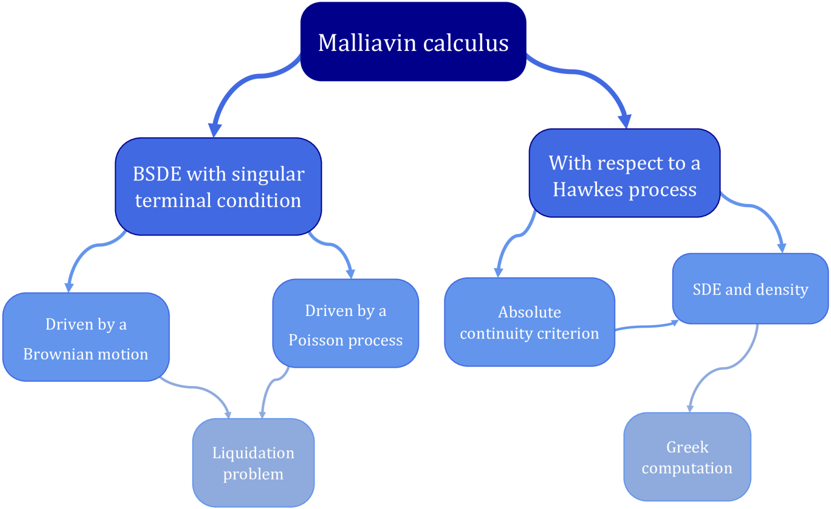
Malliavin calculus and applications in stochastic differential equations

Dorian Cacitti-Holland

Laboratoire Manceau de Mathématiques - Université du Mans

Monday, July 7, 2025

Summary



Malliavin calculus
BSDE with singular terminal condition
With respect to a Hawkes process
Driven by a Brownian motion
Driven by a Poisson process
Absolute continuity criterion
SDE and density
Liquidation problem
Greek computation

Introduction of Malliavin calculus

Malliavin calculus: To extend classical differential calculus to stochastic processes.

W.r.t. the Brownian motion:

- Integration by parts,
- Chain rule,
- Clark-Ocone formula.

W.r.t jump processes: Different constructions and interests.

- To add a jump at an instant and by chaos expansion,
- To derive w.r.t. jump times,
- To derive w.r.t. jump heights.

1. Backward stochastic differential equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\widetilde{\pi}(ds, de)$$

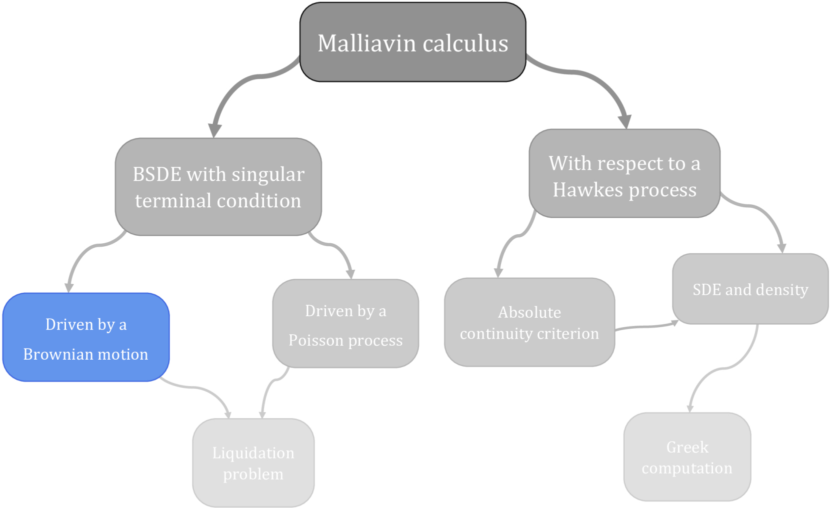
Equation: To find the unknown (Y, Z, U)

Differential: We know the dynamics of the process Y.

Stochastic: The parameters and the unknowns are stochastic processes.

Backward: To fix the terminal value $Y_T = \xi$ and (Z, U) are useful to get an adapted process Y.

1.A. BSDE driven by a Brownian motion



$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

Notations:

- $T \in \mathbb{R}_+^*$ a time horizon,
- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space,
- W a d-dimensional Brownian motion and $(\mathcal{F}_t)_{0 \leq t \leq T}$ the augmented filtration generated by W.

Parameters:

- ξ terminal condition,
- $f(\omega, s, y, z)$ progressively measurable.

Unknowns:

- Y a continuous adapted $\mathbb{R}$-valued process,
- Z a predictable $\mathbb{R}^d$-valued process s.t. $\int_0^T |Z_t|^2 dt < +\infty$ $\mathbb{P} -$ a.s..

1.A. BSDE driven by a Brownian motion

References:

- Ph. Briand et al., *L^p solutions of backward stochastic differential equations*, in: Stochastic Process. Appl. 108.1 (2003).

Existence and unicity of a solution

If $\xi, f(\cdot, 0)$ are integrable, the driver f is monotone w.r.t. y and Lipschitz w.r.t. z then there is a unique solution (Y, Z) to the BSDE.

- D. Nualart, *The Malliavin calculus and related topics*, Probability and its Applications (New York), Springer-Verlag, 2006,
- T. Mastrolia, D. Possamaï, and A. Réveillac, *On the Malliavin differentiability of BSDEs*, in: Ann. Inst. Henri Poincaré Probab. Stat. 53.1 (2017).

Formal link between Y and Z

$D_t Y_t = Z_t$ for any $0 \leq t \leq T$.

1.A. BSDE driven by a Brownian motion

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

Singular terminal condition : $\mathbb{P}(\xi = +\infty) > 0$

References :

- A. Popier, *Limit behaviour of BSDE with jumps and with singular terminal condition*, in: ESAIM: PS 20 (2016),
- T. Kruse and A. Popier, *Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting*, in: Stochastic Processes and their Applications 126.9 (2016).

Problem

We get a unique minimal supersolution (Y, Z) : $\liminf_{t \to T} Y_t \geq \xi$.

But do we have the process Y is continuous at the terminal instant ?

1.A. BSDE driven by a Brownian motion

Markovian framework:

$$f(\omega, t, y, z) = f(t, X_t(\omega), y, z), \qquad \xi = g(X_T),$$

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad 0 \leq t \leq T$$

Assumptions on X:

- σ bounded continuous w.r.t. (s, x), class C^2 w.r.t. x with bounded first derivatives,
- b bounded continuous w.r.t. (s, x), class C^1 w.r.t. x with polynomial growth derivatives,
- $\sigma\sigma^*$ uniformly elliptic and bounded second derivatives w.r.t. x.

Malliavin differentiability of the process X

- The unique solution X is in $S^\infty((0, T), \mathbb{R}^m)$,
- $X_t^i \in \mathbb{D}^{1,\infty}$ for any $t \in [0, T]$ and $i \in \{1, ..., m\}$,
- X admits a density satisfying Gaussian estimates.

1.A. BSDE driven by a Brownian motion

Assumptions on f:

- $f(t,x,0,0) \geq 0$ for any (t,x).
- f Lipschitz continuous w.r.t. z.
- f continuous and monotone w.r.t. y.
- Growth condition: for $q > 1$

$$f(t,x,y,z) - f(t,x,0,z) \leq -\eta(t,x)|y|^q$$

with $\frac{1}{\eta(s,x)} \leq C(1+|x|^\ell)$.

Assumptions on $\xi = g(X_T)$:

- $g : \mathbb{R} \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$.
- $\mathcal{S} = \{x \in \mathbb{R}^m,\ g(x) = +\infty\}$ closed.
- g locally continuously differentiable on $\mathcal{S}^c$
- $\xi = g(X_T)$ locally integrable.

Problem

$\xi = g(X_T)$ is not Malliavin differentiable.

$$\xi^n = \varphi_n(\xi) = \varphi_n(g(X_T))$$

with $(\varphi_n)_{n\in\mathbb{N}}$ a well-chosen regularizing sequence: a non-decreasing sequence of smooth non-decreasing functions such that

$$\varphi_n(u) = \begin{cases} u & \text{if} \quad u \leq n-1 \\ n & \text{if} \quad u \geq n+1 \end{cases}, \quad u \wedge (n-1) \leq \varphi_n(u) \leq u \wedge n.$$

Malliavin differentiability of the terminal condition

For any $n \in \mathbb{N}, \xi^n \in \mathbb{D}^{1,\infty}$ and $D\xi^n = G_n DX_T$ with G_n a bounded random variable.

Idea of the proof: Chain rule (Lipschitz version).

1.A. BSDE driven by a Brownian motion

Truncated BSDE:

$$Y_t^n = \xi^n + \int_t^T f^n(s, X_s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dW_s, \quad 0 \leq t \leq T$$

with $f^n(s,x,y,z) = f(s,x,y,z) - f(s,x,0,0) + \varphi_n(f(s,x,0,0))$.

Convergence of $(Y^n, Z^n)_{n \in \mathbb{N}}$ to (Y, Z).

Under our assumptions:

- Unique solution (Y^n, Z^n) in $S^p(0,T) \times H^p(0,T)$ for any $p > 1$.
- $(Y^n, Z^n)_{n \in \mathbb{N}}$ converges increasingly to (Y, Z) in $S^\infty(0, T-) \times H^\infty(0, T-)$.
- For any $0 \leq t \leq r < T$,

$$Y_t = Y_r + \int_t^r f(s, X_s, Y_s, Z_s) ds - \int_t^r Z_s dW_s.$$

- (Y, Z) is the minimal supersolution.
  In particular $\liminf_{t \to T} Y_t \geq \xi$.

1.A. BSDE driven by a Brownian motion

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T$$

Assumptions on f:

- f of class C^1 w.r.t. (x, y, z), $\frac{\partial f}{\partial y}$ locally (w.r.t. y) uniformly (w.r.t. s, x, z) bounded and, for any $i \in \{1, ..., m\}$, $\frac{\partial f}{\partial x_i}$ locally (w.r.t. y) uniformly (w.r.t. s, x, z) polynomial growth.
- Evoke that for $q > 1$

$$f(t, x, y, z) - f(t, x, 0, z) \leq -\eta(t, x)|y|^q.$$

- For $q \leq 3$, with $0 \leq \alpha \leq \frac{2(q-1)}{q+1}$,

$$|f(s, x, 0, z) - f(s, x, 0, 0)| \leq C(1 + |x|^\ell)|z|^\alpha.$$

1.A. BSDE driven by a Brownian motion

Continuity of the process Y in the terminal instant T

$\liminf_{t\to T} Y_t = \xi \quad \mathbb{P}-$ a.s.

Ideas of the proof:

- For any $\varphi \in C_c^\infty(\mathbb{R}^m, \mathbb{R})$ with $\mathrm{Supp}(\varphi) \subset \mathcal{S}^c$, by Itô's formula

$$\mathbb{E}[\varphi(X_T)Y_T^n] = \mathbb{E}[\varphi(X_t)Y_t^n] + \mathbb{E}\left[\int_t^T \Phi(s, X_s, Y_s^n)ds\right]$$
$$-\mathbb{E}\left[\int_0^T \varphi(X_s)(f(s, X_s, 0, Z_s^n) - f(s, X_s, 0, 0))ds\right]$$
$$+\mathbb{E}\left[\int_t^T Z_s^n \nabla\varphi(X_s)\sigma(s, X_s)ds\right].$$

- (Y^n, Z^n) Malliavin differentiable, $D_t Y_t^n = Z_t^n$, IBP and density of X.
- $\mathbb{E}[\varphi(X_T)\xi] \geq \mathbb{E}[\varphi(X_T)\liminf_{t\to T} Y_t]$ & a.s. $\liminf_{t\to T} Y_t \geq \xi$.

1.A. BSDE driven by a Brownian motion

Question

Do we have (Y, Z) Malliavin differentiable and DY continuous ?

Particular BSDE in liquidation problem:

$$Y_t = \xi + \int_t^T \left(-(p-1) \frac{|Y_s|^{q-1} Y_s}{\eta_s^{q-1}} ds + \gamma_s \right) ds - \int_t^T Z_s dW_s$$

Assumptions:

- $\xi = +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$,
- $\eta_t = \eta_0 + \int_0^t b_s^\eta ds + \int_0^t \sigma_s^\eta dW_s$,
- $0 < \eta_* \leq \eta_s < \eta^*$,
- b^η, σ^η bounded prog. meas.,
- γ progressively measurable,
- $0 \leq \gamma \leq \gamma^*$.

Convergence of $(Y^n, Z^n)_{n \in \mathbb{N}}$ and continuity of Y

- $(Y^n, Z^n)_{n \in \mathbb{N}}$ converges to (Y, Z),
- (Y, Z) the minimal solution of the BSDE: $\lim_{t \to T} Y_t = +\infty$.

1.A. BSDE driven by a Brownian motion

$$Y_t = (+\infty) + \int_t^T \left(-(p-1) \frac{|Y_s|^{q-1} Y_s}{\eta_s^{q-1}} ds + \gamma_s \right) ds - \int_t^T Z_s dW_s$$

Assumptions: b^η, η and γ are Malliavin differentiable with suitable standard integrability.

Malliavin derivatives and convergence

- (Y, Z) Malliavin differentiable,
- $\lim_{n \to +\infty} \sup_{0 \leq \theta \leq T} \mathbb{E} \left[\sup_{0 \leq t \leq \tau} |D_\theta Y_t - D_\theta Y_t^n|^\ell \right] = 0$, $\tau \in [0, T)$ and $\ell > 1$.

Limit behavior of Malliavin derivative

- If η is deterministic then $\lim_{t \to T} |D_\theta Y_t| = 0 =: D_\theta \xi$.
- On $\{D_\theta \eta_T \neq 0\}$, $\lim_{t \to T} |D_\theta Y_t| = +\infty \neq D_\theta \xi$.

1.A. BSDE driven by a Brownian motion

Liquidation problem: To minimize

$$J(t, A) = \mathbb{E}\left[\int_t^T (\eta_s |a_s|^p + \gamma_s |A_s|^p) ds + \xi |A_T|^p \middle| \mathcal{F}_t\right]$$

over all progressively measurable processes A that satisfy the dynamics $A_s = x + \int_t^s a_u du$ and the liquidation constraint $A_T 1_{\{\xi=+\infty\}} = 0$.

Minimizer of the functional

A minimizer of J is given by $A^*_s = x \exp\left(-\int_t^s \left(\frac{Y_u}{\eta_u}\right)^{q-1} du\right)$.

The value function is given by $v(t, x) := J(t, A^*_t) = |x|^p Y_t$.

1.A. BSDE driven by a Brownian motion

$$J(t,A) = \mathbb{E}\left[\int_t^T (\eta_s |a_s|^p + \gamma_s |A_s|^p) ds + \xi |A_T|^p \middle| \mathcal{F}_t\right]$$

$$A_s^* = x \exp\left(-\int_t^s \left(\frac{Y_u}{\eta_u}\right)^{q-1} du\right), \quad v(t,x) = |x|^p Y_t$$

Consequence

$\lim_{t \to T} v(t,x) = |x|^p \xi = v(T,x)$ and there is no extra cost.

Consequence

If $\xi = +\infty$ then the optimal quantity A^* is Malliavin differentiable and

$$D_\theta A_s^* = -(q-1) A_s^* \int_t^s \left|\frac{Y_u}{\eta_u}\right|^{q-2} \text{sign}(Y_u) D_\theta \left(\frac{Y_u}{\eta_u}\right) du.$$

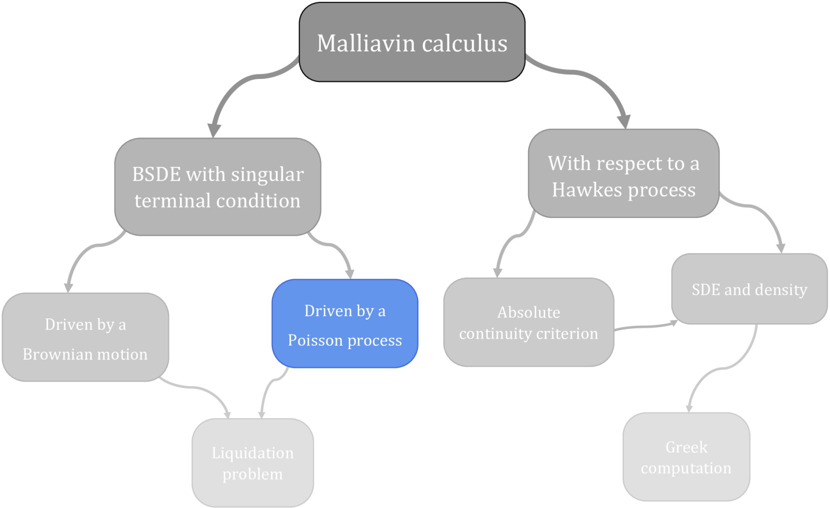
Interest: To compute the sensitivity in this liquidation problem.

1.A. BSDE driven by a Brownian motion

Two papers with L. Denis and A. Popier:

- D. Cacitti-Holland, L. Denis, and A. Popier, *Growth condition on the generator of BSDE with singular terminal value ensuring continuity up to terminal time*, in: Stochastic Processes and their Applications (2025), vol 183,
- D. Cacitti-Holland, L. Denis, A. Popier, *Malliavin derivative and sensitivity for optimal liquidation*, 2025 submitted, https://hal.science/hal-05072816.

1.B. BSDE driven by a Poisson process



$$Y_t = \xi + \int_t^T f(s, Y_s, U_s) ds - \int_t^T U_s d\widetilde{N}_s$$

Notations: N a Poisson process with intensity λ, $\widetilde{N}$ the compensated process and $(\mathcal{F}_t)_{0 \leq t \leq T}$ the filtration generated by N.

Problem

Does Y have a continuity property at the terminal instant T ?

Parameters:

- $\xi = g(N_T)$ singular terminal condition with **a right barrier**

$$g(x) = (+\infty) 1_{\{x \geq x_0\}} + \varphi(x) 1_{\{x < x_0\}},$$

- $f(s, y, u) = -y|y|$ **the quadratic case**.

Unknowns:

- Y a càdlàg progressively measurable process,
- U a predictable process.

References:

- T. Kruse and A. Popier, *Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting*, in: Stochastic Processes and their Applications 126.9 (2016),
- A. Popier, *Limit behaviour of BSDE with jumps and with singular terminal condition*, in: ESAIM: PS 20 (2016),
- G. Barles, R. Buckdahn, and É. Pardoux, *Backward stochastic differential equations and integral-partial differential equations*, in: Stochastics Stochastics Rep.60.1-2 (1997).

1.B. BSDE driven by a Poisson process

$$Y_t = g(N_T) - \int_t^T Y_s|Y_s|ds - \int_t^T U_s d\widetilde{N}_s$$

Associated IPDE:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)| = -\lambda u(t,x+1) \\ u(T,x) = g(x) \end{cases}$$

Theorem

For any $t \in [0, T]$, $Y_t = u(t, N_t) = \dfrac{1}{T-t} 1_{\{t<T\}} + g(N_T) 1_{\{t=T\}}$.

Ideas of the proof:

- $Y_t^n = u^n(t, N_t)$ with $\xi^n = n \wedge g(N_T)$.
- By Riccati's equations, $u^n(t,x)$ is obtained for $x \in [x_0, +\infty)$ then $x \in [x_0 - k - 1, x_0 - k)$ by induction on $k \in \mathbb{N}$.
- $u(t,x) = \lim_{n\to+\infty} u^n(t,x)$.

1.B. BSDE driven by a Poisson process

$$Y_t = g(N_T) - \int_t^T Y_s|Y_s|ds - \int_t^T U_s d\widetilde{N}_s$$

Consequence

$\lim_{t\to T} Y_t = +\infty \neq g(N_T)$.

Other cases

- With a term $\int_t^T Z_s dW_s$: $\lim_{t\to T} Y_t = +\infty \neq g(N_T)$.
- If $f(y) = -y|y|^{q-1}$ and $1 < q < 2$: $\lim_{t\to T} Y_t = +\infty \neq g(N_T)$.
- If $f(y) = -y|y|^{q-1}$ and $q > 2$: $\lim_{t\to T} Y_t = g(N_T)$.
- With a left barrier $g(x) = (+\infty)1_{\{x\leq x_0\}} + \varphi(x)1_{\{x>x_0\}}$: $\lim_{t\to T} Y_t = g(N_T)$.

1.B. BSDE driven by a Poisson process

$$Y_t = g(N_T) - \int_t^T Y_s |Y_s|^{q-1} ds - \int_t^T U_s d\widetilde{N}_s$$

Liquidation problem: To minimize

$$J(t, A) = \mathbb{E}\left[\int_t^T |a_s|^p ds + g(N_T)|A_T|^p \middle| \mathcal{F}_t\right], \quad A_s = x + \int_t^s a_u du$$

Minimizer of the functional

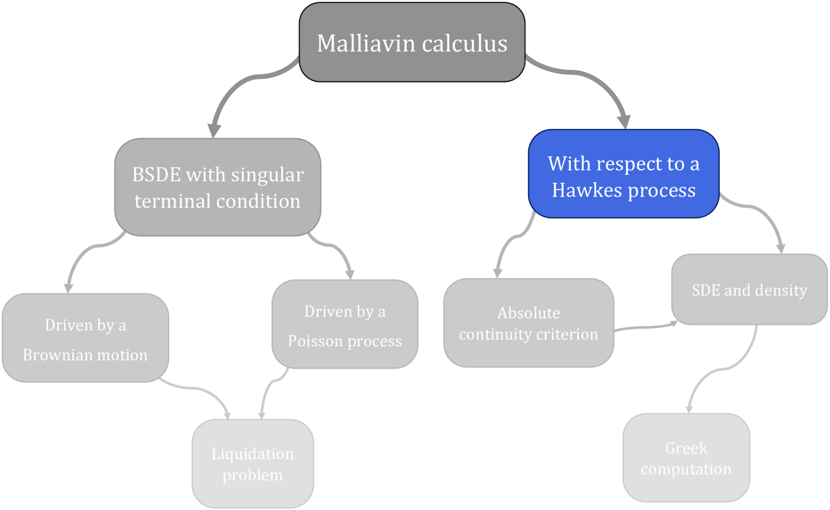
The value function is given by $v(t, x) = |x|^p Y_t$.

Consequence

There is an extra cost if and only if $p \geq 2$ i.e. $q \leq 2$.

Paper: D. Cacitti-Holland, L. Denis, and A. Popier, *Continuity problem for BSDE and IPDE with singular terminal condition*, in: Journal of Mathematical Analysis and Applications (2024).

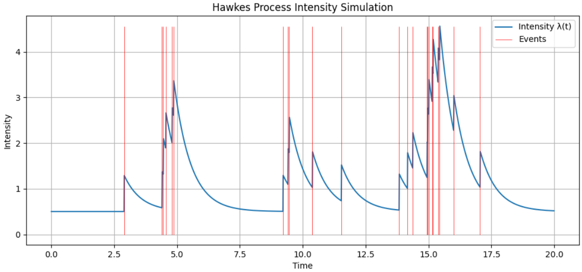
2. Malliavin calculus with respect to a Hawkes process



Hawkes process: Self-exciting point process N where past events increase the likelihood of future events.
Intensity of a Hawkes process: λ^* satisfies

$$\lambda^*(t) = \lambda + \int_{(0,t)} \mu(t-s)dN_s = \lambda + \sum_{i=1}^{N_{t-}} \mu(t - T_i).$$

Self-exciting function: $\mu \in L^1((0,+\infty),\mathbb{R}_+)$ s.t. $\int_0^{+\infty} \mu(t)dt < 1$.



References:

- P. Laub, Y. Lee, and T. Taimre, *The Elements of Hawkes Processes*, Jan. 2021,
- M. Costa et al., *Renewal in Hawkes processes with self-excitation and inhibition*, in: Advances in Applied Probability 52.3, 2020,
- E. A. Carlen and E. Pardoux, *Differential Calculus and Integration by Parts on Poisson Space*, in: Stochastics, Algebra and Analysis in Classical and Quantum Dynamics: Proceedings of the IVth French-German Encounter on Mathematics and Physics, 1990,
- N. Bouleau and L. Denis, *Dirichlet forms methods for Poisson point measures and Lévy processes*, vol. 76, Probability Theory and Stochastic Modelling, Springer, 2015.

2. Malliavin calculus with respect to a Hawkes process

Notations:

- Ω the set of càdlàg real functions on $[0, +\infty)$,
- $N_t(\omega)$ the number of jumps between 0 and $t \in [0, +\infty)$ of $\omega \in \Omega$,
- $\mathbb{P}$ the probability measure such that N is a Hawkes process with intensity

$$\lambda^*(t) = \lambda + \int_0^t \mu(t-s)dN_s, \; t \geq 0,$$

  with $\lambda \in \mathbb{R}_+^*, \mu$ class C^1 on $[0, +\infty)$ and $\|\mu\|_1 < 1$,
- $T \in (0, +\infty)$ a terminal instant,
- $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ the filtration generated by N,
- $(T_i)_{i \in \mathbb{N}^*}$ the jump instants of N.

2.A. State of the art

Existing Malliavin calculus for Hawkes processes:

- C. Hillairet, A. Réveillac, and M. Rosenbaum, *An expansion formula for Hawkes processes and application to cyber-insurance derivatives* , in: Stochastic Processes and their Applications 160, 2023,
- C. Hillairet et al., *The Malliavin-Stein method for Hawkes functionals*, in: ALEA Latin American Journal of Probability and Mathematical Statistics 19.2, 2022.

Based on representation of Hawkes process w.r.t. Poisson measure and Malliavin calculus on Poisson space. Following the Picard's approach (creation operator and chaos representation).

Idea of our construction: To perturb the jump times and differentiate w.r.t. this perturbation to get a directional derivative then a Malliavin derivative satisfying the chain rule.

2.B. Construction of a Malliavin-Hawkes calculus

Perturbation by a reparamaterization τ_ε :

$$\tau_\varepsilon(t) = t + \varepsilon \int_0^t m(s)ds = t + \varepsilon \widehat{m}(t)$$

preserving the number and the order of jump times with $\varepsilon \in \mathbb{R}_+^*$ and $m \in \mathcal{H} = \left\{ f \in L^2(0,T),\ \int_0^T f(s)ds = 0 \right\}$ Cameron-Martin space.

Directional derivative:

$$D_m F = \lim_{\varepsilon \to 0} \frac{F \circ \mathcal{T}_\varepsilon - F}{\varepsilon}, \qquad \mathcal{T}_\varepsilon(\omega) = \omega \circ \tau_\varepsilon$$

$\mathbb{D}_m^0$ the set of $F \in L^2(\Omega)$ s.t. this limit exists in $L^2(\Omega)$.

Derivative of the jump instants $\overline{T}_i := T_i \wedge T$ of the Hawkes process N

$$\overline{T}_i := T_i \wedge T \in \mathbb{D}_m^0 \text{ and } D_m \overline{T}_i = -\widehat{m}(\overline{T}_i) = -\int_0^{\overline{T}_i} m(s)ds.$$

Idea of the proof: $\mathcal{T}_\varepsilon \overline{T}_j(\omega) - \overline{T}_j(\omega) + \varepsilon \widehat{m}(\overline{T}_j)(\omega) = o(\varepsilon)$.

2.B. Construction of a Malliavin-Hawkes calculus

$$D_m F = \lim_{\varepsilon \to 0} \frac{F \circ \mathcal{T}_\varepsilon - F}{\varepsilon}$$

Smooth random variables: $\mathcal{S}$ the set of

$$F = a 1_{\{N_T = 0\}} + \sum_{n=1}^{d} f_n(T_1, \ldots, T_n) 1_{\{N_T = n\}}$$

where f_n smooth with bounded derivatives of any order.

Differentiability of smooth random variables

$\mathcal{S} \subset \mathbb{D}_m^0$ and, for any $F \in \mathcal{S}$,

$$D_m F = -\sum_{n=1}^{d} \sum_{j=1}^{n} \frac{\partial f_n}{\partial t_j}(T_1, \ldots, T_n) \widehat{m}(T_j) 1_{\{N_T = n\}}.$$

2.B. Construction of a Malliavin-Hawkes calculus

Derivative of a product

For any $F, G \in \mathcal{S}, FG \in \mathcal{S} \subset \mathbb{D}_m^0$ and $D_m(FG) = FD_mG + GD_mF$.

Chain rule

For any $\phi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ and $F_1, ..., F_n \in \mathcal{S}, \phi(F_1, ..., F_n) \in \mathcal{S} \subset \mathbb{D}_m^0$ and

$$D_m\phi(F_1, \cdots, F_n) = \sum_{j=1}^{n} \frac{\partial \phi}{\partial x_j}(F_1, \cdots, F_n) D_m F_j.$$

Idea of the proofs : To use the explicit expression of D on $\mathcal{S}$.

Integration by parts

For any $F \in \mathcal{S}$,

$$\mathbb{E}[D_m F] = \mathbb{E}\left[\left(\int_{(0,T]} (\psi(m,t) + \widehat{m}(t)\mu(T-t) + m(t))dN_t\right) F\right]$$

where $\widehat{m}$ is the previous antiderivative of m and

$$\psi(m,t) = \frac{1}{\lambda^*(t)} \int_{(0,t)} (\widehat{m}(t) - \widehat{m}(s))\mu'(t-s)dN_s.$$

Ideas of the proof: $\mathbb{P}\mathcal{T}_\varepsilon^{-1} \ll \mathbb{P}$ with explicit density G^ε which satisfies

$$\mathbb{E}[D_m F] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{\mathcal{T}_\varepsilon F - F}{\varepsilon}\right] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{G^\varepsilon - 1}{\varepsilon} F\right] = \mathbb{E}\left[\frac{\partial G^\varepsilon}{\partial \varepsilon}_{|\varepsilon=0} F\right].$$

2.B. Construction of a Malliavin-Hawkes calculus

Definition of the domain and the directional derivative

The quadratic bilinear form

$$\mathcal{E}_m(F,G) = \mathbb{E}[D_m F D_m G], \quad F, G \in \mathcal{S},$$

is closable on $L^2(\Omega)$.

We denote $(\mathbb{D}_m^{1,2}, \mathcal{E}_m)$ its closed extension and $(\mathbb{D}_m^{1,2}, D_m)$ the extension of $(\mathcal{S}, D_m)$.

The previous formulas remain valid for any $F \in \mathbb{D}_m^{1,2}$:
Derivative of a product, chain rule, integration by parts.

2.B. Construction of a Malliavin-Hawkes calculus

Domain: with $(m_i)_{i\in\mathbb{N}}$ a Hilbert basis of $\mathcal{H}$,

$$\mathbb{D}^{1,2} = \left\{ F \in \bigcap_{i\in\mathbb{N}} \mathbb{D}^{1,2}_{m_i}, \quad \mathcal{E}(F) := \sum_{i=0}^{+\infty} \| D_{m_i} F \|^2_{L^2(\Omega)} < +\infty \right\}.$$

Malliavin derivative: for any $F \in \mathbb{D}^{1,2}$,

$$DF = \sum_{i=0}^{+\infty} D_{m_i} F m_i \in L^2(\Omega; \mathcal{H}).$$

In particular $\langle DF, m\rangle = D_m F$ for any $m \in \mathcal{H}$.

Explicit expression

For any $F \in \mathcal{S}$,

$$DF = \sum_{n=1}^{d} \sum_{j=1}^{n} \frac{\partial f_n}{\partial t_j}(T_1, \cdots, T_n) \left(\frac{T_j}{T} - 1_{[0,T_j]} \right) 1_{\{N_T = n\}}.$$

Domain of the divergence operator: $\mathrm{Dom}(\delta)$ is the set of $u \in L^2(\Omega; \mathcal{H})$ such that there exists $c \in \mathbb{R}_+^*$ such that

$$\forall F \in \mathbb{D}^{1,2}, \ \left| \mathbb{E}\left[\int_0^T D_t F u_t dt \right] \right| \leq c \|F\|_{\mathbb{D}^{1,2}}$$

with

$$\|F\|^2_{\mathbb{D}^{1,2}} = \|F\|^2_{L^2(\Omega)} + \mathcal{E}(F) = \|F\|^2_{L^2(\Omega)} + \|DF\|^2_{L^2(\Omega;\mathcal{H})}.$$

Divergence operator: for any $u \in \mathrm{Dom}(\delta)$, $\delta(u)$ is the unique element in $L^2(\Omega)$ such that

$$\forall F \in \mathbb{D}^{1,2}, \quad \mathbb{E}[\delta(u)F] = \mathbb{E}[\langle u, DF \rangle_{\mathcal{H}}].$$

Explicit expression

For any u predictable process in $L^2(\Omega;\mathcal{H})$,

$$\delta(u) = \int_{(0,T]} (\psi(u,t) + \widehat{u}(t)\mu(T-t) + u(t))dN_t$$

where $\widehat{u}(t) = \int_0^t u(s)ds$ and

$$\psi(u,t) = \frac{1}{\lambda^*(t)} \int_{(0,t)} (\widehat{u}(t) - \widehat{u}(s))\mu'(t-s)dN_s.$$

We do not have the Clark-Ocone formula because $N_T \in \mathbb{D}^{1,2}$ with $DN_T = 0$ but $N_T \neq \mathbb{E}[N_T]$.

2.C. Absolute continuity criterion

Theorem

For any $F = (F_1, \cdots, F_d) \in (\mathbb{D}^{1,2})^d$ and

$$\Gamma[F] = (\Gamma[F_i, F_j])_{1 \leq i,j \leq d} = (\langle DF_i, DF_j \rangle_{\mathcal{H}})_{1 \leq i,j \leq d},$$

the image measure $F_*[\det(\Gamma[F]).\mathbb{P}]$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$:

$$F_*[\det(\Gamma[F]).\mathbb{P}] \ll \lambda_d.$$

Corollary

For any $F = (F_1, \cdots, F_d) \in (\mathbb{D}^{1,2})^d$, conditionally to $\Gamma[F] \in GL_d(\mathbb{R})$, the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$:

$$\mathbb{P}_F(\cdot \mid \Gamma[F] \in GL_d(\mathbb{R})) \ll \lambda_d.$$

SDE driven by a Hawkes process:

$$X_t = x + \int_0^t f(t, X_t)dt + \int_0^t g(t, X_{t-})dN_t$$

Assumptions:

- For any $t \in [0, T]$, the maps $f(t, \cdot), g(t, \cdot)$ are of class C^1.
- $\sup_{t,x}(|\nabla_x f(t,x)| + |\nabla_x g(t,x)|) < +\infty$.
- For any $x \in \mathbb{R}^d$, the map $g(\cdot, x)$ is differentiable.

Proposition

$X_T \in \mathbb{D}^{1,2}$ and we have an explicit expression of DX_T and $\Gamma[X_T]$.

2.D. SDE driven by a Hawkes process

$$X_t = x + \int_0^t f(t, X_t)dt + \int_0^t g(t, X_{t-})dN_t$$

Auxiliary function:

$$\varphi(t,x) = f(t, x + g(t,x)) - (I_d + \nabla_x g(t,x))f(t,x) - \frac{\partial g}{\partial t}(t,x).$$

Theorem

If $d = 1$ and $\varphi(t,x) \neq 0$ for any $(t,x) \in [0,T] \times \mathbb{R}$ then, conditionally to $\{N_T \geq 1\}$, the law X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$:

$$\mathbb{P}_{X_T}(\cdot \mid N_T \geq 1) \ll \lambda_1.$$

We also have a theorem for $d \geq 1$ with a spanning condition and conditionally to the fact that the process N admits enough jumps.

2.E. Greek computation

Dynamics of an asset price:

$$dS_t = rS_t dt + \sigma S_{t-} d\widetilde{N}_t = (r - \sigma\lambda^*(t))dt + \sigma S_{t-} dN_t, \; S_0 = x_0$$

Goal: To compute the derivatives of $\mathbb{E}[\phi(S_T)]$ w.r.t. parameters x_0, r, σ to get the sensitivity of our problem.

Expression of Delta

If $\widehat{m}(t) \neq 0$ for any $t \in (0, T)$ then

$$\frac{\partial}{\partial x_0}\mathbb{E}[1_{\{N_T>0\}}\phi(S_T)] = \mathbb{E}\left[\phi(S_T)\delta\left(m1_{\{N_T>0\}}\frac{\frac{\partial S_T}{\partial x_0}}{D_m S_T}\right)\right]$$

with explicit expressions of δ and $D_m S_T$.

Paper : D. Cacitti-Holland, L. Denis and A. Popier, *Malliavin calculus with respect to a Hawkes process*, forthcoming paper.

Perspectives

$\star$ **Simulations of Greeks in a financial model with a Hawkes process**: Choice of the function m.

$\star$ **Malliavin calculus with a non linear Hawkes process with a non constant baseline**:

$$\lambda^*(t) = \lambda_t + \gamma\left(\int_{(0,t)} \mu(t-s)dN_s\right).$$

$\star$ **Continuity problem for BSDEs with jumps and singular terminal condition**: Driven by a Poisson measure, with an infinity of jumps (positive or negative). To use a Malliavin calculus w.r.t. jump height.

$\star$ **Multidimensional BSDE with singular terminal condition** : To define the problem. Procope Project in progress.

Thank you for your attention.