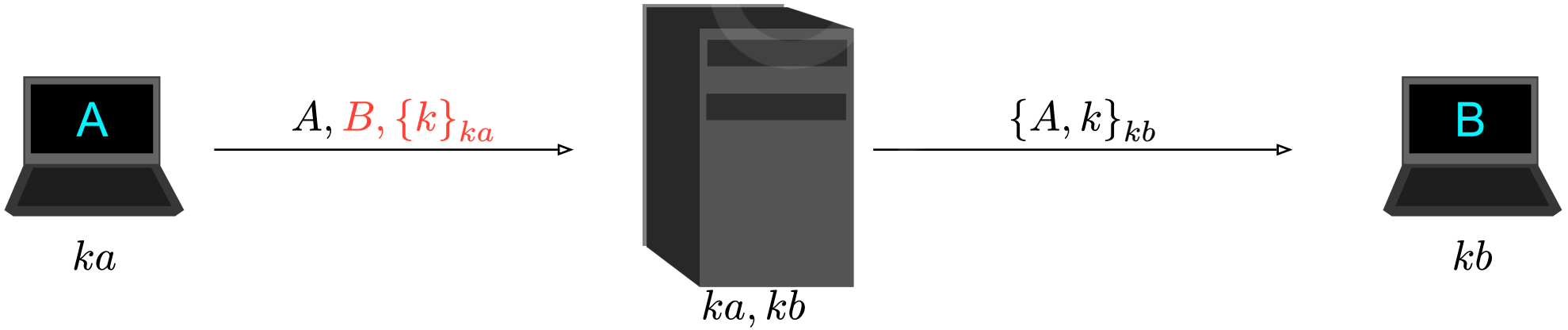

The attack modelled in Approxis

ka

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ka, kb

kb

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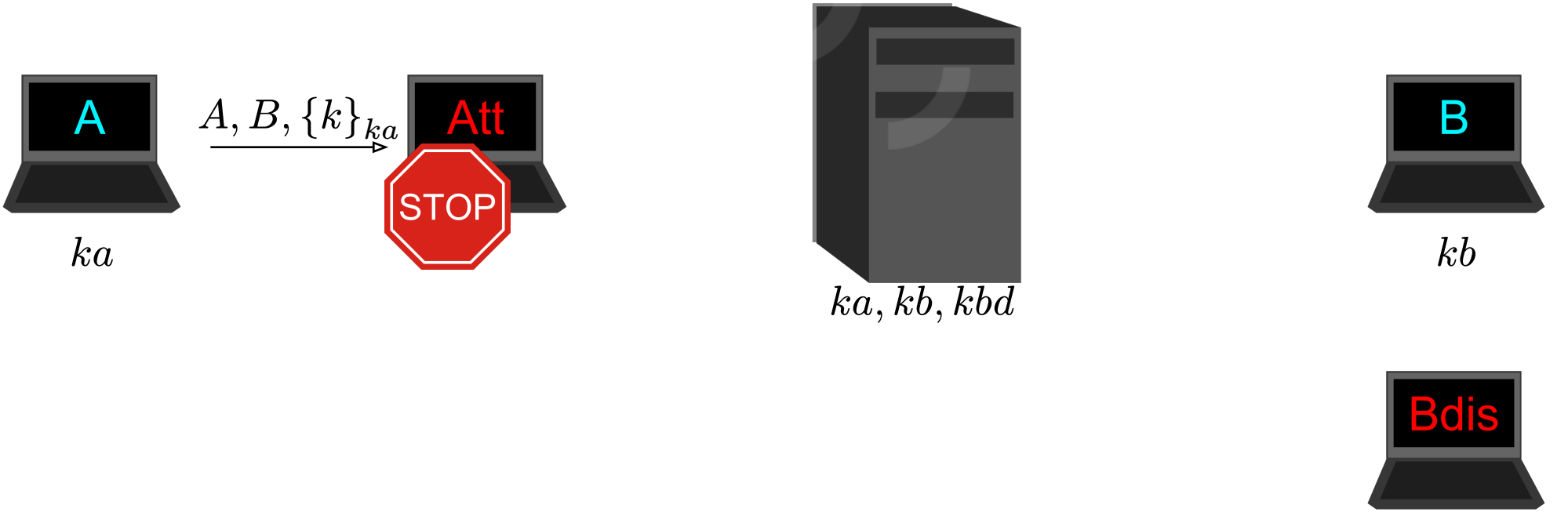

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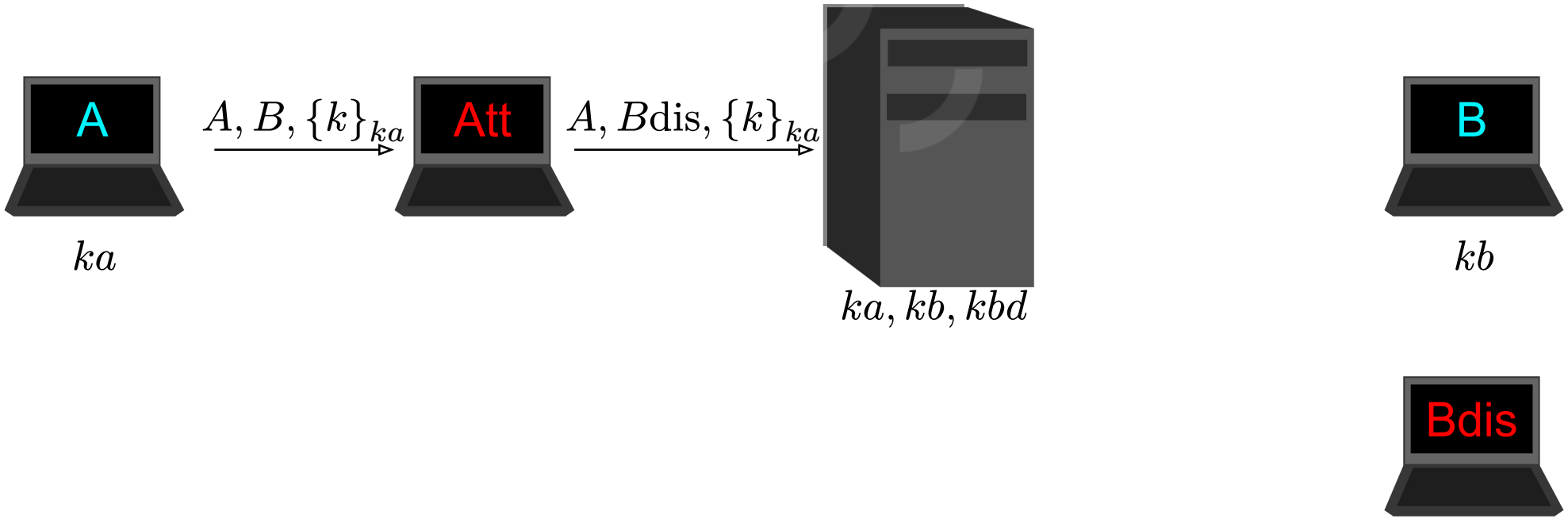

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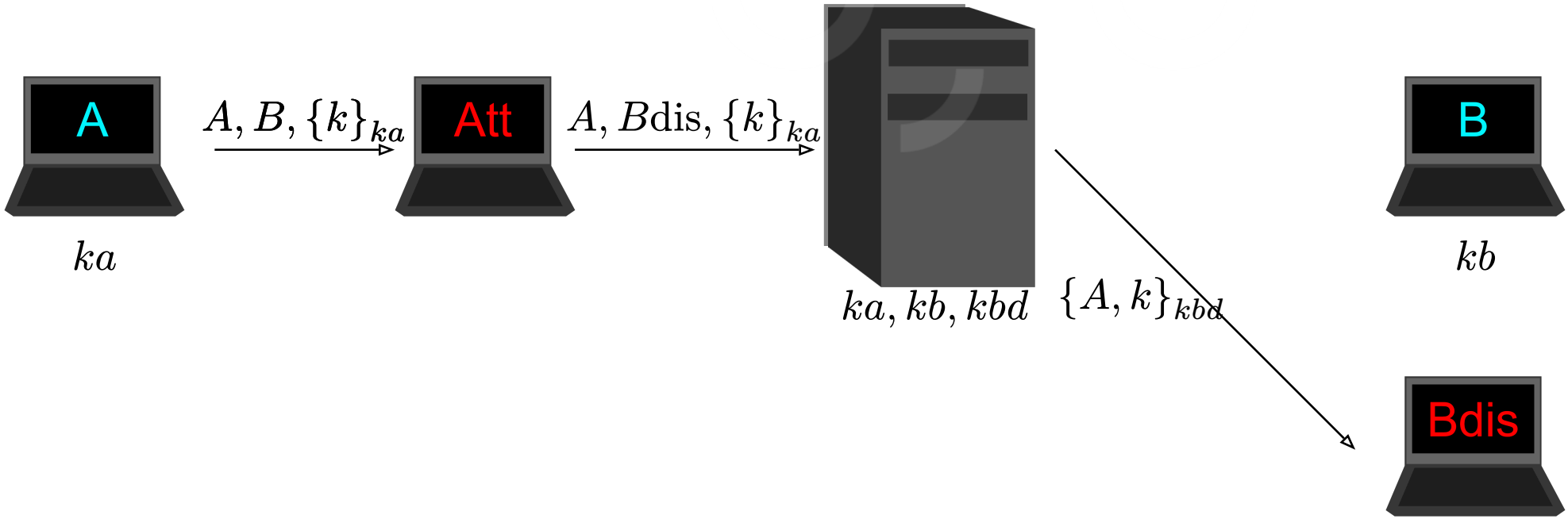

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- separating conjunction: $l \mapsto 42 * l \mapsto 42$ must lead to a contradiction.

KEMDEM

Asymmetric/Public-key encryption: $\mathrm{aenc}, \mathrm{adec}, (\mathrm{pk}, \mathrm{sk}) \leftrightsquigarrow \mathrm{keygen}$

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$\mathsf{adec2}(\mathsf{sk}, m) = \text{let } \mathrm{encmsg}, \mathrm{enckey} = m \text{ in } \mathsf{sdec}(\mathsf{adec}(\mathsf{sk}, \mathrm{enckey}), \mathrm{encmsg})$