A refinement of Horn's conjecture

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Abstract

The solved Horn conjecture gives necessary and sufficient conditions for identifying the spectrum of a sum of Hermitian matrices in the form of linear inequalities that admit a description by induction on the size of the matrices. In this poster we present this conjecture, a possible solution and a refinement of it.

Introduction

Let A and B be two square matrices of the same order. A natural question (also coming out in physics for example) is to know the relations between the eigenvalues of A, B and $A+B$. If A and B are diagonalizable and commute, then they are simultaneously diagonalizable and the spectrum of their sum is well known. Here we will study the more delicate case of Hermitian matrices (with complex coefficients). This problem and Horn's conjecture (proven in 1999) are exposed in some famous papers [1, 2, 3]. A pedagogical introduction can be found in [4].

Horn's conjecture

Notations

Let $r \in \mathbb{N}^*$. For all $i \in \mathbb{N}^*$ we denote by $[i]$ the set of integers $j \in \mathbb{N}$ such that $1 \leqslant j \leqslant i$. We denote by $\mathbb{R}^r_{\geqslant}$ the set of all $\lambda := (\lambda(i))_{i\in[r]} \in \mathbb{R}^r$ such that $\lambda(1) \geqslant \cdots \geqslant \lambda(r)$.

Hermitian matrices

A complex matrix A of order r is Hermitian if it is equal to its own conjugate transpose. By the spectral theorem, such a matrix is diagonalizable with real eigenvalues : we can see its spectrum with multiplicities as an element of $\mathbb{R}^r_{\geqslant}$. For all $\lambda \in \mathbb{R}^r_{\geqslant}$ we denote by $\mathcal{O}_\lambda$ the set of all hermitian matrices of spetrum λ (this notation comes from the fact that this set is an orbit for the conjugation by the unitary matrices subgroup).

The Kirwan cone

Now we can reformulate our question : what are the families $(\Lambda_1, \Lambda_2, \Lambda_3)$ of real tuples such that Λ_1 (resp. Λ_2) is the spectrum of an Hermitian matrix A (resp. B) and that Λ_3 is the spectrum of $-(A+B)$? We define the *Kirwan cone* as the set of all $\Lambda \in (\mathbb{R}^r_{\geqslant})^3$ such that there exists 3 hermitian matrices with a sum equal to 0 and spectrums corresponding to the 3 real sequences $\Lambda_1, \Lambda_2, \Lambda_3$:

$$\mathrm{LR}(r) := \left\{\Lambda \in (\mathbb{R}^r_{\geqslant})^3 \mid 0 \in \mathcal{O}_{\Lambda_1} + \mathcal{O}_{\Lambda_2} + \mathcal{O}_{\Lambda_3}\right\}.$$

Horn's conjecture

In 1962, Alfred Horn conjectured about the fact that a set of finite inequalities defined by induction on r is enough to describe $\mathrm{LR}(r)$.

First inequalities

Let $\Lambda \in (\mathbb{R}^r_{\geqslant})^3$. The first condition we can see comes from the trace of the equality defining the Kirwan cone : if $\Lambda \in \mathrm{LR}(r)$,

$$\sum_{j\in[r]} \Lambda_1(j) + \sum_{j\in[r]} \Lambda_2(j) + \sum_{j\in[r]} \Lambda_3(j) = 0 \tag{1}$$

and it is sufficient to describe $\mathrm{LR}(1)$. In 1912, H. Weyl exhibited other necessary inequalities : if $\Lambda \in \mathrm{LR}(r)$,

$$i+j-1 \in [r] \Rightarrow \Lambda_1(r+1-i) + \Lambda_2(r+1-j) + \Lambda_3(i+j-1) \leqslant 0. \tag{2}$$

If $r = 1$, $\Lambda \in \mathrm{LR}(r)$ if and only if (1). If $r = 2$, $\Lambda \in \mathrm{LR}(r)$ if and only if (1) and (2). We want to describe $\mathrm{LR}(r)$ with inequalities of this form.

A solution to Horn's conjecture

How to write the inequalities describing $\mathrm{LR}(r)$?

Let $\lambda \in \mathbb{R}^r$, $d \in [r]$ and $J \subset [r]$ a subset of cardinality d. We define the sum $|\lambda|_J := \sum_{j\in J} \lambda(j)$. We identify J with the unique strictly growing map $[d] \to [r]$ of image J and we define the tuples

$$\mu(J) := (J(d)-d, \ldots, J(1)-1) \in \mathbb{R}^d_{\geqslant}$$

and $\mathbb{1}_d := (1)_{k\in[d]}$ and the constant sequence equal to 1.

A solution to Horn's conjecture

A first solution was found using Schubert's calculus [5] and combinatorics [6]. The following theorem shows that the cone $\mathrm{LR}(r)$ can be described by induction on r the size of the matrices. Note that this problem also have a strong link to representation theory of Lie groups through a *saturation property* [6].

Klyachko-Knutson-Tao theorem

For all $\Lambda \in (\mathbb{R}^r_{\geqslant})^3$, Λ is in $\mathrm{LR}(r)$ if and only if the two following conditions hold :

1. equation (1) is satisfied ;
2. for all $d \in [r-1]$ and all 3-tuple $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ of subsets of $[r]$ of cardinality d such that $(\mu(\mathcal{J}_1), \mu(\mathcal{J}_2), \mu(\mathcal{J}_3) - 2(r-d)\mathbb{1}_d) \in \mathrm{LR}(d)$, $\sum_{l=1}^3 |\Lambda_l|_{\mathcal{J}_l} \leqslant 0$.

The inequalities described by Belkale

Horn's conjecture was later also proven true by a geometric method [7] which allows us to have a simple inductive description of the tuples $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ describing $\mathrm{LR}(r)$.

Intersecting tuples

For all 3-tuple $\mathcal{I} := (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3) \in [n]^3$ we denote

$$\mathrm{edim}\,\mathcal{I} := r(n-r) - \sum_{k=1}^{3} \left(r(n-r) - \sum_{j=1}^{r} \mathcal{I}_k(j) - j \right).$$

This integer has a geometric interpretation, yet it is really easy to compute. A 3-tuple $\mathcal{I} := (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3) \in [n]^3$ of subsets of cardinality r is said to be *intersecting* if $\mathrm{edim}\,\mathcal{I} \geqslant 0$ and if for all $d \in [r-1]$ and all intersecting tuple $\mathcal{J} \in [r]^3$ of subsets of cardinality r such that $\mathrm{edim}\,\mathcal{J} = 0$, $\mathrm{edim}\,\mathcal{I}\mathcal{J} \geqslant 0$. This inductive definition allows us to compute intersecting tuples of a given size.

Belkale's theorem

P. Belkale shows that $\Lambda \in (\mathbb{R}^r_{\geqslant})^3$ is in the cone $\mathrm{LR}(r)$ if and only if the two following conditions hold : equation (1) is satisfied ; for all $d \in [r-1]$ and all intersecting 3-tuple $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3) \in [r]^3$ of subsets of cardinality d, $\sum_{l=1}^3 |\Lambda_l|_{\mathcal{J}_l} \leqslant 0$.

Is there redundant inequalities ?

This last theorem gives a finite list of inequalities, called *Horn's inequalities*, to describe $\mathrm{LR}(r)$ and shows that is is a polyhedral cone. But there might be some redundant inequalities : what is the smallest number of inequalities among the ones given by this last theorem that can desribe $\mathrm{LR}(r)$?

The inequalities given by intersecting tuples $\mathcal{I}$ such that $\mathrm{edim}\,\mathcal{I} = 0$ are enough [7]. In fact, a subset of irredundant inequalies is known [8].

Spectra with repetitions

We can consider the elements of the Kirwan cone $\mathrm{LR}(r)$ satisfying the repetitions $\Lambda_1 = \Lambda_2 = \Lambda_3$: we denote

$$\mathrm{LR}'(r) := \left\{\lambda \in \mathbb{R}^r_{\geqslant} \mid (\lambda, \lambda, \lambda) \in \mathrm{LR}(r)\right\}.$$

Since $\mathrm{LR}'(r)$ can be seen as a sub-cone of $\mathrm{LR}(r)$, it is in particular described by the same Horn inequalities (given by Belkale's theorem). Do some of them become redundant when we are restricted to $\mathrm{LR}'(r)$?

Belkale's method as described in [9] can be quickly adapted to prove that this is true. The following theorem shows that it is sufficient to consider Horn inequalities that verify the same repetitions $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}_3$. The result is true in a more general framework with any finite number of spectra and with any type of repetitions [10].

A refinement of Horn's conjecture

For all $\lambda \in \mathbb{R}^r_{\geqslant}$, λ is in $\mathrm{LR}'(r)$ if and only if the two following conditions hold :

1. $\sum_{j\in[r]} \lambda(j) = 0$;
2. for all $d \in [r-1]$ and all subset $J \subset [r]$ of cardinality d such that $\mu(J) - \frac{2(r-d)}{3}\mathbb{1}_d \in \mathrm{LR}'(d)$, $|\lambda|_J \leqslant 0$.

Examples

In the tabular below, $l(r)$ is the minimal number of Horn's inequalities to describe $\mathrm{LR}(r)$ and $l'(r)$ is the number of Horn's inequalities given by the refinement of Horn's conjecture to describe $\mathrm{LR}'(r)$. We do not know if these inequalities are irredundant (i.e. if $l'(r)$ is minimal).

r	1	2	3	4	5	6	7	8	9	10
$l(r)$	2	5	20	52	156	538	2,062	8,522	37,180	168,602
$l'(r)$	2	3	4	7	10	9	16	21	18	35

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