Internship report
The Horn inequalities from a geometric point of view

Antoine Médoc, under the supervision of Prof. Paul-Émile Paradan
University of Montpellier

april-june 2023

Abstract. This report is the result of an internship at the end of the second year of a master's degree in mathematics and aims to present a geometric proof of Horn's conjecture made by Prakash Belkale. First we introduce some tools in Schubert calculus such as the Schubert varieties and their parametrization. Then we prove the Belkale's theorem about the intersecting tuples using algebraic geometry. Finally, we prove the Knutson-Tao theorem and Horn's conjecture using representation theory. See section 1.1 for an abstract with more details.

Contents

1 Introduction

1.1 About this report

About its context. This report is about an internship of eleven weeks at the end of a first year of a master's degree in fundamental mathematics at the École normale supérieure de Rennes and University of Rennes. This internship has taken place in the University of Montpellier under the supervision of professor Paul-Émile Paradan. It is about the geometric proof of the Horn's conjecture made by Prakash Belkale in [Bel05]. Yet we will only use the work of Nicole Berline, Michèle Vergne and Michael Walter [BVW18] about this proof. It will use some results about the Zariski topology, algebraic varieties and finite representations of the general linear group.

About its content. This report is mainly based on [BVW18]. We will refer to this article for some proofs.

In this first section 1 we present this report. We gives a more detailed abstract, a brief historical context and the notations used in the rest of the text.

Then in section 2 we present the main tools of Schubert calcus which are useful to Belkale's proof. We gives the main properties of integer tuples (without any geometric point of view), the definition of the Schubert position of a linear subspace with respect to a flag, the definition of a linear subspace composed of linear morphism useful to parametrize Schubert varieties, the definition of three different flag varieties and the idea to work with very convenient linear subspaces.

In a third time in section 3 we study the notion of intersecting tuples. We start with the definition of an intersecting tuple, then we give the Horn's inequalities satisfies by such a tuple, we introduce some geometric characteristics of integer tuples (the true dimension, the kernel dimension and the kernel position) and we use these new tools to prove the Belkale's theorem 3.42.

In section 4 we present the link between the Horn's tuples and the original problem concerning the eigenvalues of Hermitian matrices and the eigenvalues of their sum. We define the Kirwan cone, we study the Hersch-Zalen lemma 4.6 using the min-max theorem, we recall the main informations about the Borel-Weil construction for representations of the general linear group and we give the Knutson-Tao theorem 4.27 to conclude our work.

In the appendix 5 you can find an example of Python code to compute Horn's tuples.

Acknowledgement. I would like to express my sincere gratitude to professor Paul-Émile Paradan for its assistane at every stage of the internship and for having introduced me to these new topics. I also would like to thank all the members of the Institut Montpelliérain Alexander Grothendieck (IMAG) for their great welcome.

1.2 The Horn conjecture

1.2.1 Spectrums of Hermitian matrices

Let A and B be two square matrices of the same order. A natural question (also coming out in physics for example) is to know the relations between the eigenvalues of A and B and those of the sum $A+B$. If A and B are diagonalizable and commute then they are simultaneously diagonalizable and the spectrum of their sum is well known. In this report we will only study the case of Hermitian matrices with complex coefficient. Some of the first relations found during the twentieth century are presented in [Bha99].

Since Hermitian matrices have real eigenvalues, we will see the spectrum of these matrices as tuples with real entries ranked in increasing order. The previous question can now be reformulated as : what are the families $(\lambda_1, \lambda_2, \lambda_3)$ of real tuples such that λ_1 (resp. λ_2) is the spectrum of an Hermitian matrix A (resp. B) and that λ_3 is the spectrum of $-(A+B)$? The set of such families of tuples will be called a Kirwan cone in definition 4.2.

1.2.2 Horn's conjecture

In 1962, Alfred Horn conjectured about the fact that

- a set of finite inequalities are sufficient to describe all the tuples of possible spectrums for matrices and their sum ;
- these inequalities can be described by induction on the length of the tuples.

This conjecture is true and will be seen in the Knutson-Tao theorem 4.27 using the Belkale theorem 3.42. In addition to this, the induction description of these inequalities is simple enough to allow us to compute them easily on a computer (see section 5). The first proof of the Horn's conjecture is found in two main works : one by Anton A. Klyachko in 1998 (using the geometric invariants theory) and one by Allen Knutson and Terence Tao in 1999 (about the saturation property using combinatorics). In this report we will not be interested in this first proof but by the geometric one given later by Prakash Belkale.

1.2.3 Belkale's proof

In 2005, Prakash Belkale proposed an alternative proof of the Horn's conjecture in [Bel05]. It uses Schubert calculus and representations of the general linear group. This is the proof that we see in this report but [Bel05] is not the article we mainly use. In 2018, Nicole Berline, Michèle Vergne and Michael Walter presented Belkale's proof in [BVW18] in a different way and the present report is based on this new redaction.

The aim is to describe the Kirwan cone (definition 4.2) using inequalities parametrized by tuples of integers with an inductive description.

First, we will define two types of integer tuples : the intersecting tuples (definition 3.1) have a geometric definition using Schubert calculus and the Horn's tuples (definition 3.2) have an inductive combinatory definition. Using a well chosen dominant map between algebraic varieties (in section 3.2.1) and Harder-Narasimhan lemma 3.13, we show that any intersecting tuple is a Horn's tuple (proposition 3.19). Then, using different computations of algebraic varieties dimensions and an induction on the size of Horn's tuples, we prove that any Horn's tuple is intersecting (Belkale's theorem 3.42).

Now that we have an inductive description of the intersecting tuples, we want to use them to parametrize inequalities which can describe the Kirwan cone. These inequalities are called Horn's inequalities. From the variational principle we deduce that any real tuple in the Kirwan cone satisfies the Horn's inequalities (see Hersch-Zahlen lemma 4.6 and Klyachko lemma 4.8). Using representations of the general linear group (see the Borel-Weil construction in section 4.3) we prove Kempf-Ness lemma 4.15 and that any integer tuple satisfying the Horn's inequalities is in the Kirwan cone (see corollary 4.23). In Knutson-Tao's theorem 4.27 we finally prove this for any tuples satisfying the Horn's inequalites by density. This last theorem proves Horn's conjecture.

2 Schubert calculus

In this first section, we present the main tools of Schubert calculus we will be using in section 3.

▷ SETTINGS.
- Let $n, r, d, m \in \mathbb{N}^*$ such that $n \geqslant r \geqslant d \geqslant m$.
- Let U a $\mathbb{C}$-linear space of finite dimension n.

2.1 Tuples

We present definitions about tuples and their main properties. Although we introduce the dimension of a tuple and the expected dimension of a sequence of tuples, there is no geometry in this section. Every proof or definition is combinatorial.

2.1.1 Operations on tuples

We will use an identification between susbets of integers and strictly increasing maps. For example, this allows us to define the composition of two tuples.

▷ NOTATION.
- We denote by $[n]$ the set $[\![1, n]\!]$.
- We denote
$$\mathcal{P}_r^n := \left\{ I \subset [n] \;\middle|\; \mathrm{Card}\, I = r \right\}.$$
- We identify any subset $I \in \mathcal{P}_r^n$ with the unique strictly increasing map from $[r]$ to I and we always denote $I(0) = 0$.
- For all $I, J \in \mathcal{P}_r^n$ we denote $J \leqslant I$ the following assertion :
$$\forall i \in [r], J(i) \leqslant I(i).$$
- For all $I \in \mathcal{P}_r^n$ we denote
$$I^c := [n] \backslash I \in \mathcal{P}_{n-r}^n$$
and, for all $J \in \mathcal{P}_d^r$ we denote
$$IJ := I \circ J \in \mathcal{P}_d^n$$
the composition.

REMARK 2.1. The relation $\leqslant$ is a partial order on $\mathcal{P}_r^n$ and $[n-r+1, n]$ is the greatest element of the set.

▷ SETTINGS. Let $I \in \mathcal{P}_r^n$, $J \in \mathcal{P}_d^r$ and $K \in \mathcal{P}_m^d$.

LEMMA 2.2. Let $I \in \mathcal{P}_r^n$. We have

$$\forall i \in [\![0, r]\!],\ I(i) \geqslant i \tag{1}$$
$$\forall i \in [\![0, r-1]\!],\ 0 \leqslant I(i) - i \leqslant I(i+1) - (i+1) \leqslant n - r \tag{2}$$
$$\forall i \in [\![0, r-1]\!],\ I^c([\![I(i) - i + 1, I(i+1) - i - 1]\!]) = [\![I(i) + 1, I(i+1) - 1]\!] \tag{3}$$
$$\forall i \in [\![1, r]\!],\ 2 \leqslant I(i) - I(i-1) \Rightarrow I(i) = I^c(I(i) - i) + 1 \tag{4}$$

Proof 1. The subset I is made of r positive integers.

2. We have $I(r)-r\leqslant n-r$ and, for all $i\in[r-1]$, $I(i+1)\geqslant I(i)+1$.

3. This third statement can be proven by induction on $i\in[\![0,r-1]\!]$. We have

$$[\![1,I(1)]\!]=(I^c(1),\dots,I^c(I(1)-1),I(1))$$

hence

$$I^c([\![I(0)-0+1,I(0+1)-0-1]\!])=[\![I(0)+1,I(0+1)-1]\!]$$

(remark that if $I(1)=1$ then this is the empty set). Let $i\in[\![0,r-1]\!]$ such that

$$I^c([\![I(i)-i+1,I(i+1)-i-1]\!])=[\![I(i)+1,I(i+1)-1]\!].$$

The set $[\![I(i+1)+1,I(i+2)-1]\!]$ has $I(i+2)-I(i+1)-1$ elements wich are

$$I^c(I(i+1)-i-1+1),\dots,I^c(I(i+1)-i-1+I(i+2)-I(i+1)-1$$

and this is what we wanted to prove.

4. This is a direct consequence of the third statement.

$\square$

Remark 2.3. The sequence $(I(J^c(j))-J^c(j)+j)_{j\in[r-d]}$ is strictly increasing and takes values in $[n-d]$.

▷ Notation. We denote

$$I/J:=\{I(J^c(j))-J^c(j)+j\ ;\ j\in[r-d]\}\in\mathcal{P}_{r-d}^{n-d}$$

and

$$I^J:=I/J^c\in\mathcal{P}_d^{n-r+d}.$$

Lemma 2.4. 1. We have

$$\mathcal{P}_r^n\mathcal{P}_d^r=\mathcal{P}_d^n.$$

2. For all $I\in\mathcal{P}_r^n$ and $J\in\mathcal{P}_d^r$,

$$IJ^c=(IJ)^c(I/J).$$

Proof 1. By definition,

$$\mathcal{P}_r^n\mathcal{P}_d^r\subset\mathcal{P}_d^n.$$

Let $K\in\mathcal{P}_d^n$. There exists $i_1,\dots,i_{r-d}\in K^c$ pairwise distinct. Let

$$I:=K\cup\{i_1,\dots,i_{r-d}\}$$

and $J\in\mathcal{P}_d^r$ such that

$$K=IJ.$$

2. Let $k\in[r-d]$. We have

$$(IJ)^c=IJ^c\sqcup I^c$$

and, using lemma 2.2,

$$\begin{aligned}IJ^c(k)&=I^c(I(J^c(k))-J^c(k))+1\\&=I^c(I/J(k)-k)+1.\end{aligned}$$

hence there is $l \in [n-d]$ such that

$$I^c([I/J(k)-k]) \sqcup IJ^c([k]) = (IJ)^c([l])$$

and

$$IJ^c(k) = (IJ)^c(l).$$

In addition to that we have

$$\begin{aligned} l &= \mathrm{Card}((IJ)^c([l])) \\ &= \mathrm{Card}(I^c([I/J(k)-k]) \sqcup IJ^c([k])) \\ &= \mathrm{Card}[I/J(k)-k] + \mathrm{Card}[k] \\ &= I/J(k) \end{aligned}$$

so

$$IJ^c(k) = (IJ)^c(I/J(k)).$$

□

EXAMPLE 2.5. Assume $n = 6$, $r = 4$ and $d = 2$. Let $I := \{1,3,5,6\} \in \mathcal{P}_4^6$ and $J := \{2,4\} \in \mathcal{P}_2^4$. We have

$$\begin{cases} IJ = & \{3,6\} & \in \mathcal{P}_2^6 \\ IJ^c = & \{1,5\} & \in \mathcal{P}_2^6 \\ (IJ)^c = & \{1,2,4,5\} & \in \mathcal{P}_4^6 \end{cases}$$

hence, by lemma 2.4,

$$I/J = \{1,4\} \in \mathcal{P}_2^4.$$

2.1.2 Dimensions of tuples

DEFINITION 2.6. The dimension of I is

$$\dim I := \sum_{j=1}^{r} (I(j) - j).$$

EXAMPLES 2.7. We have

$$\dim[r] = 0 \text{ and } \dim[\![n-r+1, n]\!] = r(n-r)$$

REMARK 2.8. We have

$$\dim \mathcal{P}_r^n = [\![0, r(n-r)]\!].$$

Proof By lemma 2.2,

$$\dim \mathcal{P}_r^n \subset [\![0, r(n-r)]\!].$$

Let $d \in [\![0, r(n-r)]\!]$. There is $m \in [\![0, r]\!]$ such that

$$m(n-r) \leqslant d < (m+1)(n-r).$$

We have $0 \leqslant d - m(n-r) < n-r$ so we can define $I \in \mathcal{P}_r^n$ by

$$\forall j \in [r], I(j) = \begin{cases} n-r+j & \text{si } j \geqslant r-m+1 \\ d-m(n-r)+j & \text{si } j = r-m \\ j & \text{si } j \leqslant r-m-1 \end{cases}.$$

We have

$$\dim I = d.$$

□

LEMMA 2.9. Let $I \in \mathcal{P}_r^n$, $J \in \mathcal{P}_d^r$ and $K \subset [d]$. We have

$$\dim I/J = \dim I + \dim J - \dim IJ$$
$$\dim I^J K - \dim K = \dim IJK - \dim JK$$
$$\dim I^J = \dim IJ - \dim J$$

Proof These three equalities are proven in lemmas 3.2.13 and 3.2.14 in [BVW18]. □

▷ SETTINGS. Let $s \in \mathbb{N}^*$.

DEFINITION 2.10. The *expected dimension* of $\mathcal{I} \in (\mathcal{P}_r^n)^s$ is

$$\operatorname{edim} \mathcal{I} := r(n-r) - \sum_{k=1}^{s} (r(n-r) - \dim \mathcal{I}_k).$$

EXAMPLE 2.11. If $\mathcal{I} := ([n])_{k \in [s]}$ is seen as the unique element of $(\mathcal{P}_n^n)^s$ then

$$\operatorname{edim}(\mathcal{I}) = 0.$$

REMARKS 2.12.
- See lemma 2.15 in [BVW18] for an interesting geometrical meaning of the expected dimension of a tuple linked with the Schubert varieties defined in 2.32.
- For all $\mathcal{I} \in (\mathcal{P}_r^n)^s$ we have

$$\operatorname{edim} \mathcal{I} = (1-s)r(n-r) + \sum_{k=1}^{s} \dim \mathcal{I}_k.$$

LEMMA 2.13. Let $\mathcal{I} \in (\mathcal{P}_r^n)^s$, $\mathcal{J} \in (\mathcal{P}_d^r)^s$, $m \in [d]$ and $\mathcal{K} \in (\mathcal{P}_m^d)^s$. We have

$$\operatorname{edim} \mathcal{I}/\mathcal{J} = \operatorname{edim} \mathcal{I} + \operatorname{edim} \mathcal{J} - \operatorname{edim} \mathcal{I}\mathcal{J} \tag{1}$$
$$\operatorname{edim} \mathcal{I}^{\mathcal{J}} \mathcal{K} - \operatorname{edim} \mathcal{K} = \operatorname{edim} \mathcal{I}\mathcal{J}\mathcal{K} - \operatorname{edim} \mathcal{J}\mathcal{K} \tag{2}$$
$$\operatorname{edim} \mathcal{I}^{\mathcal{J}} = \operatorname{edim} \mathcal{I}\mathcal{J} - \operatorname{edim} \mathcal{J}. \tag{3}$$

Proof From lemma 2.9 we have

$$\begin{aligned}
\operatorname{edim} \mathcal{I}/\mathcal{J} =& (1-s)(r-d)(n-d-(r-d)) + \sum_{k=1}^{s} \dim \mathcal{I}_k/\mathcal{J}_k \\
=& (1-s)(r-d)(n-r) + \sum_{k=1}^{s} \dim \mathcal{I}_k + \dim \mathcal{J}_k - \dim \mathcal{I}_k\mathcal{J}_k \\
=& (1-s)r(n-r) + (1-s)d(r-d) - (1-s)d(n-d) \\
&+ \sum_{k=1}^{s} \dim \mathcal{I}_k + \sum_{k=1}^{s} \dim \mathcal{J}_k - \sum_{k=1}^{s} \dim \mathcal{I}_k\mathcal{J}_k \\
=& \operatorname{edim} \mathcal{I} + \operatorname{edim} \mathcal{J} - \operatorname{edim} \mathcal{I}\mathcal{J}
\end{aligned}$$

and the two other equalities are proved in exactly the same way. □

2.2 Schubert positions

Flags and Schubert positions of given linear subspaces are the main tools used in section 3.

2.2.1 Flags and filtrations on a vector space

DEFINITION 2.14.
- A *flag* on U is a sequence $E := (E(i))_{i\in[\![0,n]\!]}$ of subspaces of V such that

$$\begin{cases} \forall j \in [\![0, n-1]\!], & E(j) \subset E(j+1) \\ \forall j \in [\![0, n]\!], & \dim E(j) = j \end{cases}.$$

- Let E a flag on U. A sequence $(E(i))_{i\in[n]} \in U^n$ is a *basis adapted* to E if, for all $i \in [n]$,

$$E(i) = \operatorname{span}\{e_1, \dots, e_i\}.$$

- The *Borel subgroup* associated to a flag E on U is

$$B(E) := \{\gamma \in \mathrm{GL}(U) \mid \forall i \in [n], \gamma E(i) \subset E(i)\}.$$

REMARKS 2.15.
- For all flag E on U, the set $A(E)$ of all adapted basis to E is non empty. In addition to that,

$$\{b \in U^n \mid b \text{ is a basis of } U\} = \bigsqcup_{E \text{ flag on } U} A(E).$$

- Let E be a flag on U. We have

$$B(E) = \{\gamma \in \mathrm{GL}(U) \mid \forall i \in [n], \gamma E(i) = E(i)\} < \mathrm{GL}(U)$$

and $B(E)$ acts transitively on the set of bases adapted to E.

EXAMPLE 2.16. Let $(e_1, \dots, e_n)$ the canonical basis of $\mathbb{C}^n$. The subspace family

$$E := (\mathbb{C}^i \times \{0\}^{n-i})_{i\in[n]}$$

is a flag on $\mathbb{C}^n$, $(e_i)_{i\in[n]}$ is adapted to E and the Borel subgroup of E is the set of invertible upper triangular matrices.

The following definitions will be useful in section 2.2.3.

DEFINITION 2.17.
- A *filtration* on U is a finite sequence $E := (E(i))_i$ of linear U-subspaces such that, with l the lenght of E,

$$\begin{cases} E_0 = \{0\} \text{ and } E_l = U & \\ \forall j \in [\![0, n-1]\!], & E(j) \subset E(j+1) \\ \forall j \in [\![0, n-1]\!], & \dim E(j+1) \leqslant \dim E(j) + 1 \end{cases}.$$

- Let E a filtration on U and the unique $I \in \mathcal{P}_n^l$ such that

$$\forall j \in [\![0, n-1]\!], E(I(j+1)) \neq E(I(j)).$$

The *flag associated* to E is

$$F := (E(I(j)))_{j\in[n]}.$$

- Let E a flag on U and V a subspace of U. The *induced flag* on V (resp. on U/V) by E is the flag associated to the filtration $(E(i)\cap V)_{i\in[\![0,n]\!]}$ (resp. $(E(i)/V)_{i\in[\![0,n]\!]}$) and is denoted E^V (resp. $E_{U/V}$).

Remark 2.18. The flag associated to a filtration is a flag as in definition 2.14.

2.2.2 Schubert positions

▷ Notation. The Grassmaniann of all r-dimensional linear subspaces of U is denoted $\mathrm{Gr}(r,U)$ and we denote by $\mathrm{Gr}(U)$ the set of all linear subspaces of U.

Definition 2.19. Let E a flag on U and $V \in \mathrm{Gr}(r,U)$. The *Schubert position* of V with respect to E is the sequence $\mathrm{Pos}(V,E) \in \mathcal{P}_r^n$ such that, for all $i \in [r]$,

$$\mathrm{Pos}(V,E)(i) = \min\left\{j \in [n] \;\middle|\; \dim E(j)\cap V = i\right\}.$$

We denote

$$\mathrm{Pos}(\{0\}, E) := \emptyset.$$

Examples 2.20. Let E a flag on U, $(e_i)_{i\in[n]}$ a basis of U adapted to E.

- Assume that $n=5$. Let V be the subspace of U spanned by the vectors

  $$\begin{cases} v_1 &= e_1+e_2+e_4+e_5 \\ v_2 &= 2e_1-e_2+e_3+e_4+e_5 \\ v_3 &= e_1+e_2+2e_4+2e_5 \end{cases}.$$

  The matrix of the family (v_1,v_2,v_3) in the basis $(e_1,\dots,e_5)$ is

  $$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

  wich is equivalent to

  $$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

  From this we deduce that

  $$(\dim E(i)\cap V)_{i\in[5]} = (0,1,2,2,3)$$

  hence

  $$\mathrm{Pos}(V,E) = (2,3,5).$$

- See example 2.24.

- Let $I \in \mathcal{P}_d^r$ and
$$V := \operatorname{span}\{e_i \; ; \; i \in I\}.$$
For all $j \in [r]$,
$$E(I(j)-1) \cap V = \operatorname{span}\{e_i \; ; \; i \in I([j-1])\}$$
and
$$E(I(j)) \cap V = \operatorname{span}\{e_i \; ; \; i \in I([j])\}.$$
From that we deduce
$$\operatorname{Pos}(V, E) = I.$$

Lemma 2.21. For all $\gamma \in \mathrm{GL}(U)$,
$$\operatorname{Pos}(\gamma^{-1}U, E) = \operatorname{Pos}(U, \gamma V).$$

Proof For all $\gamma \in \mathrm{GL}(U)$ and $j \in [n]$,
$$\dim E(j) \cap \gamma^{-1}V = \dim \gamma E(j) \cap V.$$
$\square$

Lemma 2.22. Let V a r-dimensional subspace of U and $I = \operatorname{Pos}(V, E)$. There is an orthonormal basis $(v_1, \dots, v_r)$ of V such that, for all $j \in [r]$,
$$v_j \in E(I(j)).$$

Proof Let $I(0) = 0$. For all $j \in [r]$, by definition of I, there is
$$u_j \in E(I(j)) \cap V \backslash E(I(j-1)).$$
The vectors $u_1, \dots, u_r \in V$ are linearly independant, hence form a basis of V. Since E is a flag, for all $j \in [r]$,
$$E(1) + \cdots + E(i) \subset E(i).$$
Thus, applying Gram-Schmidt process to $(u_1, \dots, u_r)$ gives us $(v_1, \dots, v_r)$ as wanted in lemma 2.22. $\square$

Lemma 2.23. Let E be a flag on U and $I \in \mathcal{P}_r^n$. The following assertions are equivalent.

(i) $\operatorname{Pos}(V, E) = I$.

(ii) For all basis $(e_1, \dots, e_n)$ adapted to E, there exists a unique basis $(v_1, \dots, v_r)$ of V such that, for all $i \in [r]$,
$$v_i \in f_{I(i)} + \operatorname{span}\{f_j \; ; \; j \in [I(i)] \backslash I\}.$$

(iii) There is a basis $(e_1, \dots, e_n)$ adapted to E and a basis $(v_1, \dots, v_r)$ of V such that, for all $i \in [r]$,
$$v_i \in f_{I(i)} + \operatorname{span}\{f_j \; ; \; j \in [I(i)] \backslash I\}.$$

(iv) There exists a basis $(e_1, \dots, e_n)$ adapted to E such that $(e_{I(i)})_{i \in [r]}$ is a basis of V.

Proof
- Assume (i). Let $(e_1, \dots, e_n)$ a basis adapted to E. Let $i \in [r]$. There is
$$u_i \in E(I(i)) \cap V \backslash E(I(i) - 1)$$

and $(u_i^j)_j \in \mathbb{C}^n$ such that
$$u_i = \sum_{j=1}^{n} u_i^j e_j.$$
Since $u_i \in E(I(i))$ and $u_i \notin E(I(i)-1)$,
$$(\forall j \geqslant I(i)+1, u_i^j = 0) \text{ and } u_i^{I(i)} \neq 0$$
hence, with $\left(\tilde{u}_i^j\right)_j := \left(\frac{1}{u_i^{I(i)}} u_i^j\right)_j$ and $\tilde{u}_i := \frac{1}{u_i^{I(i)}} u_i$,
$$\tilde{u}_i = e_{I(i)} + \sum_{j=1}^{I(i)-1} \tilde{u}_i^j e_j.$$
For all $i \in [r]$, we have
$$v_i := \left(u_i - \sum_{j=1}^{i-1} \tilde{u}_i^{I(j)} u_j\right) \in e_{I(i)} + \operatorname{span}\{e_j;\ ; j \in [I(i)] \backslash I\}$$
and, since $u_1, \ldots, u_r \in V$,
$$v_i \in V.$$
For all $i \in [r]$,
$$v_i \in E(i) \backslash E(i-1)$$
so $(v_1, \ldots, v_r)$ is linearly independent thus is a basis of V. This proves the existence of such a basis. Now, let's show that it is unique.
Let $(w_1, \ldots, w_r)$ a basis of V such that, for all $i \in [r]$,
$$w_i \in e_{I(i)} + \operatorname{span}\{e_j\ ;\ j \in [I(i)] \backslash I\}.$$
Let $i \in [r]$. There is $(w_i^j)_j \in \mathbb{C}^r$ such that
$$w_i = \sum_{j=1}^{r} w_i^j v_j.$$
For all $j \in [r]$,
$$e^*_{I(j)}(w_i) = w_i^j$$
hence,
$$w_i^i = 1 \text{ and } j \neq i \Rightarrow w_i^j = 0$$
i.e.
$$w_i = v_i.$$
Finally, such a basis is unique and we have proven assertion (ii).

- As said in remarks 2.15, $A(E) \neq \emptyset$ hence $(ii) \Rightarrow (iii)$.

- Assume (iii). Let $f := (f_1, \dots, f_n)$ a basis adapted to E. There is a (unique) basis $(v_1, \dots, v_r)$ of V such that, for all $i \in [r]$,

$$v_i \in f_{I(i)} + \operatorname{span}\{f_j \ ; \ j \in [I(i)] \backslash I\}.$$

Let $e := (e_1, \dots, e_n) \in U^n$ such that

$$\begin{cases} \forall i \in [r], & e_{I(i)} = v_i \\ \forall j \in [n] \backslash I, & e_j = f_j \end{cases}.$$

By definition of $(v_1, \dots, v_r)$, $(e_{I(1)}, \dots, e_{I(r)})$ is a basis of V. We only have to show that e is a basis adapted to E. Remark that we have

$$\operatorname{Mat}_f(e) = \begin{pmatrix} 1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \in \operatorname{GL}_n(\mathbb{C})$$

hence e is a basis of U. Let $i \in [n]$. For all $j \in [i]$,

$$e_j = f_j \in E(j) \text{ or } e_j \in f_j + \operatorname{span}\{f_k \ ; \ k \in [j] \backslash I\} \subset E(j)$$

hence $e_j \in E(j) \subset E(i)$. From that we have

$$\operatorname{span}\{e_j \ ; \ j \in [i]\} \subset E(i).$$

But e is linearly independent so $(e_j)_{j \in [i]}$ is linearly independent and

$$\operatorname{span}\{e_j \ ; \ j \in [i]\} = E(i).$$

Finaly, e is a basis adapted to E and we have proven assertion (iv).

- Assume (iv). There exists a basis $(e_i)_{i \in [n]}$ adapted to E such that $(e_{I(j)})_{j \in [r]}$ is a basis of V. For all $j \in [r]$,

$$\begin{cases} E(I(j) - 1) \cap V & = \operatorname{span}\left\{e_{I(1)}, \dots, e_{I(j-1)}\right\} \\ E(I(j)) \cap V & = \operatorname{span}\left\{e_{I(1)}, \dots, e_{I(j)}\right\} \end{cases}$$

hence

$$\dim E(I(j) - 1) \cap V = j - 1 < j = \dim E(I(j)) \cap V.$$

From this we deduce that

$$\operatorname{Pos}(V, E) = I$$

i.e. assertion (i).

□

Example 2.24. Let $c := (c_i)_{i \in [4]}$ the canonical basis of $\mathbb{C}^4$ and $e := (e_i)_{i \in [4]}$ the basis of $\mathbb{C}^4$ such that

$$\operatorname{Mat}_c(e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Let E the flag on $\mathbb{C}^4$ with the adapted basis e. Let V the subspace of $\mathbb{C}^4$ with adapted basis (c_1, c_2). We have

$$\mathrm{Mat}_e(c_1, c_2) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}$$

wich is equivalent to

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

From that we deduce that $(e_i)_{i\in[4]}$ and $(e_2 - e_1, e_1 - e_3 + e_4)$ satisfies condition (iii). In addition to that, we know that $(e_1, e_2 - e_1, e_3, e_1 - e_3 + e_4)$ satisfies condition (iv). In particular,

$$\mathrm{Pos}(V, E) = (2, 4).$$

This allows us to easily compute the induced flags intruduced in definition 2.17.

LEMMA 2.25. Let $V \in \mathrm{Gr}(r, U)$ and $I := \mathrm{Pos}(V, E)$. For all $j \in [r]$ and $k \in [n-r]$,

$$(E^V)_j = E(I(j)) \cap V$$
$$(E_{U/V})_k = E(I^c(k))/V.$$

Proof There is a basis $(e_i)_{i\in[n]}$ adapted to E such that $(e_{I(j)})_{j\in[r]}$ is a basis of V. For all $i \in [n]$,

$$E(i) \cap V = \mathrm{span}\{e_j \; ; \; j \in [i] \cap I\}$$

and

$$E(i)/V = \mathrm{span}\{e_j \; ; \; j \in [i] \cap I^c\}.$$

Let $I(r+1) := I^c(n-r+1) := n+1$. Remark that for all $i \in [n]$, $j \in [r]$ and $k \in [n-r]$,

$$\mathrm{Card}[i] \cap I = j \Leftrightarrow I(j) \leqslant i < I(j+1)$$

and

$$\mathrm{Card}[i] \cap I^c = k \Leftrightarrow I^c(k) \leqslant i < I^c(k+1)$$

hence

$$\min\{l \in [r] \mid \dim E_l \cap V = j\} = I(j)$$

and

$$\min\{l \in [n-r] \mid \dim E_l/V = j\} = I^c(j).$$

□

2.2.3 Composition and quotient of Schubert positions

We have introduced compositions and quotients of tuples in section 2.1.1.

LEMMA 2.26. Let $V \in \mathrm{Gr}(r, U)$ and $W \in \mathrm{Gr}(d, V)$. Let $I := \mathrm{Pos}(V, E)$ and $J := \mathrm{Pos}(W, E^V)$. There is a basis $(e_i)_{i\in[n]}$ adapted to E such that $(e_{I(j)})_{j\in[r]}$ is a basis of V and $(e_{IJ(k)})_{k\in[d]}$.

Proof By lemma 2.23, there is a basis $(f_i)_{i\in[n]}$ adapted to E such that $(f_{I(j)})_{j\in[r]}$ is a basis of V. By lemma 2.25, $(f_{I(j)})_{j\in[r]}$ is adapted to E^V hence, by applying lemma 2.23 again, there is a (unique) basis $(w_k)_{k\in[d]}$ of W such that, for all $k\in[d]$,

$$w_k \in f_{IJ(k)} + \operatorname{span}\left\{f_{I(j)}\ ;\ j\in J^c\cap[J(k)]\right\}.$$

Remark that the sequence $(e_i)_{i\in[n]}$ defined by, for all $i\in[n]$,

$$e_i := \begin{cases} f_i & \text{if } i\notin IJ \\ v_{(IJ)^{-1}(i)} & \text{else} \end{cases}.$$

is just as we wanted. $\square$

Lemma 2.27. Let $V\in\operatorname{Gr}(r,U)$ and $W\in\operatorname{Gr}(d,V)$.

- We have

  $$\operatorname{Pos}(W,E)=\operatorname{Pos}(V,E)\operatorname{Pos}(W,E^V)$$

  and

  $$\operatorname{Pos}(V/W,E_{U/W})=\operatorname{Pos}(V,E)/\operatorname{Pos}(W,E^V).$$

- The sequence $F := ((E(i)\cap V+W)/W)_{i\in[n]}$ is a filtration on V/W and, for all $j\in[r-d]$,

  $$\operatorname{Pos}(V,E)\operatorname{Pos}(W,E^V)^c(j)=\min\left\{i\in[n]\ \middle|\ \dim F(i)=j\right\}.$$

Proof By lemma 2.26, there is a basis $(e_i)_{i\in[n]}$ adapted to E such that $(e_{I(j)})_{j\in[r]}$ is a basis of V and $(e_{IJ(k)})_{k\in[d]}$.

- By lemma 2.23 applied to the adapted basis $(e_i)_{i\in[n]}$, we have the first equality.

  We now want to prove the second equality. Since $(e_{IJ(k)})_{k\in[d]}$ is a basis of W,

  $$\begin{aligned}\forall k\in[n-s],\operatorname{span}\left\{e_{(IJ)^c l}+V\ ;\ l\in[k]\right\} &= \operatorname{span}\left\{e_i\ ;\ i\in[(IJ)^c k]\right\}/V \\ &= (E((IJ)^c(k))+V)/V\end{aligned}$$

  hence, by lemma 2.25, $(e_{(IJ)^c(k)})_{k\in[n-d]}$ is adapted to the flag $E_{U/W}$. In addition to that, $(e_{I(j)})_{j\in[r]}$ is a basis of V and, by lemma 2.4,

  $$\begin{aligned} I\backslash IJ &= IJ^c \\ &= (IJ)^c I/J\end{aligned}$$

  so $(e_{(IJ)^c I/J(k)}+V)_{k\in[r-d]}$ is a basis of V/W. Finally, by lemma 2.23,

  $$\operatorname{Pos}(V/W,E_{U/W})=I/J.$$

- This proved in lemma 3.2.11 in [BVW18].

$\square$

2.2.4 Schubert cells

▷ Settings. Let E be a flag on U.

Definition 2.28. Let $I \in \mathcal{P}_r^n$. The *Schubert cell* associated to E and I is

$$\Omega_I^0(E) := \left\{ V \in \mathrm{Gr}(r, U) \mid \mathrm{Pos}(V, E) = I \right\}.$$

Remarks 2.29.

- By example 2.20,

$$\mathrm{Pos}(\cdot, E) : \left| \begin{array}{ccc} \mathrm{Gr}(r, V) & \longrightarrow & \mathcal{P}_r^n \\ V & \longmapsto & \mathrm{Pos}(V, E) \end{array} \right.$$

is surjective i.e., for all $I \in \mathcal{P}_r^n$,

$$\Omega_I^0(E) \neq \emptyset.$$

- Schubert cells partition $\mathrm{Gr}(r, U)$ in the following way :

$$\mathrm{Gr}(r, U) = \bigsqcup_{I \in \mathcal{P}_r^n} \Omega_I^0(E).$$

- Schubert cells are called cells because they are isomorphic to linear subspaces. See corollary 2.49.

Example 2.30. For all $i \in [n]$,

$$\Omega_{[i]}^0(E) = \{E(i)\}$$

and

$$\Omega_{\{i\}} = \{\mathrm{span}\,\{v\} \ ; \ v \in E_k(i) \backslash E_k(i-1)\}.$$

Propositions 2.31. Let $I \in \mathcal{P}_r^n$.

1. For all $\gamma \in B(E)$,

$$\gamma \Omega_I^0(E) = \Omega_I^0(\gamma E)$$

2. The Schubert cell $\Omega_I^0(E)$ is a $B(E)$-orbit.

Proof

1. For all $W \in \Omega_I^0(E)$, for all $\gamma \in B(E)$, by lemma 2.21, with $V := \gamma W$,

$$\mathrm{Pos}(V, \gamma E) = \mathrm{Pos}(W, E).$$

2. Let $V, W \in \Omega_I^0(E)$. By lemma 2.23, there is $(e_i)_i, (f_i)_i$ basis adapted to E such that $(e_{I(j)})_j$ is a basis of V and $(f_{I(j)})_j$ is a basis of W. By remark 2.15, there is $\gamma \in B(E)$ such that $\gamma \cdot (e_i)_i = (f_i)_i$ hence

$$\gamma \cdot (e_{I(j)})_j = (f_{I(j)})_j$$

and, finaly, $\gamma V = W$.

□

2.2.5 Schubert varieties

▷ Settings. Let $I \in \mathcal{P}_r^n$ and E a flag on U.

Definition 2.32. The *Schubert variety* associated to I and E is the closure of the Schubert cell $\Omega_I^0(E)$ in the Grassmaniann $\mathrm{Gr}(r, U)$ seen as an algebraic variety.

Example 2.33. Because of example 2.30,

$$\Omega_{[r]}(E) = \{E(r)\}.$$

Proposition 2.34. We have

$$\Omega_I(E) = \bigsqcup_{J \leqslant I} \Omega_J^0(E).$$

Proof

- Let $V \in \Omega_I(E)$ and

$$J := \mathrm{Pos}(V, E).$$

There is a convergent sequence $(V_k)_k \in \Omega_I^0(E)^{\mathbb{N}}$ with limit V. Let $j \in [r]$. For all $k \in \mathbb{N}$ sufficiently large,

$$\dim E(I(j)) \cap V \geqslant \dim E(I(j)) \cap V_k$$

hence, since $\dim E(I(j)) \cap V_k = j$,

$$\dim E(I(j)) \cap V \geqslant j.$$

From that we deduce, for all $j \in [r]$,

$$J(j) \leqslant I(j).$$

- For all $J \in \mathcal{P}_r^n$ such that $J \leqslant I$ we denote

$$j_0(J) := \min \{ j \in [r] \mid \forall k \in [\![j+1, n]\!], I(k) = J(k) \}.$$

By induction on $s \in [r]$, let show that the assertion

$$\forall J \leqslant I, j_0(J) \leqslant s \Rightarrow \Omega_J^0(E) \subset \Omega_I(V)$$

denoted $H(s)$ is true. For all $J \leqslant I$ such that $j_0(J) = 0$, $I = J$. From that we deduce $H(0)$.

Let $s \in [\![1, r]\!]$ and assume $H(s-1)$. Let $j_0 := j_0(J)$ and $V \in \Omega_J^0(E)$. By lemma 2.23, there exists $(e_i)_{i \in [n]}$ a basis adapted to E such that $(v_j)_{j \in [r]} := (e_{J(j)})_{j \in [r]}$ is a basis of V.

Let $\varepsilon > 0$ and, for all $j \in [r]$,

$$v_j^\varepsilon := \begin{cases} v_j & \text{if } j \neq j_0 \\ v_{j_0} + \varepsilon e_{I(j_0)} & \text{else} \end{cases}.$$

Let

$$V_\varepsilon := \mathrm{span}\{v_1^\varepsilon, \ldots, v_r^\varepsilon\}.$$

Since $(e_i)_{i \in [n]}$ is linearly independant and $I(j_0) \neq J(j_0)$, $(v_j^\varepsilon)_{j \in [r]}$ is linearly independant and in particular

$$V_\varepsilon \in \mathrm{Gr}(r, U).$$

Let

$$J' := (J(1), \ldots, J(j_0 - 1), I(j_0), \ldots, I(r)).$$

We have

$$J(j_0 - 1) < J(j_0) < I(j_0)$$

hence $J' \in \mathcal{P}_r^n$. For all $i \in [n]$ we denote

$$e_i^\varepsilon := \begin{cases} e_i & \text{if } i \neq I(j_0) \\ \varepsilon e_{I(j_0)} + e_{J(j_0)} & \text{else} \end{cases}.$$

Since $(e_i)_{i\in[n]}$ is a basis adapted to E and $J(j_0) < I(j_0)$, $(e_i^\varepsilon)_{i\in[n]}$ is adapted to E. We have

$$(e^\varepsilon_{J'(j)})_{j\in[r]} = (v_j^\varepsilon)_{j\in[r]}$$

hence, by lemma 2.23,

$$V_\varepsilon \in \Omega^0_{J'}(E).$$

But

$$j_0(J') \leqslant j_0 - 1 = s - 1$$

so, using hypothesis $H(s-1)$,

$$V_\varepsilon \in \Omega_I(E).$$

Using Plucker embedding,

$$V_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} V$$

hence

$$V \in \Omega_I(E).$$

From that we deduce $H(s)$. By induction,

$$\bigcup_{J\leqslant I} \Omega^0_J(E) \subset \Omega_I(E).$$

□

Remark 2.35. For all $J \in \mathcal{P}_r^n$,

$$J \leqslant I \Rightarrow \Omega_J(E) \subset \Omega_I(E)$$

and more generally

$$\Omega_I(E) \cap \Omega_J(E) = \bigcup_{K\leqslant I,J} \Omega^0_K(E).$$

Example 2.36. By propostion 2.34,

$$\Omega_{[\![n-r+1,n]\!]}(E) = \mathrm{Gr}(r, U).$$

2.3 The set $\mathcal{L}_I(F, G)$

This set is a linear subspace of the linear morphisms $\mathcal{L}(V, Q)$ which allows us to parametrize the Schubert cells and varieties.

2.3.1 Definition

▷ Settings. Let $V \in \mathrm{Gr}(r, U)$ and $Q \in \mathrm{Gr}(n-r, U)$. Let F (resp. G) a flag on V (resp. Q). Let $I \in \mathcal{P}_r^n$.

Definition 2.37. We denote

$$\mathcal{L}_I(F, G) := \{\varphi \in \mathcal{L}(V, Q) \mid \forall j \in [r], \varphi(F(j)) \subset G(I(j) - j)\}.$$

Remarks 2.38. • Using lemma 2.2, the set $\mathcal{L}_I(F,G)$ is well defined.

- Let $(f_j)_{j\in[r]}$ a basis adapted to F. We have

$$\mathcal{L}_I(F,G) = \{\varphi \in \mathcal{L}(V,Q) \mid \forall j \in [r], \varphi(f_j) \in G(I(j)-j)\}.$$

Remarks 2.39. • For all $\varphi \in \mathcal{L}_I(F,G)$ and for all $\gamma_v \in \mathrm{GL}(V)$ and $\gamma_q \in \mathrm{GL}(Q)$,

$$\gamma_q \varphi \gamma_v^{-1} \in \mathcal{L}_I(\gamma_v F, \gamma_q G).$$

- As a direct consequence of the last remark,

$$B(G)\mathcal{L}_I(F,G)B(F) = \mathcal{L}_I(F,G).$$

Proposition 2.40. The set $\mathcal{L}_I(F,G)$ is a vector subspace of $\mathcal{L}(V,Q)$ and

$$\dim \mathcal{L}_I(F,G) = \dim I.$$

Proof Let $(f_j)_{j\in[r]}$ (resp. $(g_k)_{k\in[n-r]}$) a basis adapted to F (resp. to G). For all $j \in [r]$, for all $k \in [I(j)-j]$, there is a unique $\varphi_{j,k} \in \mathcal{L}(V,Q)$ such that

$$\forall j' \in [r], \varphi_{j,k}(f_{j'}) = \begin{cases} g_k & \text{if } j' = j \\ 0 & \text{else} \end{cases}.$$

The family $(\varphi_{j,k})_{j\in[r],k\in[I(j)-j]}$ is a basis of $\mathcal{L}_I(F,G)$. □

Lemma 2.41. Let $\varphi \in \mathcal{L}(V,Q)$, $S = \operatorname{Ker}\varphi$ and $\bar{\varphi} \in \mathcal{L}(V/S,Q)$ the corresponding injection. Let $J := \mathrm{Pos}(S,F)$. The following assertions are equivalents.

(i) $\varphi \in \mathcal{L}_I(F,G)$

(ii) $\bar{\varphi} \in \mathcal{L}_{I/J}(F_{V/S},G)$

Proof This is proven in lemma 3.2.15 in [BVW18]. □

2.3.2 A parametrization for Schubert cells and varieties

▷ Settings. Let E a flag on U and $I \in \mathcal{P}^n_r$.

Definition 2.42. Let $V \in \Omega^0_I(E)$. A complement subspace Q of V is *adapted* to E if there exists an basis $(e_i)_{i\in[n]}$ adapted to E such that

$$V = \operatorname{span}\{e_i \; ; \; i \in I\} \text{ and } Q = \operatorname{span}\{e_i \; ; \; i \in I^c\}.$$

Remark 2.43. By lemma 2.23, for all $V \in \Omega^0_I(E)$, there exists a complement subspace Q of V adapted to E.

▷ Settings. Let $V \in \Omega^0_I(E)$ and $(e_i)_{i\in[n]}$ adapted to E such that

$$V = \operatorname{span}\left\{e_{I(j)} \; ; \; j \in [r]\right\}.$$

Let

$$Q := \operatorname{span}\{e_i \; ; \; i \in I^c\}.$$

▷ Notation.

- We denote
$$\mathcal{L}_E(V,Q) := \mathcal{L}_I(E^V, E^Q).$$
- For all $\varphi \in \mathcal{L}(V,U)$, there is a unique $u_\varphi \in \mathcal{L}(U)$ such that
$$(\forall v \in V, u_\varphi(v) = v + \varphi(v)) \text{ and } (\forall q \in Q, u_\varphi(q) = q).$$
We denote
$$U_E(V,Q) := \{u_\varphi \ ; \ \varphi \in \mathcal{L}_E(V,Q)\}.$$

Remarks 2.44.

- Using lemma 2.25 and $Q \simeq U/V$,
$$\mathcal{L}_E(V,Q) \simeq \mathcal{L}_I(E^V, E_{U/V})$$
and
$$\mathcal{L}_E(V,Q) = \left\{\varphi \in \mathcal{L}(V,Q) \mid \forall j \in [r], \varphi(e_{I(j)}) \in \operatorname{span}\left\{e_{I^c(k)} \ ; \ k \in [I(j)-j]\right\}\right\}.$$
- By lemma 2.2, for all $j \in [r]$,
$$\operatorname{span}\left\{e_{I^c(k)} \ ; \ k \in [I(j)-j]\right\} = \operatorname{span}\left\{e_i \ ; \ i \in I^c, i < I(j)\right\}.$$
This will allow us to easily use lemma 2.23.
- Let $v := (e_{I(j)})_{j\in[r]}$, $q := (e_{I^c(k)})_{k\in[n-r]}$ and $e' := (v_1, \ldots, v_r, q_1, \ldots, q_{n-r})$. For all $\varphi \in \mathcal{L}_E(V,Q)$,
$$\operatorname{Mat}_{e'}(u_\varphi) = \begin{pmatrix} \mathrm{I}_r & 0 \\ \operatorname{Mat}_{v,q}(\varphi) & \mathrm{I}_{n-r} \end{pmatrix}.$$

Lemma 2.45. Let $\gamma \in \mathrm{GL}(U)$.

1. We have
$$\mathcal{L}_{\gamma E}(\gamma V, \gamma Q) = \gamma \mathcal{L}_E(V,Q)\gamma^{-1}.$$
2. For all $\varphi \in \mathcal{L}_E(V,Q)$,
$$\gamma u_\varphi \gamma^{-1} = u_{\gamma\varphi\gamma^{-1}}$$
and
$$U_{\gamma E}(\gamma V, \gamma Q) = \gamma U_E(V,Q)\gamma^{-1}.$$

Proof

1. The basis $(\gamma e_i)_{i\in[n]}$ is adapted to the flag γE and we have
$$\gamma V = \operatorname{span}\left\{\gamma e_{I(j)} \ ; \ j \in [r]\right\} \text{ and } \gamma Q = \operatorname{span}\left\{\gamma e_{I^c(k)} \ ; \ k \in [n-r]\right\}.$$
By remarks 2.44, for all $\varphi \in \mathcal{L}(\gamma V, \gamma Q)$,
$$\begin{aligned}\varphi \in \mathcal{L}_{\gamma E}(\gamma V, \gamma Q) &\Leftrightarrow \forall j \in [r], \varphi(\gamma e_{I(j)}) \in \operatorname{span}\left\{\gamma e_{I^c(k)} \ ; \ k \in [I(j)-j]\right\} \\ &\Leftrightarrow \forall j \in [r], \gamma^{-1}\varphi(\gamma e_{I(j)}) \in \operatorname{span}\left\{e_{I^c(k)} \ ; \ k \in [I(j)-j]\right\} \\ &\Leftrightarrow \gamma^{-1}\varphi\gamma \in \mathcal{L}_E(V,Q).\end{aligned}$$
2. Let $\varphi \in \mathcal{L}_E(V,Q)$ and $\tilde{\varphi} := \gamma\varphi\gamma^{-1}$ wich is an element of $\mathcal{L}_{\gamma E}(\gamma V, \gamma Q)$. For all $v \in V$ and $q \in Q$
$$(\gamma u_\varphi \gamma^{-1})(\gamma v) = \gamma v + \tilde{\varphi}(\gamma v) \text{ and } (\gamma u_\varphi \gamma^{-1})(\gamma q) = \gamma q$$

hence

$$\gamma u_\varphi \gamma^{-1} = u_{\gamma\varphi\gamma^{-1}}.$$

Using this and the first point of the lemma,

$$U_{\gamma E}(\gamma V, \gamma Q) = \gamma U_E(V, Q)\gamma^{-1}.$$

$\square$

LEMMA 2.46. The set $U_E(V, Q)$ is a unipotent subgroup of $B(E)$ and

$$\begin{array}{ccc} (\mathcal{L}_E(V, Q), +) & \longrightarrow & (U_E(V, Q), \circ) \\ \varphi & \longmapsto & u_\varphi \end{array}$$

is a group isomorphism.

Proof By the second point of remarks 2.44, $U_E(V, Q)$ is a unipotent subgroup of $\mathrm{GL}(U)$ and the given map is a group isomorphism.

To conclude, we only have to show that $U_E(V, Q) \subset B(E)$. Let $\varphi \in \mathcal{L}_E(V, Q)$. For all $i \in I^c$, $e_i \in Q$ so

$$u_\varphi(e_i) = e_i \in E(i).$$

For all $j \in [r]$, with $i = I(j)$, $e_i \in V$ so

$$\begin{aligned} u_\varphi(e_i) &= e_i + \varphi(e_i) \\ &\in E(i) + \operatorname{span}\left\{e_{I^c(k)} \ ; \ k \in [I(j) - j]\right\} \\ &\in E(i) + E(i - 1) \\ &\in E(i). \end{aligned}$$

From this we deduce that $u_\varphi \in B(E)$. $\square$

REMARK 2.47. Since $U_E(V, Q) \subset B(E)$, we know from propositions 2.31 that $\Omega_I^0(E)$ is stable under the left action of $U_E(V, Q)$.

LEMMA 2.48. The group action $U_E(V, Q) \curvearrowright \Omega_I^0(E)$ coming from the action $\mathrm{GL}(U) \curvearrowright \Omega_I^0(E)$ is simply transitive.

Proof This is a consequence of lemma 2.23.

- Let $W \in \Omega_I^0(E)$. By lemma 2.23, there exists a unique basis $(w_j)_{j\in[r]}$ such that, for all $j \in [r]$,

  $$w_j \in e_{I(j)} + \operatorname{span}\{e_i \ ; \ i \in I^c, i < I(a)\}.$$

  There is a unique $\varphi \in \mathcal{L}(V, Q)$ such that, for all $j \in [r]$,

  $$\varphi(e_{I(j)}) = w_j - e_{I(j)}.$$

  By definition of $(w_j)_j$, for all $j \in [r]$,

  $$\varphi(e_{I(j)}) \in \operatorname{span}\{e_i \ ; \ i \in I^c, i < I(a)\}$$

  hence $\varphi \in \mathcal{L}_E(V, Q)$. Because u_φ sends the basis $(e_{I(j)})_j$ of V on the basis $(w_j)_j$ of W,

  $$u\varphi(V) = W.$$

  From this we deduce that $U_E(V, Q) \curvearrowright \Omega_I^0(E)$ is transitive.

- Let $\varphi, \psi \in \mathcal{L}_E(V, Q)$ such that

$$W := u_\varphi(V) = u_\psi(V).$$

Since $(e_{I(j)})_{j\in[r]}$ is a basis of V,

$$(w_j)_{j\in[r]} := (u_\varphi(e_{I(j)}))_{j\in[r]} \text{ and } (w'_j)_{j\in[r]} := (u_\psi(e_{I(j)}))_{j\in[r]}$$

are basis of W. Remark that for all $j \in [r]$,

$$\begin{aligned} w_j &= e_{I(j)} + \varphi(e_{I(j)}) \\ &\in e_{I(j)} + \operatorname{span}\{e_i \; ; \; i \in I^c, i < I(j)\} \end{aligned}$$

hence, by lemma 2.23,

$$\operatorname{Pos}(W, E) = I.$$

But we also have, for all $j \in [r]$,

$$w'_j \in e_{I(j)} + \operatorname{span}\{e_i \; ; \; i \in I^c, i < I(a)\}$$

hence, by lemma 2.23 again,

$$(w_j)_j = (w'_j)_j$$

i.e. u_φ agree to u_ψ on V. By definition, u_φ agree to u_ψ on Q. Finally,

$$u_\varphi = u_\psi.$$

Finally, $U_E(V, Q) \curvearrowright \Omega_I^0(E)$ is simply transitive.

□

Corollary 2.49. Let $V \in \Omega_I^0(E)$ and Q a complement space of V adapted to E. We have

$$\mathcal{L}_E(V, Q) \simeq U_E(V, Q) \simeq U_E(V, Q)V = \Omega_I^0(E).$$

Proof The first isomorphism comes from lemma 2.46. The second isomorphism and the equality come from lemma 2.48. □

2.3.3 Consequences

▷ Settings. Let E a flag on U, $V \in \operatorname{Gr}(r, U)$ and Q a complement space of V adapted to E. Let F (resp. G) a flag on V (resp. Q). Let $I \in \mathcal{P}_r^n$.

Proposition 2.50. We have

$$T_V \Omega_I^0(E) \simeq \mathcal{L}_E(V, Q)$$

and

$$\dim \Omega_I(E) = \dim \Omega_I^0(E) = \dim \mathcal{L}_E(V, Q) = \dim I.$$

Proof The isomorphism comes from corollary 2.49. The first dimension equality comes from the definition of Schubert varieties. The second equality comes from the isomorphism. The last equality comes from proposition 2.40. □

Proposition 2.51. Let S a vector subspace of V and $J := \mathrm{Pos}(S, F)$. We have

$$\mathcal{L}_{I/J}(F_{V/S}, G)\mathcal{L}_J(F^S, F_{V/S}) \subset \mathcal{L}_{I^J}(F^S, G).$$

Proof This is proven in lemma 3.2.15 in [BVW18]. □

2.4 About flag varieties

We will consider the set of all flags on U and two of its subvarieties. Computing their dimensions gives us useful equations in section 3.

2.4.1 All flags

▷ Settings. Let E be a flag on U.

Definition 2.52. The set of all flags on U is denoted by $\mathrm{Flag}(U)$.

Remark 2.53. The action of $\mathrm{GL}(U)$ on $\mathrm{Flag}(U)$ is transitive hence $\mathrm{Flag}(U)$ is an irreducible variety.

Proposition 2.54. We have, with B_n the set of all invertible upper triangular complex matrices and $E \in \mathrm{Flag}(U)$,

$$\mathrm{Flag}(U) \simeq \mathrm{GL}(U)/B(E) \simeq \mathrm{GL}_n / B_n$$

and in particular

$$\dim \mathrm{Flag}(U) = \frac{n(n-1)}{2}.$$

Proof The action of $\mathrm{GL}(U)$ on $\mathrm{Flag}(U)$ is transitive and $B(E)$ is the stabilizer of E so

$$\mathrm{Flag}(U) \simeq \mathrm{GL}(U)/B(E).$$

The choice of a basis adapted to E gives us

$$\mathrm{Flag}(U) \simeq \mathrm{Flag}(\mathbb{C}^n) \text{ and } B(E) \simeq B_n.$$

□

2.4.2 Flags defined by a given position

▷ Settings. Let $I \in \mathcal{P}_r^n$, $V \in \mathrm{Gr}(r, U)$ and E a flag on U.

Definition 2.55.
- We denote

$$\mathrm{Flag}_I^0(V, U) := \{E \in \mathrm{Flag}(U) \mid \mathrm{Pos}(V, E) = I\}$$

  and $\mathrm{Flag}_I(V, U)$ its closure in $\mathrm{Flag}(U)$.

- The parabolic subgroup associated to V is

$$P(V, U) := \{\gamma \in \mathrm{GL}(U) \mid \gamma V \subset V\}.$$

- For all $E \in \mathrm{Flag}(U)$ we denote
$$G_I(V,E) := \left\{\gamma \in \mathrm{GL}(U) \mid \gamma E \in \mathrm{Flag}_I^0(V,U)\right\}.$$

Remarks 2.56. • The parabolic subgroup $P(V,U)$ is a subgroup of $\mathrm{GL}(U)$ and, for all $\gamma \in \mathrm{GL}(U)$,
$$P(\gamma V,U) = \gamma P(V,U)\gamma^{-1}.$$

- We have
$$\mathrm{Gr}(r,U) \simeq \mathrm{GL}(U)/P(V,U)$$
and
$$\dim P(V,U) = r^2 + n(n-r)$$
hence $\dim \mathrm{Gr}(r,U) = r(n-r)$.

Propositions 2.57. Let $E \in \mathrm{Flag}(U)$.

1. The set $G_I(V,E)$ is not empty.
2. For all $\alpha,\beta \in \mathrm{GL}(U)$,
$$G_I(\alpha V,\beta E) = \alpha G_I(V,E)\beta^{-1}.$$
3. If $V \in \Omega_I^0(E)$,
$$G_I(V,E) = P(V,U)B(E).$$
4. Assume $V \in \Omega_I^0(E)$ and let Q a complement subspace of V adapted to E. We have
$$G_I(V,E) = P(V,U)U_E(V,Q).$$
5. We have
$$\dim G_I(V,E) = \dim P(V,U) + \dim I.$$

Proof 1. This comes from example 2.20.

2. This comes from lemma 2.21 and the fact that, for all $\gamma \in \mathrm{GL}(U)$,
$$\gamma \in G_I(\alpha V,\beta E) \Leftrightarrow \mathrm{Pos}(\alpha V,\gamma\beta E) = I.$$
3. For all $\gamma \in \mathrm{GL}(U)$, using propositions 2.31,
$$\Omega_I^0(\gamma E) = \gamma B(E)V$$
hence
$$\begin{aligned}\gamma \in G_I(V,E) &\Leftrightarrow V \in \gamma B(E)V\\ &\Leftrightarrow \gamma B(E)\cap P(V,U) \neq \emptyset\\ &\Leftrightarrow \gamma \in P(V,U)B(E).\end{aligned}$$
4. We use the method of the previous point. For all $\gamma \in \mathrm{GL}(U)$, using lemma 2.48,
$$\Omega_I^0(\gamma E) = \gamma U_E(V,Q)V$$

hence

$$\begin{aligned}\gamma \in G_I(V,E) &\Leftrightarrow V \in \gamma U_E(V,Q)V \\ &\Leftrightarrow \gamma U_E(V,Q) \cap P(V,U) \neq \emptyset \\ &\Leftrightarrow \gamma \in P(V,U)U_E(V,Q).\end{aligned}$$

5. There exists $(e_i)_{i\in[n]}$ an adapted basis to E such that $v := (e_i)_{i\in I}$ is a basis of V and $q := (e_i)_{i\in I^c}$ is a basis of Q. For all $\varphi \in \mathcal{L}_E(V,Q)$ we will identify φ with the matrix $\mathrm{Mat}_{v,q}(\varphi)$. Using the fourth point,

$$\begin{aligned}G_I(V,E) &= P(V,U)U_E(V,Q) \\ &\simeq \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \mathrm{I}_r & 0 \\ \varphi & \mathrm{I}_{n-r} \end{pmatrix} \ ;\ a \in \mathrm{GL}(r), d \in \mathrm{GL}(n-r), \varphi \in \mathcal{L}_E(V,Q) \right\} \\ &\simeq \left\{ \begin{pmatrix} a' & b \\ c' & d \end{pmatrix} \ ;\ d \in \mathrm{GL}(n-r), a' - bd^{-1}c' \in \mathrm{GL}(r), d^{-1}c' \in \mathcal{L}_E(V,Q) \right\}\end{aligned}$$

hence

$$\begin{aligned}\dim G_I(V,E) &= \dim \mathrm{GL}(r) + \dim \mathrm{GL}(n-r) + \dim \mathcal{L}_E(V,Q) \\ &= \dim P(V,U) + \dim I.\end{aligned}$$

□

Remark 2.58. From the second point of propositions 2.57 we know that

$$\begin{array}{ccc} P(V,U) \times G_I(V,E) & \longrightarrow & G_I(V,E) \\ (g,\gamma) & \longmapsto & g\gamma \end{array} \quad \text{and} \quad \begin{array}{ccc} B(E) \times G_I(V,E) & \longrightarrow & G_I(V,E) \\ (g,\gamma) & \longmapsto & \gamma g^{-1} \end{array}$$

are group actions.

Lemma 2.59. For all $\gamma \in \mathrm{GL}(U)$,

$$\begin{aligned}\gamma \,\mathrm{Flag}_I^0(V,U) &= \mathrm{Flag}_I^0(\gamma V,U) \\ \gamma \,\mathrm{Flag}_I(V,U) &= \mathrm{Flag}_I(\gamma V,U)\end{aligned}$$

Proof The first equality is a direct application of lemma 2.21. The second one is

$$\overline{\gamma \,\mathrm{Flag}_I^0(V,U)} = \overline{\mathrm{Flag}_I^0(\gamma V,U)}.$$

□

Remark 2.60. In particular, $\mathrm{Flag}_I^0(V,U)$ and $\mathrm{Flag}_I(V,U)$ are stable under the action of $P(V,U)$ coming from $\mathrm{GL}(U) \curvearrowright \mathrm{Flag}(U)$.

The set $G_I(V,E)$ has been introduced in order to study the set $\mathrm{Flag}_I^0(V,U)$ using the following lemma.

Lemma 2.61. Let $E \in \mathrm{Flag}(U)$.

1. We have
$$\mathrm{Flag}_I^0(V,U) \simeq G_I(V,E)/B(E).$$

2. If $V \in \Omega_I^0(E)$,
$$\mathrm{Flag}_I^0(V,U) = P(V,U)E.$$

3. We have

$$\begin{aligned}\dim \operatorname{Flag}^0_I(V,U) = \dim \operatorname{Flag}_I(V,U) &= \dim \operatorname{Flag}(V) + \dim \operatorname{Flag}(Q) + \dim I\\ &= \dim \operatorname{Flag}(U) - \dim \operatorname{Gr}(r,U) + \dim I\end{aligned}$$

Proof 1. The set $G_I(V,E)$ is defined to verify

$$G_I(V,E)E = \operatorname{Flag}^0_I(V,U) \tag{$*$}$$

hence, using proposition 2.54,

$$\operatorname{Flag}^0_I(V,U) \simeq G_I(V,E)/B(E)$$

2. In addition to the previous equation $(*)$ we have

$$B(E)E = \{E\}.$$

From this, using propositions 2.57, we deduce

$$\operatorname{Flag}^0_I(V,U) = P(V,U)E.$$

3. Since $\operatorname{Flag}_I(V,U)$ is the closure of $\operatorname{Flag}^0_I(V,U)$, they are of the same dimension. In addition to this, using the first point,

$$\begin{aligned}\dim \operatorname{Flag}^0_I(V,U) &= \dim G_I(V,E) - \dim B(E)\\ &= \dim P(V,U) + \dim I - \dim B(E)\\ &= \dim \operatorname{GL}(U) - \dim \operatorname{Gr}(r,U) + \dim I - \dim B(E)\\ &= \dim \operatorname{Flag}(U) - \dim \operatorname{Gr}(r,U) + \dim I.\end{aligned}$$

Yet from 2.56 and proposition 2.54 we know that

$$\dim \operatorname{Flag}(U) - \dim \operatorname{Gr}(r,U) = \dim \operatorname{Flag}(V) + \dim \operatorname{Flag}(Q)$$

hence

$$\dim \operatorname{Flag}^0_I(V,U) = \dim \operatorname{Flag}(V) + \dim \operatorname{Flag}(Q) + \dim I.$$

□

2.4.3 Flags giving a precise morphism

▷ Settings. Let $I \in \mathcal{P}^n_r$, $V \in \operatorname{Gr}(r,U)$, $F \in \operatorname{Flag}(V)$ and $\varphi \in \mathcal{L}(V,Q)$ an injective linear morphism. This implies that

$$n \geqslant 2r.$$

Definition 2.62. We denote

$$\operatorname{Flag}^0_I(F,\varphi) := \{G \in \operatorname{Flag}(Q) \mid \varphi \in \mathcal{L}_I(F,G)\}.$$

Remark 2.63. For all $I, J \in \mathcal{P}^n_r$,

$$I \leqslant J \Rightarrow \operatorname{Flag}^0_I(F,\varphi) \subset \operatorname{Flag}^0_J(F,\varphi).$$

LEMMA 2.64. 1. We have

$$\mathrm{Flag}^0_I(F,\varphi) \neq \emptyset \Leftrightarrow \forall i \in [r], I(i) \geqslant 2i.$$

2. If $\mathrm{Flag}^0_I(V,Q)$ is nonempty, it is a smooth irreducible subvariety of $\mathrm{Flag}(Q)$ and

$$\dim \mathrm{Flag}^0_I(F,\varphi) = \dim \mathrm{Flag}(Q) + \dim I - (\dim V)(\dim Q).$$

Proof 1. First, assume there exists $G \in \mathrm{Flag}^0_I(F,\varphi)$. For all $i \in [r]$, since φ is injective,

$$\begin{aligned} I(i) &= \dim \varphi(F(I(i))) \\ &\leqslant \dim G(I(i)-i) \\ &\leqslant I(i)-i. \end{aligned}$$

On the other hand, assume now that for all $i \in [r]$ we have

$$I(i) - i \geqslant i.$$

Let $(v_i)_{i\in[r]}$ a basis adapted to F and $(q_i)_{i\in[r]} := (\varphi(v_i))_{i\in[r]}$. There exists $(q_{r+i})_{i\in[n-2r]}$ such that $(q_i)_{i\in[n-r]}$ is a basis of Q. Let G be the flag on Q associated to the basis q. For all $i \in [r]$,

$$\begin{aligned} \varphi(F(i)) &= \mathrm{span}\,\{\varphi(v_j)\ ;\ j \in [i]\} \\ &= G(i) \\ &\subset G(I(i)-i) \end{aligned}$$

hence $G \in \mathrm{Flag}^0_I(F)$.

2. This is proven in lemma 3.3.10 in [BVW18].

□

2.4.4 Remarks about cells

For all $V \in \mathrm{Gr}(r,U)$, $E \in \mathrm{Flag}(U)$ and $I \in \mathcal{P}^n_r$,

$$V \in \Omega^0_I(E) \Leftrightarrow E \in \mathrm{Flag}^0_I(V,U).$$

We have already seen in remarks 2.29 that fixing a flag on U gives us a decomposition of the Grassmaniann in Schubert cells

$$\mathrm{Gr}(r,U) = \bigsqcup_{I\in\mathcal{P}^n_r} \Omega^0_I(E).$$

In a similar way, fixing a linear subspace $V \in \mathrm{Gr}(r,U)$ gives us the decomposition

$$\mathrm{Flag}(U) = \bigsqcup_{I\in\mathcal{P}^n_r} \mathrm{Flag}^0_I(V,U).$$

It is although not the case for the flags introduced in section 2.4.3. For example, for all $F \in \mathrm{Flag}(V)$ and all injective map $\varphi \in \mathcal{L}(V,Q)$, by remark 2.63,

$$\begin{aligned} \bigcup_{I\in\mathcal{P}^n_r} \mathrm{Flag}^0_I(F,\varphi) &= \mathrm{Flag}^0_{[\![n-r+1,n]\!]}(F,\varphi) \\ &= \{G \in \mathrm{Flag}(Q) \mid \varphi(F(r)) \subset G(n-r)\} \\ &= \mathrm{Flag}(Q). \end{aligned}$$

3 Intersecting tuples

In this section, we introduce the notion of intersecting tuples and Horn's tuples and prove Belkale's theorem 3.42 (showing that these two definitions are equivalent).

3.1 Definition

▷ SETTINGS.
- Let $n, r, d, m \in \mathbb{N}^*$ such that $n \geqslant r \geqslant d \geqslant m$. Let $s \in \mathbb{N}^*$.
- Let U a $\mathbb{C}$-linear space of finite dimension n.

▷ NOTATION. For all $\mathcal{I} \in (\mathcal{P}_r^n)^s$ and $E \in \mathrm{Flag}(U)^s$ we denote

$$\Omega_{\mathcal{I}}(E) := \bigcap_{k=1}^{s} \Omega_{\mathcal{I}_k}(E_k)$$

and, for all $V \in \mathrm{Gr}(r, U)$, $Q \in \mathrm{Gr}(n-r, U)$, $F \in \mathrm{Flag}(V)^s$ and $G \in \mathrm{Flag}(Q)^s$,

$$\mathcal{L}_{\mathcal{I}}(F, G) := \bigcap_{k=1}^{s} \mathcal{L}_{\mathcal{I}_k}(F_k, G_k).$$

DEFINITION 3.1. An tuple $\mathcal{I} \in (\mathcal{P}_r^n)^s$ is *intersecting* if, for all $E \in \mathrm{Flag}(U)^s$

$$\Omega_{\mathcal{I}}(E) \neq \emptyset.$$

We denote

$$\mathrm{Intersecting}(r, n, s) := \left\{\mathcal{I} \in (\mathcal{P}_r^n)^s \mid \mathcal{I} \text{ is intersecting}\right\}$$

and

$$\mathrm{Intersecting}(r, n, s)^0 := \left\{\mathcal{I} \in \mathrm{Intersecting}(r, n, s) \mid \operatorname{edim} \mathcal{I} = 0\right\}.$$

DEFINITION 3.2. We define

$$\mathrm{Horn}(1, n, s) := \left\{\mathcal{I} \in (\mathcal{P}_1^n)^s \mid \operatorname{edim} \mathcal{I} \geqslant 0\right\}$$
$$\mathrm{Horn}^0(1, n, s) := \left\{\mathcal{I} \in (\mathcal{P}_1^n)^s \mid \operatorname{edim} \mathcal{I} = 0\right\}.$$

and, by induction, we define $\mathrm{Horn}(r, n, s)$ as the set

$$\left\{\mathcal{I} \in (\mathcal{P}_r^n)^s \mid \operatorname{edim} \mathcal{I} \geqslant 0 \text{ and } \forall d \in [r-1], \forall \mathcal{J} \in \mathrm{Horn}^0(d, r, s), \operatorname{edim} \mathcal{I}\mathcal{J} \geqslant 0\right\}$$

and we denote

$$\mathrm{Horn}^0(r, n, s) := \left\{\mathcal{I} \in \mathrm{Horn}(r, n, s) \mid \operatorname{edim} \mathcal{I} = 0\right\}.$$

EXAMPLE 3.3. Let $\mathcal{I} := (\{i_1\}, \ldots, \{i_s\}) \in \mathrm{Horn}(1, n, s)$. Let $E \in \mathrm{Flag}(U)^s$. By proposition 2.34 and example 2.30, for all $k \in [s]$,

$$\begin{aligned}
\Omega_{\{i_k\}}(E_k) &= \bigcup_{J \leqslant \{i_k\}} \Omega_J^0(E_k) \\
&= \bigcup_{i \in [i_k]} \Omega_{\{i\}}^0(E_k) \\
&= \bigcup_{i \in [i_k]} \{\mathrm{span}\{v\} \ ; \ v \in E_k(i) \backslash E_k(i-1)\} \\
&= \{\mathrm{span}\{v\} \ ; \ v \in E(i_k) \backslash \{0\}\}
\end{aligned}$$

hence

$$\Omega_{\mathcal{I}}(E) = \left\{ \operatorname{span}\{v\}\ ;\ v \in \left(\bigcap_{k\in[s]} E_k(i_k) \right) \setminus \{0\} \right\}.$$

But

$$\dim \prod_{k\in[s]} \mathbb{C}^n/E_k(i_k) = \sum_{k=1}^{s}(n-i_k)$$

and

$$0 \leqslant \operatorname{edim}\mathcal{I} = (n-1) - \sum_{k=1}^{s}(n-i_k)$$

so

$$\dim \prod_{k\in[s]} \mathbb{C}^n/E_k(i_k) < \dim \mathbb{C}^n.$$

For all $k \in [s]$ we denote by π_k the canonical projection $\mathbb{C}^n \twoheadrightarrow \mathbb{C}^n/E_k(i_k)$. From this follows that the map

$$\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \prod_{k\in[s]} \mathbb{C}^n/E_k(i_k) \\ v & \longmapsto & (\pi_k(v))_k \end{array}$$

is not injective, i.e.

$$\bigcap_{k\in[s]} E_k(i_k) \neq \{0\}$$

i.e.

$$\Omega_{\mathcal{I}}(E) \neq \emptyset.$$

From this we deduce that $\mathcal{I}$ is intersecting. Finally,

$$\operatorname{Horn}(r,n,s) \subset \operatorname{Intersecting}(r,n,s).$$

Proposition 3.4. We have

$$\operatorname{Intersecting}(r,n,s) \circ \operatorname{Intersecting}(d,r,s) \subset \operatorname{Intersecting}(d,n,s).$$

Proof Let $\mathcal{I} \in \operatorname{Intersecting}(r,n,s)$ and $\mathcal{J} \in \operatorname{Intersecting}(d,r,s)$. Let $E \in \operatorname{Flag}(U)^s$. There exists $V \in \Omega_{\mathcal{I}}(E)$ and $W \in \Omega_{\mathcal{J}}(E^V)$. From proposition 2.34 we know that, for all $k \in [s]$,

$$\begin{aligned} \operatorname{Pos}(V, E_k) &\leqslant \mathcal{I}_k \\ \operatorname{Pos}(W, E_k^V) &\leqslant \mathcal{J}_k \end{aligned}$$

hence

$$\begin{aligned} \operatorname{Pos}(W, E_k) &= \operatorname{Pos}(V, E_k)\operatorname{Pos}(S, E_k^V) \\ &\leqslant \operatorname{Pos}(V, E_k)\mathcal{J}_k \\ &\leqslant \mathcal{I}_k\mathcal{J}_k. \end{aligned}$$

Using proposition 2.34 again,

$$W \in \Omega_{\mathcal{I}\mathcal{J}}(E).$$

From this we deduce that $\mathcal{I}\mathcal{J}$ is intersecting. □

3.2 Horn inequalities

3.2.1 Using dominance

▷ NOTATION. We denote

$$\omega^0_{\mathcal{I}} : \left| \begin{array}{ccc} \mathrm{GL}(U) \times \prod_{k=1}^s \mathrm{Flag}^0_{\mathcal{I}_k}(V,U) & \longrightarrow & \mathrm{Flag}(U)^s \\ (\gamma, E) & \longmapsto & (\gamma E_k)_k \end{array} \right.$$

and

$$\omega_{\mathcal{I}} : \left| \begin{array}{ccc} \mathrm{GL}(U) \times \prod_{k=1}^s \mathrm{Flag}_{\mathcal{I}_k}(V,U) & \longrightarrow & \mathrm{Flag}(U)^s \\ (\gamma, E) & \longmapsto & (\gamma E_k)_k \end{array} \right. .$$

REMARK 3.5. The continuous map $\omega_{\mathcal{I}}$ is an extension of the continuous map $\omega^0_{\mathcal{I}}$ and

$$\mathrm{Im}\omega^0_{\mathcal{I}} \subset \mathrm{Im}\omega_{\mathcal{I}}.$$

LEMMA 3.6. We have

$$\begin{aligned} \mathrm{Im}\omega^0_{\mathcal{I}} &= \left\{ E \in \mathrm{Flag}(U)^s \mid \Omega^0_{\mathcal{I}}(E) \neq \emptyset \right\} \\ \mathrm{Im}\omega_{\mathcal{I}} &= \left\{ E \in \mathrm{Flag}(U)^s \mid \Omega_{\mathcal{I}}(E) \neq \emptyset \right\}. \end{aligned}$$

Proof

- Let $E \in \mathrm{Flag}(U)^s$. Using lemma 2.59 we have

$$\begin{aligned} E \in \mathrm{Im}\omega^0_{\mathcal{I}} &\Leftrightarrow \exists \gamma \in \mathrm{GL}(U), \forall k \in [s], E_k \in \gamma\, \mathrm{Flag}^0_{\mathcal{I}_k}(V,U) \\ &\Leftrightarrow \exists \gamma \in \mathrm{GL}(U), \forall k \in [s], V \in \Omega^0_{\mathcal{I}_k}(\gamma^{-1}E_k) \\ &\Leftrightarrow \exists \gamma \in \mathrm{GL}(U), \gamma V \in \Omega^0_{\mathcal{I}}(E) \\ &\Leftrightarrow \exists V' \in \mathrm{Gr}(r,U), V' \in \Omega^0_{\mathcal{I}}(E). \end{aligned}$$

- This is exactly the same proof without the zeros in exponents.

□

LEMMA 3.7. The following assertions are equivalent.

(i) The tuple $\mathcal{I}$ is intersecting.

(ii) The map $\omega_{\mathcal{I}}$ is surjective.

(iii) The map $\omega_{\mathcal{I}}$ is dominant.

(iv) The map $\omega^0_{\mathcal{I}}$ is dominant.

Proof

- The equivalence (i) $\Leftrightarrow$ (ii) is a direct consequence of lemma 3.6.
- From

$$\mathrm{Im}\omega^0_{\mathcal{I}} \subset \mathrm{Im}\omega_{\mathcal{I}}$$

we know that (iv) $\Rightarrow$ (iii). Let A be the domain of $\omega^0_{\mathcal{I}}$. The domain of $\omega_{\mathcal{I}}$ is the closure $\overline{A}$ and $\omega_{\mathcal{I}}$ is continous hence

$$\begin{aligned} \mathrm{Im}\omega_{\mathcal{I}} &= \omega_{\mathcal{I}}\left(\overline{A}\right) \\ &\subset \overline{\omega_{\mathcal{I}}(A)} \\ &= \overline{\mathrm{Im}\omega^0_{\mathcal{I}}}. \end{aligned}$$

From this we deduce (iii) $\Rightarrow$ (iv).

- The implication (ii) $\Rightarrow$ (iii) comes from the fact that a surjective map is always dominant.
- To conclude the proof of the lemma it now remains to show that (iii) $\Rightarrow$ (ii). Assume (iii). Using lemma 2.59, we can consider the action of $P(V,U)$ on the domain of $\omega_{\mathcal{I}}$ given by, for all $b \in P(V,U)$ and all $(\gamma, E) \in \mathrm{GL}(U) \times \prod_{k=1}^{s} \mathrm{Flag}_{\mathcal{I}_k}(V,U)$,

$$b \cdot (\gamma, E) := (b^{-1}\gamma, \gamma E)$$

  and remark that

$$\omega_{\mathcal{I}}(b \cdot (\gamma, E)) = \omega_{\mathcal{I}}(\gamma, E).$$

  Let $\bar{\omega}_{\mathcal{I}}$ be the factorisation of $\omega_{\mathcal{I}}$ through this group action. Since $\omega_{\mathcal{I}}$ and $\bar{\omega}_{\mathcal{I}}$ have the same image, $\bar{\omega}_{\mathcal{I}}$ is also dominant. Yet its domain is compact so its image is closed. We deduce that $\bar{\omega}_{\mathcal{I}}$ is surjective, likewise $\omega_{\mathcal{I}}$.

□

Proposition 3.8. If $\mathcal{I}$ is intersecting,

$$\operatorname{edim} \mathcal{I} \geqslant 0.$$

Proof We use the same notation as in the proof of lemma 3.24. We denote by X the domain of $\bar{\omega}_{\mathcal{I}}$ and by Y its image. Since $\mathcal{I}$ is intersecting,

$$\begin{aligned} 0 &\leqslant \dim X - \dim \mathrm{Flag}(U)^s \\ &= \dim \mathrm{GL}(U)/P(V,U) + \sum_{k=1}^{s} \mathrm{Flag}_{\mathcal{I}_k}(V,U) - \sum_{k=1}^{s} \dim \mathrm{Flag}(U) \\ &\overset{2.61}{=} \dim \mathrm{Gr}(r,U) + \sum_{k=1}^{s} (\dim \mathcal{I}_k - \dim \mathrm{Gr}(r,U)) \\ &= \operatorname{edim} \mathcal{I}. \end{aligned}$$

□

Example 3.9. From proposition 3.8 we have

$$\mathrm{Intersecting}(1,n,s) \subset \mathrm{Horn}(1,n,s)$$

hence, using example 3.3,

$$\mathrm{Intersecting}(1,n,s) = \mathrm{Horn}(1,n,s).$$

This is the base case of the induction proof of Belkale's theorem 3.42.

3.2.2 Using slopes

Definition-proposition 3.10. There exists a subset $\mathrm{Good}(U,s) \subset \mathrm{Flag}(U)^s$ which satisfies the following properties.

1. The set $\mathrm{Good}(U,s)$ is a nonempty Zariski-open subset of $\mathrm{Flag}(U)^s$.
2. We have

$$\mathrm{Intersecting}(r,n,s) = \left\{ \mathcal{I} \in (\mathcal{P}_r^n)^s \mid \exists E \in \mathrm{Good}(U,s), \Omega_{\mathcal{I}}^0(E) \neq \emptyset \right\}.$$

3. Let $\mathcal{I} \in (\mathcal{P}_r^n)^s$ intersecting. For all $E, E' \in \mathrm{Good}(U,s)$, $\Omega^0_{\mathcal{I}}(E)$ and $\Omega^0_{\mathcal{I}}(E)$ have the same number of irreducible components and each one of them is of dimension $\operatorname{edim}\mathcal{I}$.

4. For all $\mathcal{I} \in (\mathcal{P}_r^n)^s$ intersecting and $E \in \mathrm{Good}(U,s)$, $\Omega^0_{\mathcal{I}}(E)$ is dense in $\Omega_{\mathcal{I}}(E)$.

Proof This is proven in lemma 4.3.1 of [BVW18]. □

Remark 3.11. From definition-proposition 3.10 we know that

$$\mathrm{Intersecting}(r,n,s) = \{\mathrm{Pos}(V,E)\ ;\ V \in \mathrm{Gr}(r,U)\}.$$

Definition 3.12. Let $\theta \in (\mathbb{R}^r)^s$. The *slope* associated to θ is defined, for all $d \in [r]$ and all $\mathcal{J} \in (\mathcal{P}_d^r)^s$, by

$$\mu_\theta(\mathcal{J}) := \sum_{k=1}^{s} \sum_{j \in \mathcal{J}_k} \theta_k(j)$$

and, for all $V \in \mathrm{Gr}(d,U)$ and all $F \in \mathrm{Flag}(V)^s$, by

$$\mu_\theta(V,F) := \mu_\theta(\mathrm{Pos}(V,F)).$$

Harder-Narasimhan lemma 3.13 refers to a classic method used in algebraic geometry. Here, it allows us to compute expected dimensions in a convenient way.

Lemma 3.13 *(Harder-Narasimhan).* Let $\theta \in (\mathbb{R}^r)^s$ and $F \in \mathrm{Flag}(V)^s$ such that, for all $k \in [s]$, θ_k is increasing. There exists a unique subspace $W_* \in \mathrm{Gr}(V)$ such that

$$\left\{\begin{array}{llll} & \mu_\theta(W_*,F) & = \min_{W\in \mathrm{Gr}(V), W\neq\{0\}} \mu_\theta(W,F) & =: m_* \\ 0 < & \dim W_* & = \max_{W \in \mathrm{Gr}(V), \mu_\theta(W,F)=m_*} \dim W & =: d_* \end{array}\right. .$$

Proof

- We have

$$\{\mathrm{Pos}(W,F)\ ;\ W \in \mathrm{Gr}(V), W \neq \{0\}\} \subset \bigcup_{d\in[r]} (\mathcal{P}_d^r)^s$$

hence this subset is finite and we can consider the minimum m_* : there exists $W'_* \in \mathrm{Gr}(V)$ such that

$$\dim W'_* \geqslant 1 \text{ and } m_* = \mu_\theta(W'_*).$$

From this we know that

$$\emptyset \neq \{\dim W\ ;\ W \in \mathrm{Gr}(V), \mu_\theta(W,F) = m_*\} \subset [r]$$

and there exists $W_* \in \mathrm{Gr}(V)$ such that

$$1 \leqslant \dim W'_* \leqslant d_* = \dim(W_*).$$

We have just proven the existence of such a linear subspace of V.

- It remains to prove that it is unique. Let $W \in \mathrm{Gr}(V)$ such that

$$\left\{\begin{array}{lll} & \mu_\theta(W,F) & =: m_* \\ 0 < & \dim W & =: d_* \\ & W & \neq W_* \end{array}\right. .$$

Let
$$\mathcal{J} := \mathrm{Pos}(W_*, F) \text{ and } \mathcal{K} := \mathrm{Pos}(W_* \cap W, F^{W_*}).$$
Using 2.27,
$$\mathrm{Pos}(W \cap W_*, F) = \mathcal{J}\mathcal{K}.$$
We denote
$$d := \dim W \cap W_*.$$
Remark that, if $d = 0$, $\mathcal{K}$ is $(\emptyset)_{k \in [s]}$. We have
$$\begin{aligned}\mu_\theta(\mathcal{J}\mathcal{K}) &= \mu_\theta(W \cap W_*, F)\\ &\geqslant m_*\\ &= \mu_\theta(W_*, F)\\ &= \mu_\theta(\mathcal{J}).\end{aligned}$$
On the other hand, for all $k \in [s]$,
$$\mathcal{J}_k = \mathcal{J}_k\mathcal{K}_k \cup \mathcal{J}_k \cup \mathcal{K}_k^c$$
hence
$$\begin{aligned}\mu_\theta(\mathcal{J}) &= \frac{1}{d_*}\left(d\mu_\theta(\mathcal{J}\mathcal{K}) + (d - d_*)\mu_\theta(\mathcal{J}\mathcal{K}^c)\right)\\ \mu_\theta(\mathcal{J}) &\geqslant \frac{d}{d_*}\mu_\theta(\mathcal{J}) + \frac{d_* - d}{d_*}\mu_\theta(\mathcal{J}\mathcal{K}^c)\\ \frac{d_* - d}{d_*}\mu_\theta(\mathcal{J}) &\geqslant \frac{d_* - d}{d_*}\mu_\theta(\mathcal{J}\mathcal{K}^c).\end{aligned}$$
But, since $W \neq W_*$,
$$d < d_*$$
and
$$m_* = \mu_\theta(\mathcal{J}) \geqslant \mu_\theta(\mathcal{J}\mathcal{K}^c). \tag{$*$}$$

- Let
$$\mathcal{L} := \mathrm{Pos}(W + W_*, F) \text{ and } \mathcal{M} := \mathrm{Pos}(W, F^{W+W'}).$$
Using lemma 2.27 again,
$$\mathrm{Pos}(W, F) = \mathcal{L}\mathcal{M}$$
hence
$$m_* = \mu_\theta(\mathcal{L}\mathcal{M}).$$
Since $W \neq W_*$,
$$d_* < \dim(W + W_*) := d$$
hence, using the fact that d_* is a maximum,
$$m_* < \mu_\theta(W_* + W)$$
i.e.
$$\mu_\theta(\mathcal{L}\mathcal{M}) < \mu_\theta(\mathcal{L}).$$

Just as in the last point,

$$\begin{aligned} m_* <& \mu_\theta(\mathcal{L}) \\ &= \frac{d_*}{d'}\mu_\theta(\mathcal{LM}) + \frac{d'-d_*}{d'}\mu_\theta(\mathcal{LM}^c) \\ &= \frac{d_*}{d'}m_* + \frac{d'-d_*}{d'}\mu_\theta(\mathcal{LM}^c) \end{aligned}$$

hence

$$\mu_\theta(\mathcal{LM}) = m_* < \mu_\theta(\mathcal{LM}^c). \tag{$**$}$$

- Let $k \in [s]$. For all $i \in [r]$ we denote

$$\begin{aligned} G_*(i) &:= ((F(i) \cap W_*) + W \cap W_*)/(W \cap W_*) \\ G'(i) &:= ((F(i) \cap (W_* + W)) + W)/W. \end{aligned}$$

Remark that

$$d_* - d = \dim W_*/(W_* \cap W) = \dim(W_* + W)/W = d' - d_*.$$

Using lemma 2.27, G_* is a filtration on $W_*/(W_* \cap W)$, G' is a filtration on $(W_* + W)/W$ and, for all $j \in [d_* - d] = [d' - d_*]$,

$$\begin{aligned} (\mathcal{JK}^c)_k(j) &= \min\left\{i \in [r] \;\middle|\; \dim G_*(i) = j\right\} \\ (\mathcal{LM}^c)_k(j) &= \min\left\{i \in [r] \;\middle|\; \dim G'(i) = j\right\}. \end{aligned}$$

Yet, for all $i \in [r]$,

$$\begin{aligned} \dim G_*(i) =& \dim((F(i) \cap W_*) + W)/W \\ &\leqslant \dim((F(i) \cap (W_* + W)) + W)/W \\ &= \dim G'(i) \end{aligned}$$

hence, for all $j \in [d_* - d]$,

$$(\mathcal{JK}^c)_k(j) \geqslant (\mathcal{LM}^c)_k(j).$$

From this we deduce

$$(\mathcal{LK}^c)_k \geqslant (\mathcal{LM}^c)_k$$

and, since θ_k is increasing,

$$\sum_{i \in (\mathcal{LK}^c)_k} \theta_k(i) \geqslant \sum_{i \in (\mathcal{LM}^c)_k} \theta_k(i).$$

- From this last point we know that

$$\mu_\theta(\mathcal{LM}^c) \geqslant \mu_\theta(\mathcal{LM}^c)$$

hence, from equations $(*)$ and $(**)$,

$$m_* > m_*$$

which is absurd. Finally, W_* is unique.

□

Definition 3.14. • We the set of *dominant weights* is

$$\Lambda_+(r) := \{\lambda \in \mathbb{Z}^r \mid \lambda_1 \geqslant \cdots \geqslant \lambda_r\}.$$

- Let $I \in \mathcal{P}_r^n$. The *weight associated to I* is

$$w(I) := (n - r + i - I(i))_{i\in[r]}.$$

- Let $\mathcal{I} \in (\mathcal{P}_r^n)^s$. The weight associated to $\mathcal{I}$ is

$$w(\mathcal{I}) = (w(\mathcal{I}_1), \ldots, w(\mathcal{I}_{s-1}), (i - \mathcal{I}_s(i))_{i\in r}).$$

Remark 3.15. The name "weight" refers to representation theory we use in section 4.

Lemma 3.16. We have

$$\{\lambda \in \Lambda_+(r) \mid n - r \geqslant \lambda(1) \geqslant \lambda(r) \geqslant 0\} = \{w(I) \; ; \; I \in \mathcal{P}_r^n\}$$

and

$$\{\lambda \in \Lambda_+(r)^s \mid \forall k \in [s-1], n - r \geqslant \lambda_k(1) \geqslant \lambda_k(r) \geqslant 0 \geqslant \lambda_s(1)\} = \{w(\mathcal{I}) \; ; \; \mathcal{I} \in (\mathcal{P}_r^n)^s\}$$

Proof • Using lemma 2.2, for all $I \in \mathcal{P}_r^n$,

$$n - r \geqslant w(I)(1) \geqslant \cdots \geqslant w(I)(r) \geqslant 0.$$

- Let $\lambda \in \Lambda_+(r)$ such that

$$n - r \geqslant \lambda(1) \geqslant \lambda(r) \geqslant 0.$$

We have

$$I := \{n - r - \lambda(i) + i\} \in \mathcal{P}_r^n$$

and

$$\lambda = w(I).$$

□

Lemma 3.17. Let $\mathcal{I} \in (\mathcal{P}_r^n)^s$, $d \in [r]$ and $\mathcal{J} \in (\mathcal{P}_d^r)^s$. We have

$$\operatorname{edim} \mathcal{I}\mathcal{J} - \operatorname{edim} \mathcal{J} = d\mu_{-w(\mathcal{I})}(\mathcal{J})$$

and particularly

$$\operatorname{edim} \mathcal{I} = -\sum_{k=1}^{s}\sum_{i=1}^{r} w(\mathcal{I})_k(i).$$

Proof This calculus is presented in the proof of lemma 4.3.9 of [BVW18]. □

Lemma 3.18. If

$$\forall d \in [r], \forall \mathcal{J} \in \text{Intersecting}^0(d, r, s), \operatorname{edim} \mathcal{I}\mathcal{J} \geqslant 0$$

then

$$\forall d \in [r], \forall \mathcal{J} \in \text{Intersecting}(d, r, s), \operatorname{edim} \mathcal{I}\mathcal{J} \geqslant \operatorname{edim} \mathcal{J}.$$

Proof Because $\mathcal{I}$ is of expected dimension zero, the conclusion of the lemma holds for $d = r$. Let $d \in [r-1]$ and $\mathcal{J} \in (\mathcal{P}_d^r)^s$. Assume

$$\operatorname{edim} \mathcal{I}\mathcal{J} < \operatorname{edim} \mathcal{J}$$

i.e., using lemma 3.17,

$$\mu_{-w(\mathcal{I})}(\mathcal{J}) < 0.$$

Let $F \in \operatorname{Good}(V, s)$. Since $\mathcal{J}$ is intersecting we have

$$\Omega_{\mathcal{J}}(F) \neq \emptyset$$

and, by remark 3.11, there exists a nonzero $W \in \operatorname{Gr}(V)$ such that

$$\mathcal{J} = \operatorname{Pos}(W, F).$$

Using Harder-Narasimhan lemma 3.13, there exists a unique nonzero W_* of minimal slope m_* with respect to $-w(\mathcal{I})$ and maximal dimension d_*. Let

$$\mathcal{J}_* := \operatorname{Pos}(W_*, F).$$

For all $W' \in \Omega^0_{\mathcal{J}_*}(F)$,

$$\dim W' = d_* \text{ and } \mu_{-w(\mathcal{I})}(W', F) = m_*$$

hence, since W_* is unique,

$$\Omega^0_{\mathcal{J}_*}(F) = \{W_*\}.$$

From this and definition-proposition 3.10 we have

$$\operatorname{edim} \mathcal{J}_* = 0$$

and

$$\operatorname{edim} \mathcal{I}\mathcal{J}_* = \operatorname{edim} \mathcal{I}\mathcal{J}_* - \operatorname{edim} \mathcal{J}_*$$

i.e., using lemma 3.17,

$$\operatorname{edim} \mathcal{I}\mathcal{J}_* = d_* m_* < 0.$$

This is in contradiction with the hypothesis on $\mathcal{I}$. □

PROPOSITION 3.19. Let $\mathcal{I} \in \operatorname{Intersecting}(r, n, s)$, $d \in [r]$ and $\mathcal{J} \in \operatorname{Intersecting}(d, r, s)$. We have

$$\operatorname{edim} \mathcal{I}\mathcal{J} \geqslant \operatorname{edim} \mathcal{J}.$$

Proof For all $d' \in [r]$ and all $\mathcal{J}' \in \operatorname{Intersecting}(d', r, s)$, using proposition 3.4 $\mathcal{I}\mathcal{J}'$ is intersecting and, using proposition 3.8,

$$\operatorname{edim} \mathcal{I}\mathcal{J}' \geqslant 0.$$

Particularly, $\mathcal{I}$ satisfies the hypothesis of lemma 3.18 and

$$\operatorname{edim} \mathcal{I}\mathcal{J} \geqslant \operatorname{edim} \mathcal{J}.$$

□

From this we can prove that the intersecting tuples satisfies the Horn inequalities. The remaining of this section is an introduction to the tools necessary to prove the reciprocal in subsection 3.4.

3.3 True dimension, kernel dimension and kernel position of a tuple

3.3.1 Introduction

▷ SETTINGS. Let $\mathcal{I} \in (\mathcal{P}_r^n)^s$ and

$$V := \mathbb{C}^r \times \{0\}^{n-r} \text{ and } Q := \{0\}^r \times \mathbb{C}^{n-r}.$$

▷ NOTATION. We denote

$$\mathrm{B}(V, Q, s) := \mathrm{Flag}(V)^s \times \mathrm{Flag}(Q)^s$$

and

$$\mathrm{P}(V, Q, \mathcal{I}) := \left\{(F, G, \varphi) \in \mathrm{Flag}(V)^s \times \mathrm{Flag}(Q)^s \times \mathcal{L}(V, Q) \mid \varphi \in \mathcal{L}_{\mathcal{I}}(F, G)\right\}.$$

LEMMA 3.20. The set $\mathrm{P}(V, Q, \mathcal{I})$ is a closed subvariety of $\mathrm{Flag}(V)^s \times \mathrm{Flag}(Q)^s \times \mathcal{L}(V, Q)$.

Proof This is lemma 5.2.1 in [BVW18]. □

3.3.2 True dimension

DEFINITION 3.21. The true dimension of $\mathcal{I}$ is

$$\operatorname{tdim} \mathcal{I} := \min_{(F,G) \in \mathrm{B}(V,Q,s)} \dim \mathcal{L}_{\mathcal{I}}(F, G).$$

EXAMPLE 3.22. If $s = 1$,

$$\mathcal{L}_{\mathcal{I}}(F, G) = \mathcal{L}_{\mathcal{I}_1}(F_1, G_1)$$

and

$$\operatorname{tdim} \mathcal{I} = \dim \mathcal{I}_1 = \operatorname{edim} \mathcal{I}.$$

▷ NOTATION. For all $g \in \mathrm{GL}(V)^s$ and $h \in \mathrm{GL}(Q)^s$,

$$\Delta_{\mathcal{I},g,h} : \left| \begin{array}{ccc} \mathcal{L}(V, Q) \times \prod_{k \in [s]} \mathcal{L}_{\mathcal{I}_k}(F_0, G_0) & \longrightarrow & \mathcal{L}(V, Q)^s \\ (\xi, (\varphi_k)_{k \in [s]}) & \longmapsto & (\xi + h_k \varphi_k g_k^{-1})_{k \in [s]} \end{array} \right. .$$

LEMMA 3.23. We have

$$\min_{k \in [s]} \dim \mathcal{I}_k \geqslant \operatorname{tdim} \mathcal{I} = \min_{(g,h) \in \mathrm{GL}(V)^s \times \mathrm{GL}(Q)^s} \dim \operatorname{Ker} \Delta_{\mathcal{I},g,h} \geqslant \operatorname{edim} \mathcal{I}.$$

Proof

- Using remark 2.53,

$$\mathrm{B}(V, G, s) = \{((g_k F_0)_k, (h_k G_0)_k) \ ; \ (g, h) \in \mathrm{GL}(V)^s \times \mathrm{GL}(Q)^s\}$$

hence

$$\operatorname{tdim} \mathcal{I} = \min_{(g,h) \in \mathrm{GL}(V)^s \times \mathrm{GL}(Q)^s} \dim \mathcal{L}_{\mathcal{I}}((g_k F_0)_k, (h_k G_0)_k).$$

Yet, for all $g \in \mathrm{GL}(V)^s$ and $h \in \mathrm{GL}(Q)^s$, the map

$$\begin{array}{ccc} \bigcap_{k=1}^s h_k \mathcal{L}_{\mathcal{I}_k}(F_0, G_0) g_k^{-1} & \longrightarrow & \operatorname{Ker} \Delta_{\mathcal{I},g,h} \\ \xi & \longmapsto & (\xi, (-h_k^{-1} \xi g_k)_k) \end{array}$$

is a linear isomorphism and, using remarks 2.39, we know that for all $k \in [s]$ we have

$$h_k \mathcal{L}_{\mathcal{I}_k}(F_0, G_0) g_k^{-1} = \mathcal{L}_{\mathcal{I}_k}(g_k F_0, h_k G_0)$$

hence

$$\dim \operatorname{Ker} \Delta_{\mathcal{I},g,h} = \dim \mathcal{L}_{\mathcal{I}}((g_k F_0)_k, (h_k G_0)_k).$$

From this we deduce that

$$\operatorname{tdim} \mathcal{I} = \min_{(g,h) \in \mathrm{GL}(V)^s \times \mathrm{GL}(Q)^s} \dim \operatorname{Ker} \Delta_{\mathcal{I},g,h}.$$

In addition to this, for all $g \in \mathrm{GL}(V)^s$ and $h \in \mathrm{GL}(Q)^s$, for all $k \in [s]$,

$$\begin{aligned} \dim \mathcal{I}_k &= \dim \mathcal{L}_{\mathcal{I}_k}(g_k F_0, h_k G_k) \\ &\geqslant \dim \mathcal{L}_{\mathcal{I}}((g_k F_0)_k, (h_k G_0)_k) \\ &\geqslant \operatorname{tdim} \mathcal{I} \end{aligned}$$

hence

$$\min_{k \in [s]} \dim \mathcal{I}_k \geqslant \operatorname{tdim} \mathcal{I}.$$

- Let $g \in \mathrm{GL}(V)^s$ and $h \in \mathrm{GL}(Q)^s$. By the rank-nullity theorem,

$$\begin{aligned} \dim \operatorname{Ker} \Delta_{\mathcal{I},g,h} &= \dim \mathcal{L}(V,Q) \times \prod_{k \in [s]} \mathcal{L}_{\mathcal{I}_k}(F_0, G_0) - \operatorname{rk} \Delta_{\mathcal{I},g,h} \\ &\geqslant \dim \mathcal{L}(V,Q) + \sum_{k=1}^{s} \dim \mathcal{L}_{\mathcal{I}_k}(F_0, G_0) - \dim \mathcal{L}(V,Q)^s \\ &= r(n-r) + \sum_{k=1}^{s} \dim \mathcal{I}_k - sr(n-r) \\ &= \operatorname{edim} \mathcal{I}. \end{aligned}$$

From this we deduce the inequality we wanted.

□

LEMMA 3.24. The following assertions are equivalent.

(i) $\mathcal{I}$ is intersecting.

(ii) There exists $g \in \mathrm{GL}(V)^s$ and $h \in \mathrm{GL}(Q)^s$ such that $\Delta_{\mathcal{I},g,h}$ is surjective.

(iii) We have

$$\operatorname{tdim} \mathcal{I} = \operatorname{edim} \mathcal{I}.$$

Proof

- The equivalence (i) $\Leftrightarrow$ (ii) is proved in lemma 5.1.1 of [BVW18].
- As we have seen in the proof of lemma 3.23,

$$\dim \mathcal{L}(V,Q) \times \prod_{k \in [s]} \mathcal{L}_{\mathcal{I}_k}(F_0, G_0) - \dim \mathcal{L}(V,Q)^s = \operatorname{edim} \mathcal{I}$$

hence, for all $g \in \mathrm{GL}(V)^s$ and $h \in \mathrm{GL}(Q)^s$, by the rank-nullity theorem,

$$\dim \operatorname{Ker} \Delta_{g,h} - \operatorname{edim} \mathcal{I} = \dim \mathcal{L}(V,Q)^s - \operatorname{rk} \Delta_{g,h}.$$

From this we deduce (ii) $\Rightarrow$ (i) and, using lemma 3.23, that there exists $g \in \mathrm{GL}(V)^s$ and $h \in \mathrm{GL}(Q)^s$ such that

$$\operatorname{tdim}\mathcal{I} - \operatorname{edim}\mathcal{I} = \dim \mathcal{L}(V,Q)^s - \operatorname{rk}\Delta_{g,h}.$$

Thus (i) $\Rightarrow$ (ii).

□

Example 3.25. Using lemma 3.24 and lemma 3.23, if $\operatorname{tdim}\mathcal{I} = 0$ and $\operatorname{edim}\mathcal{I} \geqslant 0$ then $\mathcal{I}$ is intersecting.

▷ Notation. We denote

$$\begin{aligned} \mathrm{P_t}(V,Q,\mathcal{I}) &:= \left\{(F,G,\varphi) \in \mathrm{P}(V,Q,\mathcal{I}) \;\middle|\; \dim \mathcal{L}_{\mathcal{I}}(F,G) = \operatorname{tdim}\mathcal{I}\right\} \\ \mathrm{B_t}(V,Q,\mathcal{I}) &:= \left\{(F,G) \in \mathrm{B}(V,Q,s) \mid \exists\varphi \in \mathcal{L}(V,Q), (F,G,\varphi) \in \mathrm{P_t}(V,Q,\mathcal{I})\right\}. \end{aligned}$$

Remark 3.26. We have

$$\mathrm{B_t}(V,Q,\mathcal{I}) = \left\{(F,G) \in \mathrm{B}(V,Q,s) \;\middle|\; \dim \mathcal{L}_{\mathcal{I}}(F,G) = \operatorname{tdim}\mathcal{I}\right\}.$$

Lemma 3.27.

1. The set $\mathrm{B_t}(V,Q,\mathcal{I})$ is a nonempty Zariski-open subset of $\mathrm{B}(V,Q,s)$.
2. The set $\mathrm{P_t}(V,Q,\mathcal{I})$ is a nonempty Zariski-open subset of $\mathrm{P}(V,Q,\mathcal{I})$ and an irreducible smooth variety.

Proof This is in [BVW18], lemma 5.2.1. □

Corollary 3.28. We have

$$\dim \mathrm{P_t}(V,Q,\mathcal{I}) = s(\dim \operatorname{Flag} V + \dim \operatorname{Flag} Q) + \operatorname{tdim}\mathcal{I}.$$

Proof This is in [BVW18], lemma 5.2.1. □

3.3.3 The kernel dimension

Definition 3.29. The *kernel dimension* of $\mathcal{I}$ is

$$\operatorname{kdim}\mathcal{I} := \min\left\{\dim \operatorname{Ker}\varphi \;;\; (F,G,\varphi) \in \mathrm{P_t}(\mathcal{I})\right\}.$$

Proposition 3.30. If $\operatorname{edim}\mathcal{I} \geqslant 0$ and $\operatorname{kdim}\mathcal{I} = r$,

$$\operatorname{tdim}\mathcal{I} = \operatorname{edim}\mathcal{I} = 0$$

and $\mathcal{I}$ is intersecting.

Proof Assume that $\operatorname{edim}\mathcal{I} \geqslant 0$ and $\operatorname{kdim}\mathcal{I} = r$. Let $(F,G) \in \mathrm{B_t}(V,Q,\mathcal{I})$. For all $\varphi \in \mathcal{L}_{\mathcal{I}}(F,G)$,

$$\dim \operatorname{Ker}\varphi \geqslant \operatorname{kdim}\mathcal{I} = r$$

hence

$$\varphi = 0.$$

From this we deduce

$$\operatorname{tdim}\mathcal{I} = \dim \mathcal{L}_{\mathcal{I}}(F,G) = 0.$$

As seen in example 3.25, $\mathcal{I}$ is then intersecting. □

Proposition 3.31. If $\operatorname{kdim} \mathcal{I} = 0$, $\mathcal{I}$ is intersecting.

Proof Assume that the kernel dimension of $\mathcal{I}$ is null. The set

$$\mathrm{P_k} := \left\{ (F, G, \varphi) \in \mathrm{P}(V, Q, \mathcal{I}) \;\middle|\; \dim \operatorname{Ker} \varphi = 0 \right\}$$

is a nonempty Zariski-open subset of $\mathrm{P}(V, Q, \mathcal{I})$ and, since $\operatorname{kdim} \mathcal{I} = 0$,

$$\mathrm{P_k} \cap \mathrm{P_t}(V, Q, \mathcal{I}) \neq \emptyset.$$

For all element (F, G, φ) in this intersection, φ is injective and in particular, for all $k \in [s]$ and $i \in [r]$,

$$\mathcal{I}_k(i) - i \geqslant i.$$

Let

$$\mathcal{L}_0(V, Q) := \{ \varphi \in \mathcal{L}(V, Q) \mid \operatorname{Ker} \varphi = \{0\} \}$$

which is a nonempty Zariski-open subset of $\mathcal{L}(V, Q)$ and

$$\pi : \left| \begin{array}{ccc} \mathrm{P_k} & \longrightarrow & \operatorname{Flag}(V)^s \times \mathcal{L}_0(V, Q) \\ (F, G, \varphi) & \longmapsto & (F, \varphi) \end{array} \right. .$$

Through this map, $\mathrm{P_k}$ is a fiber bundle over $\operatorname{Flag}(V)^s \times \mathcal{L}_0(V, Q)$ hence $\mathrm{P_k}$ is irreducible and

$$\dim \mathrm{P_t}(V, Q, \mathcal{I}) = \dim \mathrm{P_k}\,. \tag{$*$}$$

We have

$$\dim \operatorname{Flag}(V)^s \times \mathcal{L}_0(V, Q) = s \dim \operatorname{Flag} V + r(n - r). \tag{$**$}$$

For all $(F, \varphi) \in \operatorname{Flag}(V)^s \times \mathcal{L}_0(V, Q)$,

$$\begin{aligned} \pi^{-1}(F, \varphi) &\simeq \{ G \in \operatorname{Flag}(Q)^s \mid (F, G, \varphi) \in \mathrm{P}(V, Q, \mathcal{I}) \} \\ &\simeq \prod_{k=1}^{s} \{ G_k \in \operatorname{Flag}(Q) \mid \varphi \in \mathcal{L}_{\mathcal{I}_k}(F_k, G_k) \} \\ &= \prod_{k=1}^{s} \operatorname{Flag}^0_{\mathcal{I}_k}(F_k, \varphi) \end{aligned}$$

hence, using lemma 2.64,

$$\dim \operatorname{Flag}(Q) + \dim I - r(n - r) = s \dim \operatorname{Flag}(Q) - sr(n - r) + \sum_{k=1}^{s} \dim \mathcal{I}_k.$$

From this and equations $(*), (**)$ we have

$$\begin{aligned} \dim \mathrm{P_t}(V, Q, \mathcal{I}) &= \dim \mathrm{P_k} \\ &= s \dim \operatorname{Flag} V + r(n - r) + s \dim \operatorname{Flag} Q - sr(n - r) + \sum_{k=1}^{s} \dim \mathcal{I}_k \\ &= s(\dim \operatorname{Flag} V + \dim \operatorname{Flag} Q) + \operatorname{edim} \mathcal{I}. \end{aligned}$$

hence, by corollary 3.28,

$$\operatorname{edim} \mathcal{I} = \operatorname{tdim} \mathcal{I}$$

and by lemma 3.24 $\mathcal{I}$ is intersecting. $\square$

▷ Notation. We denote

$$\begin{aligned}
\mathrm{P_{kt}}(V,Q,\mathcal{I}) &:= \left\{(F,G,\varphi) \in \mathrm{P_t}(V,Q,\mathcal{I}) \;\middle|\; \dim \operatorname{Ker}\varphi = \operatorname{kdim}\mathcal{I}\right\},\\
\mathrm{B_{kt}}(V,Q,\mathcal{I}) &:= \left\{(F,G) \in \mathrm{B_t}(V,Q,s) \mid \exists \varphi \in \mathcal{L}(V,Q), (F,G,\varphi) \in \mathrm{P_{kt}}(V,Q,\mathcal{I})\right\}.
\end{aligned}$$

Lemma 3.32. 1. The set $\mathrm{B_{kt}}(V,Q,\mathcal{I})$ is a nonempty Zariski-open subset of $\mathrm{B}(V,Q,s)$.

2. The set $\mathrm{P_{kt}}(V,Q,\mathcal{I})$ is a nonempty Zariski-open subset of $\mathrm{P_t}(V,Q,\mathcal{I})$ and an irreducible smooth variety.

Proof This is in [BVW18], lemma 5.2.7. □

3.3.4 The kernel position

▷ Settings. This this subsubsection we assume

$$1 \leqslant \operatorname{kdim}\mathcal{I} =: d \leqslant r-1.$$

Lemma 3.33. For all $k \in [s]$, the map

$$\begin{array}{ccc}
[d] & \longrightarrow & [r]\\
j & \longmapsto & \min\left\{\operatorname{Pos}(\operatorname{Ker}\varphi, F_k)(j) \;;\; (F,G,\varphi) \in \mathrm{P_{kt}}(V,Q,\mathcal{I})\right\}
\end{array}$$

is strictly increasing.

Proof Let J_k be this map. Let $j \in [d-1]$. For all $(F,G,\varphi) \in \mathrm{P_{kt}}(V,Q,\mathcal{I})$,

$$\begin{aligned}
J_k(j) &\leqslant \operatorname{Pos}(\operatorname{Ker}\varphi, F_k, G_k)(j)\\
&\leqslant \operatorname{Pos}(\operatorname{Ker}\varphi, F_k, G_k)(j+1) - 1
\end{aligned}$$

hence

$$J_k(j) < J_k(j+1).$$

□

This allows us to define the kernel position of $\mathcal{I}$ just as below.

Definition 3.34. The *kernel position* of $\mathcal{I}$ is the tuple $\operatorname{kPos}(\mathcal{I}) \in (\mathcal{P}^r_d)^s$ such that, for all $k \in [s]$ and all $j \in [d]$,

$$(\operatorname{kPos}(\mathcal{I}))_k(j) = \min\left\{\operatorname{Pos}(\operatorname{Ker}\varphi, F_k)(j) \;;\; (F,G,\varphi) \in \mathrm{P_{kt}}(V,Q,\mathcal{I})\right\}.$$

Remark 3.35. For all S linear subspace of V and $F \in \operatorname{Flag}(V)$,

$$\operatorname{kPos}(\mathcal{I}) = \operatorname{Pos}(S,F) \Rightarrow \operatorname{kdim}\mathcal{I} = \dim S.$$

▷ Notation. We denote $d := \operatorname{kdim}\mathcal{I}$ and

$$\begin{aligned}
\mathrm{P_{kpt}}(V,Q,\mathcal{I}) &:= \left\{(F,G,\varphi) \in \mathrm{P_{kt}}(V,Q,\mathcal{I}) \;\middle|\; \operatorname{Pos}(\operatorname{Ker}\varphi, F) = \operatorname{kPos}(\mathcal{I})\right\}\\
\mathrm{B_{kpt}}(V,Q,\mathcal{I}) &:= \left\{(F,G) \in \mathrm{B}(V,Q,s) \mid \exists \varphi \in \mathcal{L}(V,Q), (F,G,\varphi) \in \mathrm{P_{kpt}}(V,Q,\mathcal{I})\right\}.
\end{aligned}$$

Lemma 3.36. 1. The set $\mathrm{B_{kpt}}(V,Q,\mathcal{I})$ is a nonempty Zariski-open subset of $\mathrm{B}(V,Q,s)$.

2. The set $\mathrm{P_{kpt}}(V,Q,\mathcal{I})$ is a nonempty Zariski-open subset of $\mathrm{P_{kt}}(V,Q,\mathcal{I})$ and an irreducible smooth variety.

Proof This is lemma 5.2.9 in [BVW18]. $\square$

Remark 3.37. We recall lemmas 3.27, 3.32 and lemma 3.36 in this small diagram.

$$\begin{array}{ccccccccc} \emptyset \neq & \mathrm{P}_{\mathrm{kpt}} & \overset{\text{open}}{\subset} & \mathrm{P}_{\mathrm{kt}} & \overset{\text{open}}{\subset} & \mathrm{P}_{\mathrm{t}} & \overset{\text{open}}{\subset} & \mathrm{P} & \overset{\text{closed}}{\subset} & \mathrm{Flag}(V)^s \times \mathrm{Flag}(Q)^s \times \mathcal{L}(V,Q) \\ & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \emptyset \neq & \mathrm{B}_{\mathrm{kpt}} & \overset{\text{open}}{\subset} & \mathrm{B}_{\mathrm{kt}} & \overset{\text{open}}{\subset} & \mathrm{B}_{\mathrm{t}} & \overset{\text{open}}{\subset} & \mathrm{B} & = & \mathrm{Flag}(V)^s \times \mathrm{Flag}(Q)^s \end{array}$$

Proposition 3.38. If $1 \leqslant \operatorname{kdim} \mathcal{I} \leqslant r-1$, $\mathrm{kPos}(\mathcal{I})$ is intersecting.

Proof Let $\mathcal{J} := \mathrm{kPos}(\mathcal{I})$. By lemma 3.36, $\mathrm{B}_{\mathrm{kpt}}(V, Q, \mathcal{I})$ is a nonempty Zariski-open subset of $\mathrm{B}(V, Q, s)$ thus is Zariksi-dense and its image D by the continuous map

$$\begin{array}{ccc} \mathrm{B}(V,Q,s) & \longrightarrow & \mathrm{Flag}(V)^s \\ (F,G) & \longmapsto & F \end{array}$$

is Zariski-dense in $\mathrm{Flag}(V)^s$. For all $F \in D$ there exists $G \in \mathrm{Flag}(Q)^s$ and $\varphi \in \mathcal{L}(V, Q)$ such that

$$(F, G, \varphi) \in \mathrm{P}_{\mathrm{kpt}}(V, Q, \mathcal{I})$$

and in particular

$$\operatorname{Ker} \varphi \in \Omega^0_{\mathcal{J}}(F).$$

From this and lemma 3.6,

$$D \subset \mathrm{Im}\omega^0_{\mathcal{J}}$$

hence, using lemma 3.7, $\mathcal{J}$ is intersecting. $\square$

Lemma 3.39. If $1 \leqslant \operatorname{kdim} \mathcal{I} \leqslant r-1$,

$$\operatorname{tdim} \mathcal{I} = \operatorname{edim} \mathrm{kPos}(\mathcal{I}) + \operatorname{edim} \mathcal{I}/\mathrm{kPos}(\mathcal{I}).$$

Proof This corollary 5.2.13 in [BVW18]. $\square$

Lemma 3.40. Let $\mathcal{I} \in (\mathcal{P}^n_r)^s$ such that $1 \leqslant \operatorname{kdim} \mathcal{I} \leqslant r-1$. We have

$$\operatorname{edim}(\mathrm{kPos}\,\mathcal{I}) \leqslant \operatorname{tdim} \mathcal{I}^{\mathrm{kPos}\,\mathcal{I}}.$$

Proof Let $\mathcal{J} := \mathrm{kPos}(\mathcal{I})$ and $d := \operatorname{kdim} \mathcal{I}$.

- Let $(F, G, \varphi) \in \mathrm{P}_{\mathrm{kp}}(\mathcal{I})$. Let $S := \operatorname{Ker} \varphi$ and $\bar{\varphi} : V/S$ the injective map induced by φ. We have

  $$\mathcal{J} = \mathrm{Pos}(S, F)$$

  so, by lemma 2.41,

  $$\bar{\varphi} \in \mathcal{L}_{\mathcal{I}/\mathcal{J}}(F_{V/S}, G)$$

  and by proposition 2.51 we can define

  $$\begin{array}{ccc} \mathcal{L}_{\mathcal{J}}(F^S, F_{V/S}) & \longrightarrow & H_{\mathcal{I}\mathcal{J}}(F^S, G) \\ \psi & \longmapsto & \bar{\varphi} \circ \psi \end{array}.$$

  But $\bar{\varphi}$ is injective so this last map is injective and we have

  $$\dim \mathcal{L}_{\mathcal{J}}(F^S, F_{V/S}) \leqslant \mathcal{L}_{\mathcal{I}\mathcal{J}}(F^S, G).$$

But

$$\begin{aligned}\operatorname{edim}\mathcal{J} &\leqslant \operatorname{tdim}\mathcal{J}\\ &\leqslant \dim \mathcal{L}_{\mathcal{J}}(F^S, F_{V/S})\end{aligned}$$

hence

$$\operatorname{edim}\mathcal{J} \leqslant \mathcal{L}_{\mathcal{I}\mathcal{J}}(F^S, G) \tag{$*$}$$

and, if $\dim \mathcal{L}_{\mathcal{I}\mathcal{J}}(F^S, G) \leqslant \operatorname{tdim}\mathcal{I}^{\mathcal{J}}$, then the lemma is proven.

- Let

$$\mathcal{K} := \left\{ (S, \tilde{F}, G) \in \operatorname{Gr}(d, V) \times \bigcup_{S' \in \operatorname{Gr}(d,V)} \operatorname{Flag}(S')^s \times \operatorname{Flag}(Q)^s \mid \tilde{F} \in \operatorname{Flag}(S)^s \right\}$$

which is non empty. It is the fiber bundle over $\operatorname{Gr}(d, V)$ with fiber over any $S \in \operatorname{Gr}(d, V)$ given by $\operatorname{Flag}(S)^s \times \operatorname{Flag}(Q)^s$ hence $\mathcal{K}$ is an irreducible algebraic variety. For all $(F, G, \varphi) \in \mathrm{P}_{\mathrm{kp}}(\mathcal{I})$,

$$\dim \operatorname{Ker}\varphi = d$$

so we can define the morphism

$$\pi : \left| \begin{array}{ccc} \mathrm{P}_{\mathrm{kp}}(V, Q, \mathcal{I}) & \longrightarrow & \mathcal{K} \\ (F, G, \varphi) & \longmapsto & (\operatorname{Ker}\varphi, F^{\operatorname{Ker}\varphi}, G) \end{array} \right. .$$

To conclude the proof of the lemma, the idea is now to find an element $(S, \tilde{F}, G) \in \mathcal{K}$ such that $\dim \mathcal{L}_{\mathcal{I}\mathcal{J}}(\tilde{F}, G) \leqslant \operatorname{tdim}\mathcal{I}^{\mathcal{J}}$ and that there is a preimage of this element by π.

- We want to find a good set of candidates in $\mathcal{K}$. Let

$$\mathcal{K}_{\mathrm{t}} := \left\{ (S, \tilde{F}, G) \in \mathcal{K} \mid \dim \mathcal{L}_{\mathcal{I}\mathcal{J}}(\tilde{F}, G) = \operatorname{tdim}\mathcal{I}^{\mathcal{J}} \right\} \subset \mathcal{K}.$$

With the surjective morphism

$$p : \left| \begin{array}{ccc} \mathcal{K} & \longrightarrow & \bigcup_{S' \in \operatorname{Gr}(d,V)} \operatorname{Flag}(S')^s \times \operatorname{Flag}(Q)^s \\ (S, F, G) & \longmapsto & (F, G) \end{array} \right.$$

we have

$$\mathcal{K}_{\mathrm{t}} = \bigcup_{S \in \operatorname{Gr}(d,V)} p^{-1}(\mathrm{B}_{\mathrm{t}}(S, Q, \mathcal{I}^{\mathcal{J}}))$$

so, using lemma 3.27, $\mathcal{K}_{\mathrm{t}}$ is a nonempty Zariski-open subset of $\mathcal{K}$. But $\mathcal{K}$ is irreducible so $\mathcal{K}_{\mathrm{t}}$ is dense in $\mathcal{K}$.

- We now want to prove that $\pi^{-1}(\mathcal{K}_{\mathrm{t}})$ is nonempty. First we will prove that π is in fact dominant. Let

$$q : \left| \begin{array}{ccc} \mathrm{P}_{\mathrm{kp}}(V, Q, \mathcal{I}) & \longrightarrow & \operatorname{Flag}(Q)^s \\ ((F, G, \varphi) & \longmapsto & G \end{array} \right. .$$

According to lemma 3.36, $\mathrm{B}_{\mathrm{kpt}}(V, Q, \mathcal{I})$ is a nonempty Zariski-open subset of $\mathrm{B}(V, Q, s)$ hence there exists a nonempty Zariski-open subset $O \subset \operatorname{Flag}(Q)^s$ such that

$$O \subset q(\mathrm{P}_{\mathrm{kpt}}(V, Q, \mathcal{I})).$$

From this we deduce that

$$\mathcal{K}_O := \left\{(S, \tilde{F}, G) \in \mathcal{K} \mid G \in O\right\} \subset \mathcal{K}$$

is a nonempty Zariski-open subset of $\mathcal{K}$ and, since $\mathcal{K}$ is irreducible, $\mathcal{K}_O$ is Zariski-dense in $\mathcal{K}$. Let's show that the image $\pi(\mathrm{P}_{\mathrm{kp}}(\mathcal{I}))$ contains this subset. Let $(S, \tilde{F}, G) \in \mathcal{K}_O$. Let $(F', G, \varphi') \in q^{-1}(\{O\})$ and

$$S' := \operatorname{Ker} \varphi'.$$

We have

$$\dim S' = d = \dim S \text{ and } S, S' \subset V$$

so there exists $g \in \mathrm{GL}(V)$ such that

$$S = gS'.$$

Let

$$F := gF' \text{ and } \varphi := \varphi' \circ (g')^{-1}.$$

Remark the following points.

- We have $(F, G) \in \mathrm{B}(V, Q, s)$.
- We have

  $$\operatorname{Ker} \varphi = g \operatorname{Ker} \varphi' = S.$$

  and, by remarks 2.39,

  $$\varphi \in \mathcal{L}_{\mathcal{I}}(gF', G) = \mathcal{L}_{\mathcal{I}}(F, G).$$

- We have

  $$\begin{aligned} \mathrm{kPos}(\mathcal{I}) &= \mathrm{Pos}(\operatorname{Ker} \varphi', F') \\ &= \mathrm{Pos}(g^{-1} \operatorname{Ker} \varphi, F') \\ &= \mathrm{Pos}(\operatorname{Ker} \varphi, F). \end{aligned}$$

From this we deduce

$$(F, G, \varphi) \in \mathrm{P}_{\mathrm{kp}}(V, Q, \mathcal{I}).$$

Since $\tilde{F}, F^S \in \mathrm{Flag}(S)^s$, there exists $h' \in \mathrm{GL}(S)^s$ such that

$$h' F^S = \tilde{F}.$$

Let T a complement subspace of S in V. Let $h \in \mathrm{GL}(V)^s$ the unique sequence of isomorphisms such that, for all $k \in [s]$, $s \in S$ and $t \in T$,

$$h_k(s) = h'_k(s) \text{ and } h_k(t) = t.$$

To conclude, we now want to prove that (hF, G, φ) is mapped to $(S, \tilde{F}, G)$ by π.

- We have $(hF, G) \in \mathrm{B}(V, Q, s)$.

– We know that

$$\begin{cases} S = \operatorname{Ker}\varphi \\ \mathcal{J} = \operatorname{Pos}(S,F) \\ \varphi \in \mathcal{L}_{\mathcal{I}}(F,G) \end{cases}$$

hence, with $\bar{\varphi} \in \mathcal{L}(V/S, Q)$ the injection induced by φ, using lemma 2.41,

$$\bar{\varphi} \in \mathcal{L}_{\mathcal{I}/\mathcal{J}}(F_{V/S}, G).$$

We have

$$T \simeq V/S$$

hence, for all $k \in [s]$, h_k acts trivially on V/S. Using this last statement and lemma 2.25, for all $k \in [s]$ and for all $j \in [r-d]$,

$$\begin{aligned} (h_k F_k)_{V/S}(j) &= ((h_k F_k)(\mathcal{J}^c(j)) + S)/S \\ &= h_k(F_k(\mathcal{J}^c(j)) + S)/S \\ &= F_k(\mathcal{J}^c(j)) + S)/S \\ &= (F_k)_{V/S}(j) \end{aligned}$$

hence

$$(hF)_{V/S} = F_{V/S}$$

and

$$\bar{\varphi} \in \mathcal{L}_{\mathcal{I}/\mathcal{J}}((hF)_{V/S}, G).$$

Yet, in addition to that, for all $k \in [s]$, $h_k S = S$ hence

$$\operatorname{Pos}(S, hF) = \operatorname{Pos}(S,F) = \mathcal{J}. \tag{$**$}$$

Using lemma 2.41 again,

$$\varphi \in \mathcal{L}_{\mathcal{I}}(hF, G).$$

– Equation $(**)$ also tells us that

$$\operatorname{Pos}(\operatorname{Ker}\varphi, hF) = \mathrm{kPos}(\mathcal{I}).$$

From this we deduce

$$(hF, G, \varphi) \in \mathrm{P}_{\mathrm{kp}}(V, Q, \mathcal{I}).$$

For all $k \in [s]$ and all $j \in [d]$, using $(**)$, the fact that the injective map h_k stabilises S and lemma 2.25,

$$\begin{aligned} (h_k F_k)^S(j) &= (h_k F_k)(\mathcal{J}(j)) \cap S \\ &= h_k(F_k(\mathcal{J}(j)) \cap S) \\ &= h_k(F_k^S(j)) \\ &= \tilde{F}_k(j). \end{aligned}$$

From this we deduce

$$(hF)^S = \tilde{F}$$

and

$$(S, \tilde{F}, G) = \pi(hF, G, \varphi).$$

Thus

$$\mathcal{K}_O \subset \pi(\mathrm{P}_{\mathrm{kp}}(V, Q, \mathcal{I})).$$

Yet we know that $\mathcal{K}_O$ is Zariski-dense in $\mathcal{K}$ hence π is dominant.

- We know that π is dominant and that $\mathcal{K}_t$ is a Zariksi-open and Zariski-dense subset of $\mathcal{K}$ hence
$$\pi^{-1}(\mathcal{K}_t) \neq \emptyset$$
i.e. there exists $(F, G, \varphi) \in \mathrm{P}_{\mathrm{kp}}(V, Q, \mathcal{I})$ such that
$$\dim \mathcal{L}_{\mathcal{I}\mathcal{J}}(F^{\operatorname{Ker}\varphi}, G) = \operatorname{tdim} \mathcal{I}^{\mathcal{J}}.$$
Using $(*)$, we have
$$\operatorname{edim} \mathcal{J} \leqslant \operatorname{tdim} \mathcal{I}^{\mathcal{J}}.$$

□

Corollary 3.41 is the relation used in the induction proving Belkale's theorem 3.42.

Corollary 3.41. Let $\mathcal{I} \in (\mathcal{P}_r^n)^s$ such that
$$1 \leqslant \operatorname{kdim} \mathcal{I} \leqslant r - 1.$$
We have
$$0 \leqslant \operatorname{tdim} \mathcal{I} - \operatorname{edim} \mathcal{I} \leqslant \operatorname{tdim} \mathcal{I}^{\operatorname{kPos} \mathcal{I}} - \operatorname{edim} \mathcal{I}^{\operatorname{kPos} \mathcal{I}}.$$

Proof Let $\mathcal{J} := \operatorname{kPos} \mathcal{I}$. We have
$$\operatorname{tdim} \mathcal{I} - \operatorname{edim} \mathcal{I} \overset{3.23}{\geqslant} 0$$
and
$$\begin{aligned} \operatorname{tdim} \mathcal{I} - \operatorname{edim} \mathcal{I} &\overset{3.39}{=} \operatorname{edim} \mathcal{J} + \operatorname{edim} \mathcal{I}/\mathcal{J} - \operatorname{edim} \mathcal{I} \\ &\overset{2.13}{=} \operatorname{edim} \mathcal{J} + \operatorname{edim} \mathcal{I} + \operatorname{edim} \mathcal{J} - \operatorname{edim} \mathcal{I}\mathcal{J} - \operatorname{edim} \mathcal{I} \\ &= \operatorname{edim} \mathcal{J} - (\operatorname{edim} \mathcal{I}\mathcal{J} - \operatorname{edim} \mathcal{J}) \\ &\overset{2.13}{=} \operatorname{edim} \mathcal{J} - \operatorname{edim} \mathcal{I}^{\mathcal{J}} \\ &\overset{3.40}{\leqslant} \operatorname{tdim} \mathcal{I}^{\mathcal{J}} - \operatorname{edim} \mathcal{I}^{\mathcal{J}}. \end{aligned}$$

□

3.4 Belkale's theorem

3.4.1 A brief resume of what we know so far

In the next section we present a proof of Belkale's theorem 3.42 by an induction based on several equations and propositions that we proved previously about the different dimensions of a tuple.

Most of them came from a geometrical point of view ; in example 3.9 we have seen the base case of the induction, in lemma 3.18 and proposition 3.19 we have seen some inequations on the expected dimension of specific tuples, in lemmas 3.30 and 3.31 we have seen that tuples with extremal kernel dimensions are intersecting, in corollary 3.41 we have seen a powerful relation between the kernel and expected dimensions of different tuples (which is at the heart of the induction), in lemma 3.38 we have seen that the kernel position is intersecting, in proposition 3.4 we have seen that the composition of two intersecting tuples is still intersecting and in lemma 3.24 we have seen a useful caracterisation of intersecting tuples using their true and expected dimensions (this lemma will greatly work with the inequation in corollary 3.41).

Others only came from a combinatorial point of view ; in lemma 2.13 we have seen some equations about the expected dimension of a composition and a quotient of given tuples.

3.4.2 The theorem

THEOREM 3.42 *(Belkale).* For all $r \in [n]$ and $s \geqslant 2$,

$$\text{Intersecting}(r, n, s) = \text{Horn}(r, n, s).$$

Proof We will prove this result by induction on $r \in \mathbb{N}^*$. Let $s \geqslant 2$. For all $r \in \mathbb{N}^*$ we denote by $H(r)$ the assertion

$$\forall d \in [r], \forall n \geqslant r, \text{Intersecting}(d, n, s) = \text{Horn}(d, n, s).$$

- From example 3.9 we deduce $H(1)$.
- Let $r \geqslant 2$. Assume $H(r-1)$. Let $n \geqslant r$. Using $H(r-1)$,

  $$\text{Horn}(r, n, s) = \left\{\mathcal{I} \in (\mathcal{P}_r^n)^s \mid \forall d \in [r], \forall \mathcal{J} \in \text{Intersecting}^0(d, r, s), \operatorname{edim} \mathcal{I}\mathcal{J} \geqslant 0\right\}$$

  hence, by proposition 3.19,

  $$\text{Intersecting}(r, n, s) \subset \text{Horn}(r, n, s) \tag{$*$}$$

  and, by lemma 3.18,

  $$\text{Horn}(r, n, s) = \{\mathcal{I} \in (\mathcal{P}_r^n)^s \mid \forall d \in [r], \forall \mathcal{J} \in \text{Intersecting}(d, r, s), \operatorname{edim} \mathcal{I}\mathcal{J} \geqslant \operatorname{edim} \mathcal{J}\}. \tag{$**$}$$

  Let $\mathcal{I} \in \text{Horn}(r, n, s)$ and

  $$d := \operatorname{kdim} \mathcal{I}.$$

  Using lemmas 3.30 and 3.31, if $d \in \{0, r\}$, $\mathcal{I}$ is intersecting. Assume

  $$d \in [r-1].$$

  Let $\mathcal{J} := \text{kPos}(\mathcal{I})$. Using corollary 3.41,

  $$0 \leqslant \operatorname{tdim} \mathcal{I} - \operatorname{edim} \mathcal{I} \leqslant \operatorname{tdim} \mathcal{I}^{\mathcal{J}} - \operatorname{edim} \mathcal{I}^{\mathcal{J}}. \tag{$***$}$$

  Let's prove that $\mathcal{I}^{\mathcal{J}}$ satisfies the Horn inequalities. Let $m \in [d-1]$ and $\mathcal{K} \in \text{Horn}(m, d, s)$. Because of $H(r-1)$, K is intersecting. But, using lemma 3.38, $\mathcal{J}$ is also intersecting and, by proposition 3.4, $\mathcal{J}\mathcal{K}$ is intersecting. It follows from equation $(**)$ that

  $$\operatorname{edim} \mathcal{I}(\mathcal{J}\mathcal{K}) - \operatorname{edim} \mathcal{J}\mathcal{K} \geqslant 0$$

  i.e., using lemma 2.13,

  $$\operatorname{edim} \mathcal{I}^{\mathcal{J}}\mathcal{K} - \operatorname{edim} K \geqslant 0.$$

  From this we deduce that

  $$\mathcal{I}^{\mathcal{J}} \in \text{Horn}(d, n - r + d, s).$$

But $d \leqslant r-1$ hence, using $H(r-1)$, $\mathcal{I}^{\mathcal{J}}$ is intersecting. It follows from lemma 3.24 that

$$\operatorname{tdim} \mathcal{I}^{\mathcal{J}} - \operatorname{edim} \mathcal{I}^{\mathcal{J}} = 0$$

hence, using equation $(***)$,

$$\operatorname{tdim} \mathcal{I} = \operatorname{edim} \mathcal{I}$$

i.e., using lemma 3.24 again, $\mathcal{I}$ is intersecting. From this we deduce

$$\operatorname{Horn}(r, n, s) \subset \operatorname{Intersecting}(r, n, s).$$

But we have seen the converse in equation $(*)$ so

$$\operatorname{Horn}(r, n, s) = \operatorname{Intersecting}(r, n, s).$$

Finally, $H(r)$ is true.

- By induction on $r \in \mathbb{N}^*$, Belkale's theorem is true.

□

3.4.3 Computation of intersecting tuples

The inductive description of Horn's tuples allow us to easily find them with a computer. For an example of a Python code returning the set $\operatorname{Horn}(r, n, s)$, see appendix 5. We do not claim any efficiency or optimisation about this algorithm.

4 Kirwan cones

In this last section, we come back to Horn's conjecture and prove it in Knutson-Tao's theorem 4.27.

4.1 Introduction

▷ NOTATION. • We denote by $U(r)$ the set of the unitary matrices of order r :

$$U(r) := \{u \in \mathrm{GL}(r) \mid uu^* = \mathrm{I}_r\}.$$

- We denote
$$C_+(r) := \{\lambda \in \mathbb{R}^r \mid \lambda_1 \geqslant \cdots \geqslant \lambda_r\}$$
and, for all $g \in \mathrm{GL}(r)$ and $\lambda \in C_+(r)$,
$$g \cdot \lambda := g\lambda g^{-1}.$$
- For all $\lambda \in C_+(r)$ we denote
$$\lambda^* := (-\lambda_n, \dots, -\lambda_1) \in C_+(r).$$
- For all X Hermitian matrix, we denote by $\lambda(X) \in C_+(r)$ its (real) spectrum with multiplicites and ranked in decreasing order. We denote by $\mathcal{O}_\lambda$ the set of all Hermitian matrices of order r and spectrum λ.

REMARK 4.1. We have

$$\{A \in \mathrm{M}_r(\mathbb{C}) \mid A^* = A\} = \bigsqcup_{\lambda \in C_+(r)} \mathcal{O}_\lambda$$

and, for all $\lambda \in C_+(r)$,
$$\mathcal{O}_\lambda = U(r) \cdot \lambda.$$

DEFINITION 4.2. The *Kirwan cone* associated to (r, s) is

$$\mathrm{Kir}(r,s) := \left\{\xi \in C_+(r)^s \mid \exists X \in \prod_{k=1}^{s} \mathcal{O}_{\xi_k}, \sum_{k=1}^{s} X_k = 0\right\}.$$

REMARK 4.3. For all $\xi \in \mathrm{Kir}(r, s)$,

$$\sum_{k=1}^{s} |\xi_k| = 0.$$

Proof There is $(X_1, \dots, X_s) \in \prod_{k=1}^{s} \mathcal{O}_{\xi_k}$ such that

$$\sum_{k=1}^{s} X_k = 0.$$

In addition to this,
$$\sum_{k=1}^{s} |\xi_k| = \mathrm{tr}\left(\sum_{k=1}^{s} X_k.\right)$$

□

4.2 Hersch-Zahlen lemma

▷ NOTATION. For all $J \in \mathcal{P}_d^r$, we denote $T_J \in \mathrm{M}_r(\mathbb{C})$ the vector $v \in \mathbb{R}_r$ such that

$$\forall j \in J, v_j = 1$$
$$\forall j \in J^c, v_j = 0$$

and we indentify it with the diagonal matrix in $\mathrm{M}_r(\mathbb{C})$ associated to v.

REMARK 4.4. For all $\xi \in \mathbb{R}^r$ and $J \in \mathcal{P}_d^r$,

$$\sum_{j \in J} \xi_j = \langle T_J, \xi \rangle .$$

DEFINITION 4.5. Let X an Hermitian matrix. An *eigenflag* F_X on $\mathbb{C}^r$ is a flag on $\mathbb{C}^r$ such that there exists an orthonormal eigenbasis $(f_X(i))_{i \in [r]}$ of X adapted to F_X such that, for all $i \in [r]$, $f_X(i)$ is associated to the eigenvalue $\lambda(X)(i)$.

LEMMA 4.6 *(Hersch-Zahlen).* Let $\xi \in C_+(r)$ and $X \in \mathcal{O}_\xi$ with an eigenflag F_X. Let $J \in \mathcal{P}_d^r$.

1. The set
$$A^0(X, F_X, J) := \left\{ \mathrm{tr}(P_S X) \ ; \ S \in \Omega_J^0(F_X) \right\}$$
is a subset of $\mathbb{R}$, has a minimum and
$$\min A^0(X, F_X, J) = \sum_{j \in J} \xi(j).$$

2. The set
$$A(X, F_X, J) := \{ \mathrm{tr}(P_S X) \ ; \ S \in \Omega_J(F_X) \}$$
is a subset of $\mathbb{R}$, has a minimum and
$$\min A(X, F_X, J) = \sum_{j \in J} \xi(j).$$

Proof This lemma is an application of the variational principle.

1. There exists an orthonormal eigenbasis $(f_X(i))_{i \in [r]}$ of X adapted to F_X such that, for all $i \in [r]$, $f_X(i)$ is associated to the eigenvalue $\xi(i)$. Let $S \in \Omega_J^0(F_X)$. By lemma 2.22, there is an orthonormal basis $(s_1, \ldots, s_d)$ of S such that, for all $i \in [d]$,
$$s_i \in F_X(J(i)).$$
There is $s_{d+1}, \ldots, s_r$ such that $s := (s_1, \ldots, s_r)$ is an orthonomal basis of $\mathbb{C}^r$. We have
$$\begin{aligned} \mathrm{tr}(P_S X) &= \mathrm{tr}(\mathrm{Mat}_s(P_S) \, \mathrm{Mat}_s(X)) \\ &= \mathrm{tr}(T_{[d]} \, \mathrm{Mat}_s(X)) \\ &= \sum_{i=1}^d [\mathrm{Mat}_s(X)]_{i,i} \\ &= \sum_{i=1}^d \langle s_i, X s_i \rangle \end{aligned}$$

hence $\mathrm{tr}(P_S X) \in \mathbb{R}$. Yet X is Hermitian and, for all $i \in [d]$,

$$s_i \in F_X(J(i)) = \mathrm{span}\,\{f_X(j)\ ;\ j \in [J(i)]\} \text{ and } \forall j \in [J(i)], \xi(j) \geqslant \xi(J(i)).$$

By the variational principle,

$$\begin{aligned} \mathrm{tr}(P_S X) &\geqslant \sum_{i=1}^{d} \xi(J(i)) \\ &\geqslant \sum_{j \in J} \xi(j). \end{aligned}$$

From that we deduce that

$$\inf A^0(X, F_X, J) \geqslant \sum_{j \in J} \xi(j).$$

Using example 2.20,

$$S' := \mathrm{span}\,\{f_X(j)\ ;\ \in J\} \in \Omega^0_J(F_X).$$

The set $(f_X(J(i)))_{i \in [d]}$ is an orthonormal basis of S' such that, for all $i \in [d]$,

$$f_X(J(i)) \in F_X(J(i))$$

hence

$$\begin{aligned} \mathrm{tr}(P_{S'} X) &= \sum_{i=1}^{d} \langle f_X(J(i)), X f_X(J(i)) \rangle \\ &= \sum_{j \in J} \xi(j). \end{aligned}$$

Finally, $A^0(X, F_X, J)$ has a minimum and

$$\min A^0(X, F_X, J) = \sum_{j \in J} \xi(j).$$

2. Let $S \in \Omega_J(F_X)$. By remark 2.29, there is $I \in \mathcal{P}^r_d$ such that $S \in \Omega^0_I(F_X)$ thus, by the first point of lemma 4.6,

$$\mathrm{tr}(P_S X) \in \mathbb{R}.$$

Since the map

$$P : \left| \begin{array}{ccc} \mathrm{Gr}(d, V) & \longrightarrow & \mathfrak{gl}_r \\ S & \longmapsto & P_S \end{array} \right.$$

is continuous,

$$t : \left| \begin{array}{ccc} \mathrm{Gr}(d, V) & \longrightarrow & \mathbb{R} \\ S & \longmapsto & \mathrm{tr}(P_S X) \end{array} \right.$$

is also continuous. Yet we have just proven that t has a minimum equal to $\sum_{j \in J} \xi(j)$ on $\Omega^0_J(F_X)$ and, by definition,

$$\Omega_J(F_X) = \overline{\Omega^0_J(F_X)}.$$

Finally, t has a minimum on $\Omega_J(F_X)$ and

$$\min_{\Omega_J(F_X)} t = \sum_{j \in J} \xi(j).$$

$\square$

Corollary 4.7. Let $\xi_1, \dots, \xi_s \in C_+(r)$, $(X_1, \dots, X_s) \in \prod_{k=1}^s \mathcal{O}_{\xi_k}$ and $\mathcal{J} \in (\mathcal{P}_d^r)^s$ such that, with $\mathcal{F} := (F_{X_1}, \dots, F_{X_k})$,

$$\sum_{k=1}^s X_k = 0 \text{ and } \Omega_{\mathcal{J}}(\mathcal{F}) \neq \emptyset.$$

We have

$$\sum_{k=1}^s \langle T_{J_k}, \xi_k \rangle \leqslant 0.$$

Proof There is $S \in \Omega_{\mathcal{J}}(\mathcal{F})$. For all $k \in [s]$, by lemma 4.6,

$$\langle T_{J_k}, \xi_k \rangle \leqslant \operatorname{tr}(P_S X_k).$$

Hence

$$\begin{aligned} \sum_{k=1}^s \langle T_{J_k}, \xi_k \rangle &\leqslant \operatorname{tr}\left(P_S \sum_{k=1}^s X_k\right) \\ &\leqslant 0. \end{aligned}$$

□

Corollary 4.8 *(Klyachko).* Let $\xi \in \operatorname{Kir}(r, s)$. We have

$$\sum_{k=1}^s |\xi_k| = 0$$

and, for all $d \in [r-1]$, for all $J \in \operatorname{Intersecting}(d, r, s)$,

$$\sum_{k=1}^s \langle T_{J_k}, \xi_k \rangle \leqslant 0.$$

Proof This is the sum of remark 4.3 and corollary 4.7. □

4.3 The Borel-Weil construction

This section is a quick reminder about the Borel-Weil construction without any proof. We use this construction in section 4.4.

▷ Notation.
- We denote by $H(r)$ the set of all invertible diagonal matrices of order r and we identify it with $(\mathbb{C}^*)^r$ through

$$\begin{array}{ccc} (\mathbb{C}^*)^r & \longrightarrow & H(r) \\ (t_1, \dots, t_r) & \longmapsto & \operatorname{diag}(t_1, \dots, t_r) \end{array}.$$

  We denote by $B(r)$ the group of invertible upper-triangular matrices of order r and by $N(r)$ the group composed of all the elements of $B(r)$ with only ones on the diagonal.

- We recall that the set of *dominant weights* is

$$\Lambda_+(r) := \{\lambda \in \mathbb{Z}^r \mid \lambda_1 \geqslant \dots \geqslant \lambda_r\}.$$

- For all $\lambda \in \Lambda_+(r)$ we denote

$$\chi_\lambda : \left| \begin{array}{ccc} B(r) & \longrightarrow & \mathbb{C} \\ (b_{i,j})_{(i,j)\in[r]^2} & \longmapsto & \prod_{i=1}^r b_{i,i}^{\lambda_i} \end{array} \right.$$

the associated character.

- We denote by $\det_r$ the determinant representation of $\mathrm{GL}(r)$.

- Let $\lambda \in \Lambda_+(r)$ and

$L(\lambda) := \{s : \mathrm{GL}(r) \to \mathbb{C} \mid s \text{ holomorphic and } \forall g \in \mathrm{GL}(r), \forall b \in B(r), s(gb) = s(g)\chi_{\lambda^*}(b)\}$

and, for all $g \in \mathrm{GL}(r)$ and $s \in L(\lambda)$,

$$g \cdot s : \left| \begin{array}{ccc} \mathrm{GL}(r) & \longrightarrow & \mathbb{C} \\ h & \longmapsto & s(g^{-1}h) \end{array} \right. .$$

- We denote
$$\mathbf{1}_r := (1)_{i\in[r]}.$$

Remark 4.9. The set $\Lambda_+(r)$ is a semi-group and is a subset of the cone $C_+(r)$.

Theorem 4.10 *(Borel-Weil).* The representation $L(\lambda)$ of $\mathrm{GL}(r)$ is irreducible with highest weight λ.

Theorem 4.11. Let $\lambda \in \Lambda_+(r)$. There exists a unique irreducible representation of $\mathrm{GL}(r)$ with highest weight λ up to isomorphism.

Corollary 4.12. For all $\lambda \in \Lambda_+(r)$ and $k \in \mathbb{Z}$,

$$L(\lambda + k\mathbf{1}_r) \simeq L(\lambda) \otimes \det_k^r.$$

▷ Notation. Let $\lambda \in \Lambda_+(r)$ and $\lambda' \in \Lambda(r')$. We denote by $L(\lambda, \lambda')$ the set of all maps $s : \mathrm{GL}(r) \times \mathrm{GL}(r') \to \mathbb{C}$ such that, for all $g \in \mathrm{GL}(r)$, $g' \in \mathrm{GL}(r')$, $b \in B(r)$, $b' \in B(r')$,

$$s(gb, g'b') = s(g, g')\chi_{\lambda^*}(b)\chi_{\lambda'^*}(b').$$

Remark 4.13. The application

$$\begin{array}{ccc} L(\lambda) \times L(\lambda') & \longrightarrow & L(\lambda, \lambda') \\ (s, s') & \longmapsto & ss' \end{array}$$

is bilinear and bijective. From this we deduce that

$$L(\lambda) \otimes L(\lambda') \simeq L(\lambda, \lambda').$$

4.4 The Knutson-Tao theorem

4.4.1 The Kempf-Ness's lemma

▷ Notation. Let $\lambda \in \Lambda_+(r)$.

- We denote by $\langle \cdot, \cdot \rangle$ an $U(r)$-invariant Hermitian product on the vector space $L(\lambda)$.

- We denote by $v_\lambda \in L(\lambda)$ a unitary vector such that
$$L(\lambda)^{N(r)} = \mathbb{C}v_\lambda.$$
- We denote by $\rho_\lambda : \mathfrak{gl}(r) \to \mathfrak{gl}(L(\lambda))$ the Lie algebra representation associated to $L(\lambda)$.
- For a vector space W we denote by $\mathbb{P}(W)$ its projective space and by
$$\begin{array}{ccc} W & \longrightarrow & \mathbb{P}(W) \\ w & \longmapsto & [w] \end{array}$$
the canonical projection.

Lemma 4.14. Let $\lambda \in C_+(r)$.

1. We have
$$\mathrm{GL}(r) \cdot [v_\lambda] = U(r) \cdot [v_\lambda].$$
2. We have
$$U(r)_{v_\lambda} = U(r)_\lambda$$
and the associated $U(r)$-equivariant diffeomorphism
$$\begin{array}{ccc} \mathcal{O}_\lambda & \longrightarrow & \mathrm{GL}(R) \cdot [v_\lambda] \\ u \cdot \lambda & \longmapsto & u \cdot [v_\lambda] \end{array}.$$
3. For all $m \in \mathfrak{gl}(r)$ and $u \in U(r)$,
$$\mathrm{tr}((u \cdot \lambda)m) = \langle u \cdot v_\lambda, \rho_\lambda(m)(u \cdot v_\lambda) \rangle .$$

Proof This lemma correspond to equations 2.1 and 2.2 in [BVW18]. □

Lemma 4.15 *(Kempf-Ness).* Let $\lambda \in \Lambda(r)^s$ such that
$$\left(\bigotimes_{k=1}^{s} L(\lambda_k) \right)^{\mathrm{GL}(r)} \neq \{0\} .$$
We have
$$\lambda \in \mathrm{Kir}(r, s).$$

Proof This is proposition 2.3 in [BVW18]. □

4.4.2 Invariants

▷ Settings. Let $\mathcal{I} \in (\mathcal{P}_r^n)^s$ such that
$$\mathrm{edim}\, \mathcal{I} = 0.$$
Let b a base of $\mathcal{L}(V, Q) \times \prod_{k \in [s]} \mathcal{L}_{\mathcal{I}_k}(F_0, G_0)$ and b' a base of $\mathcal{L}(V, Q)^s$.

Remark 4.16. In the proof of lemma 3.23 we have seen that
$$\dim \left(\mathcal{L}(V, Q) \times \prod_{k \in [s]} \mathcal{L}_{\mathcal{I}_k}(F_0, G_0) \right) = \dim \mathcal{L}(V, Q)^s + \mathrm{edim}\, \mathcal{I}.$$

This allows us to define $\delta_{\mathcal{I}}$ below.

Definition 4.17. The *determinant function* associated to $\mathcal{I}$ is

$$\delta_{\mathcal{I}} : \left| \begin{array}{ccc} \mathrm{GL}(r)^s \times \mathrm{GL}(n-r)^s & \longrightarrow & \mathbb{C} \\ (g,h) & \longmapsto & \det \mathrm{Mat}_{b,b'}(\Delta_{\mathcal{I},g,h}) \end{array} \right. .$$

Remark 4.18. The map $\delta_{\mathcal{I}}$ is polynomial hence holomorphic.

Lemma 4.19. The following assertions are equivalent.

i The map $\delta_{\mathcal{I}}$ is nonzero.

ii The tuple $\mathcal{I}$ is intersecting.

Proof For all $(g,h) \in \mathrm{GL}(r) \times \mathrm{GL}(n-r)$, $\delta_{\mathcal{I}}(g,h) \neq 0$ if and only if $\Delta_{\mathcal{I},g,h}$ is bijective, if and only if it is surjective. Using lemma 3.24, we have the announced result. ☐

▷ Notation. For all $(g,h) \in \mathrm{GL}(r) \times \mathrm{GL}(n-r)$ we denote

$$\sigma_{g,h} : \left| \begin{array}{ccc} \mathcal{L}(V,Q) & \longrightarrow & \mathcal{L}(V,Q) \\ \varphi & \longmapsto & h\varphi g^{-1} \end{array} \right. .$$

Remark 4.20. This defines a group action $\mathrm{GL}(r) \times \mathrm{GL}(n-r) \curvearrowright \mathcal{L}(V,Q)$.

Lemma 4.21.

1. For all $(g,h) \in \mathrm{GL}(r) \times \mathrm{GL}(n-r)$,

$$\det(\sigma_{g,h}) = \det(g)^{-(n-r)} \det(h)^r.$$

2. Let $I \in \mathcal{P}_r^n$, $(g,h) \in B(r) \times B(n-r)$ and $\sigma_{g,h}^I$ the endomorphism induced by $\sigma_{g,h}$ on $\mathcal{L}_I(F_0, G_0)$. We have

$$\det\left(\sigma_{g,h}^I\right) = \chi_{w(I)-(n-r)\mathbf{1}_r}(g)\chi_{w(I^c)}(h).$$

Proof

1. Through the identification $\mathcal{L}(V,Q) = V^* \otimes Q$, $\sigma_{g,h}$ is identified with the only endormorphism u of $V^* \otimes Q$ such that, for all $(l,q) \in V^* \times Q$,

$$u(l \otimes q) = (lg^{-1}) \otimes (hq).$$

From this we deduce the first point of the lemma.

2. This is lemma 6.2.3 in [BVW18].

☐

Proposition 4.22. We have

$$\delta_{\mathcal{I}} \in \bigotimes_{k=1}^{s} \left(L(\lambda(\mathcal{I}_k)^*) \otimes L(\lambda(\mathcal{I}_k^c)^* - r\mathbf{1}_{n-r}) \right).$$

and, for all $(g,h) \in \mathrm{GL}(r) \times \mathrm{GL}(n-r)$,

$$(g,h) \cdot \delta_{\mathcal{I}} = (\det g)^{(n-r)(s-1)} (\det h)^{r(1-s)}.$$

Proof This calculus is explained in the proof of theorem 6.2.4 in [BVW18]. ☐

Corollary 4.23. If $\mathcal{I}$ is intersecting,

$$\left(\det_r^{(s-1)(n-r)} \otimes \bigotimes L(\lambda(\mathcal{I}_k))\right)^{\mathrm{GL}(r)} \neq \{0\}\,.$$

Proof Using proposition 4.22,

$$\left(\delta_{\mathcal{I}} \in \det_r^{(s-1)(n-r)} \otimes \bigotimes L(\lambda(\mathcal{I}_k))\right)^{\mathrm{GL}(r)}.$$

□

4.4.3 Knutson-Tao's theorem

▷ Notation. For all $\mathcal{I} \in (\mathcal{P}_r^n)^s$ we denote

$$c(\mathcal{I}) := \dim \left(\det_r^{(s-1)(n-r)} \otimes \bigotimes L(\lambda(\mathcal{I}_k))\right)^{\mathrm{GL}(r)}$$

and, for all $\lambda \in \Lambda_+(r)$, we denote

$$c(\lambda) := \dim \left(\bigotimes_{k=1}^{s} L(\lambda_k)\right)^{\mathrm{GL}(r)}.$$

Remarks 4.24.
- Lemma 4.15 means that, for all $\lambda \in \Lambda_+(r)$,

$$c(\lambda) \geqslant 1 \Rightarrow \lambda \in \mathrm{Kir}(r,s).$$

- Corollary 4.23 means that, for all $\mathcal{I} \in (\mathcal{P}_r^n)^s$ intersecting and of expected dimension zero,

$$c(\mathcal{I}) \geqslant 1.$$

Corollary 4.25. Let $\mathcal{I} \in (\mathcal{P}_r^n)^s$ intersecting and of expected dimension zero. We have

$$c(w(\mathcal{I})) = c(\mathcal{I}) \geqslant 1.$$

Proposition 4.26. Let $\lambda \in \Lambda_+(r)$ such that

$$\begin{cases} & \sum_{k=1}^{s} |\lambda_k| & = 0 \\ \forall d \in [r-1], \forall \mathcal{J} \in \mathrm{Horn}^0(d,r,s), & \sum_{k=1}^{s} \langle T_{\mathcal{J}_k}, \lambda_k \rangle & \leqslant 0. \end{cases}$$

We have

$$c(\lambda) \geqslant 1.$$

Proof This is proposition 6.3.2 in [BVW18]. □

The first point of Kuntson-Tao theorem 4.27 proves the Horn conjecture ; for all $r, s \in \mathbb{N}^*$, there exists a finite set of inequalities (Horn's inequalities) describing the Kirwan cone $\mathrm{Kir}(r,s)$ and these inequalities have an inductive description with respect to the dimension r.

Theorem 4.27 *(Knutson-Tao).* 1. *Horn inequalities.* We have

$$\mathrm{Kir}(r,s) = \left\{\xi \in C_+(r)^s \mid \sum_{k=1}^{s} |\xi_k| = 0 \text{ and } \forall d \in [r-1], \forall J \in \mathrm{Horn}^0(d,r,s), \sum_{k=1}^{s} \langle T_{J_k}, \xi_k \rangle \leqslant 0\right\}.$$

2. *Saturation property.* We have

$$\mathrm{Kir}(r,s) \cap (\mathbb{Z}^r)^s = \left\{\lambda \in \Lambda_+(r)^s \mid c(\lambda) > 0\right\}.$$

Proof 1. We denote

$$\mathcal{K}(r,s) := \left\{\xi \in C_+(r)^s \mid \sum_{k=1}^{s} |\xi_k| = 0 \text{ and } \forall d \in [r-1], \forall J \in \mathrm{Horn}^0(d,r,s), \sum_{k=1}^{s} \langle T_{J_k}, \xi_k \rangle \leqslant 0\right\}.$$

Using Klyachko's lemma 4.8 and Belkale's theorem 3.42,

$$\mathrm{Kir}(r,s) \subset \mathcal{K}(r,s).$$

Let's show the converse.

- First, we want to show that $\mathrm{Kir}(r,s)$ is closed (in the euclidean topology). Let $(\lambda^i)_i \in \mathrm{Kir}(r,s)^{\mathbb{N}}$ such that and $\lambda \in (\mathbb{R}^r)^s$ such that

$$\lambda^i \longrightarrow \lambda.$$

For all $i \in \mathbb{N}$ there exists $X^i \in \prod_{k=1}^{s} \mathcal{O}_{\lambda_k^i}$ such that,

$$\sum_{k=1}^{s} X_k^i = 0.$$

Yet, for all $k \in [s]$,

$$\forall i \in \mathbb{N}, \mathcal{O}_{\lambda_k^i} = U(r) \cdot \lambda_k^i$$

and $(\lambda_k^i)_i$ is bounded hence there exists $\varphi_k : \mathbb{N} \to \mathbb{N}$ strictly increasing such that $(X_k^{\varphi_k(i)})_i$ converge to an Hermitian matrix X_k. Thus there exists $\varphi : \mathbb{N} \to \mathbb{N}$ strictly increasing such that

$$(X_k^{\varphi(i)})_k \to (X_k)_k.$$

Finally,

$$\sum_{k=1}^{s} X_k = 0 \text{ and } (X_k)_k \in \prod_{k=1}^{s} \mathcal{O}_{\lambda_k}$$

hence $\lambda \in \mathrm{Kir}(r,s)$. From this we deduce that $\mathrm{Kir}(r,s)$ is closed.

- Using the definition of $\mathrm{Kir}(r,s)$, this set is invariant under rescaling by any $x \in \mathbb{R}_+$. Yet, using proposition 4.26 and Kempf-Ness lemma 4.15,

$$\mathcal{K}(r,s) \cap (\mathbb{Z}^r)^s \subset \mathrm{Kir}(r,s)$$

hence

$$\mathcal{K}(r,s) \cap (\mathbb{Q}^r)^s \subset \mathrm{Kir}(r,s).$$

But $\mathcal{K}(r,s)$ is a polyhedral cone hence

$$\overline{\mathcal{K}(r,s) \cap (\mathbb{Q}^r)^s} = \mathcal{K}(r,s)$$

and

$$\mathcal{K}(r,s) \subset \mathrm{Kir}(r,s).$$

Finally,

$$\mathrm{Kir}(r,s) = \mathcal{K}(r,s).$$

2. By Kempf-Ness lemma 4.15,

$$\{\lambda \in \Lambda_+(r)^s \mid c(\lambda) > 0\} \subset \mathrm{Kir}(r,s) \cap (\mathbb{Z}^r)^s.$$

Let $\lambda \in \mathrm{Kir}(r,s) \cap (\mathbb{Z}^r)^s$. By Klyachko's lemma 4.8 and Belkale's theorem 3.42, λ satisfies the hypothesis of 4.26 hence

$$c(\lambda) > 0$$

and, using Kempf-Ness lemma 4.15,

$$\lambda \in \mathrm{Kir}(r,s) \cap (\mathbb{Z}^r)^s.$$

□

COROLLARY 4.28. Let $\lambda \in \Lambda_+(r)^s$. The following assertions are equivalent.

(i) $c(\lambda) > 0$

(ii) $\exists N \in \mathbb{N}^*, c(N\lambda) > 0$

(iii) $\forall N \in \mathbb{N}^*, c(N\lambda) > 0$

Proof

- Assume (i) and let $N \in \mathbb{N}^*$. We have $N\lambda \in \Lambda_+(r)^s$. By theorem 4.27, $\lambda \in \mathrm{Kir}(r,s)$. Hence $N\lambda \in \mathrm{Kir}(r,s)$ and, using theorem 4.27 again, $c(N\lambda) > 0$. We have (iii).
- In the same way, since $\mathrm{Kir}(r,s)$ is a cone, $(ii) \Rightarrow (i)$.
- The implication $(iii) \Rightarrow (ii)$ is easy.

□

5 Appendix

This is an example of Python code computing the set $\mathrm{Horn}(r, n, s)$ by induction. We do not claim any efficiency or optimisation about this algorithm.

```
from itertools import combinations, product

s=3 #Corresponds to the parameter s used : we will consider sequences of s tuples.

def crochet(r): #Returns the set [r]
    return [i+1 for i in range(r)]

def nb_of(x,l): #Returns the number of x in the list l
    s=0
    for y in l:
        if y==x:
            s+=1
    return s

def is_a_permutation(l1,l2): #Returns True if the list l1 is a permutation of the list l2
    for x in l1:
        if nb_of(x,l1)!=nb_of(x,l2):
            return False
    return True

def dim(I): #Returns the dimension of a tuple I seen in Python as a list of integers
    return sum([I[i]-(i+1) for i in range(len(I))])

def compo(I,J): #Returns the composition of to sequences of tuples I and J
    d=len(J[0])
    return [[I[k][J[k][i]-1] for i in range(d)] for k in range(s)]

def edim(I,n): #Returns the expected dimension of a sequence of s tuples
    r=len(I[0])
    return sum([dim(I[k]) for k in range(s)])-(s-1)*r*(n-r)

def candidats(r,n): #Returns two lists of sequences of s r-tuples : the ones such that edim=0 and the
↪   ones such that edim>0
    enull=[]
    epos=[]
    souscand=product(list(combinations(crochet(n),r)),repeat=s)
    for x in souscand:
        e=edim(x,n)
        if e>0:
            epos.append(x)
        if e==0:
            enull.append(x)
    return enull,epos

def horn(r,n): #Returns the set Horn(r,n,s)
    candn,candp=candidats(r,n) #All of the candidates are such that edim>=0
    rn,rp=[],[] #We will put the verified candidates such that edim=0 and such that edim>0
    if r==1: #Base case
        return candn,candp
    else: #We use the inductive description
        h=[horn(d,r)[0] for d in range(1,r)] #These are the Horn tuples such that edim=0 for d=1,...,r-1
        #Warning : h[d] are elements of Horn(d+1,r,s)
        for I in candn:
            b=True #b tells us if the candidate I is verified or not
            d=0
            while b and d<r-1:
```

```
            for J in h[d]:
                if edim(compo(I,J),n)<0:
                    b=False
            d+=1
        if b: #Then the candidate is verified and we can add it to the list rn
            rn.append(I)
    for I in candp:
        b=True #b tells us if the candidate I is verified or not
        d=0
        while b and d<r-1:
            for J in h[d]:
                if edim(compo(I,J),n)<0:
                    b=False
            d+=1
        if b: #Then the candidate is verified and we can add it to the list rp
            rp.append(I)
    return rn,rp

def sans_permutation(h): #Returns the element of the list h up to a permutation
    rn,rp=h[0],h[1]
    a,b=len(rn),len(rp)
    rnb=[True for i in range(a)]
    rpb=[True for i in range(b)]
    repn,repp=[],[]
    for i in range(a):
        if rnb[i]:
            repn.append(rn[i])
            for j in range(i+1,a):
                if is_a_permutation(rn[i],rn[j]):
                    rnb[j]=False
    for i in range(b):
        if rpb[i]:
            repp.append(rp[i])
            for j in range(i+1,b):
                if is_a_permutation(rp[i],rp[j]):
                    rpb[j]=False
    return repn,repp
```

Now we give some examples of the results this algorithm can return.

```
In [1]: horn(2,3)
Out[1]:
([((1, 2), (2, 3), (2, 3)),
  ((1, 3), (1, 3), (2, 3)),
  ((1, 3), (2, 3), (1, 3)),
  ((2, 3), (1, 2), (2, 3)),
  ((2, 3), (1, 3), (1, 3)),
  ((2, 3), (2, 3), (1, 2))],
 [((1, 3), (2, 3), (2, 3)),
  ((2, 3), (1, 3), (2, 3)),
  ((2, 3), (2, 3), (1, 3)),
  ((2, 3), (2, 3), (2, 3))])

In [2]: sans_permutation(horn(2,3))
Out[2]:
([((1, 2), (2, 3), (2, 3)), ((1, 3), (1, 3), (2, 3))],
 [((1, 3), (2, 3), (2, 3)), ((2, 3), (2, 3), (2, 3))])

In [3]: h=horn(5,10)

In [4]: len(h[0])+len(h[1])
```

```
Out[4]: 718738

In [5]: h=horn(5,11)

In [6]: len(h[0])+len(h[1])
Out[6]: 3640866
```

From this we know that $\mathrm{Horn}(2,3,3)$ is made of the tuples

$$\begin{gathered}(\{1,2\},\{2,3\},\{2,3\})\\(\{1,3\},\{1,3\},\{2,3\})\\(\{1,3\},\{2,3\},\{1,3\})\\(\{2,3\},\{1,2\},\{2,3\})\\(\{2,3\},\{1,3\},\{1,3\})\\(\{2,3\},\{2,3\},\{1,2\})\end{gathered}$$

of expected dimension 0 and of the tuples

$$\begin{gathered}(\{1,3\},\{2,3\},\{2,3\})\\(\{2,3\},\{1,3\},\{2,3\})\\(\{2,3\},\{2,3\},\{1,3\})\\(\{2,3\},\{2,3\},\{2,3\}).\end{gathered}$$

of strictly positive expected dimension. We also know that

$$\begin{aligned}\mathrm{Card}\,\mathrm{Horn}(5,10,3)&=718,738\\\mathrm{Card}\,\mathrm{Horn}(5,11,3)&=3,640,866.\end{aligned}$$

6 References

[Bel05] Prakash Belkale. “Geometric Proofs of Horn and Saturation Conjectures”. In: *Journal of Algebraic Geometry* 15 (2005).

[Bha99] Rajendra Bhatia. “Algebraic geometry solves an old matrix problem”. In: *Mathematics and Statistics* (Dec. 1999), pp. 101–105.

[BVW18] Nicole Berline, Michèle Vergne, and Michael Walter. “The Horn inequalities from a geometric point of view”. In: *L’Enseignement Mathématique* 63.3 (Sept. 2018), pp. 403–470. DOI: 10.4171/lem/63-3/4-7. URL: https://doi.org/10.4171%5C%2Flem%5C%2F63-3%5C%2F4-7.