Limit behavior of the solution of a backward stochastic differential equation with singular terminal condition

Dorian Cacitti-Holland

30th birthday of LMM

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Introduction of the equations

The Markovian BSDE driven by a Brownian motion W

$$\begin{cases} Y_t &= g(X_T) + \int_t^T F(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \\ X_t &= x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \end{cases}$$

where $T \in \mathbb{R}_+^*, g : \mathbb{R}^m \longrightarrow \overline{\mathbb{R}}_+, F : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ and $x_0 \in \mathbb{R}, b : [0, T] \times \mathbb{R}^m \longrightarrow \mathbb{R}^m, \sigma : [0, T] \times \mathbb{R}^m \longrightarrow \mathbb{R}^{m \times d}$ are the parameters, and $(X, Y, Z) : \Omega \times [0, T] \longrightarrow \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d$ the unknown.

Introduction of the equations

The Markovian BSDE driven by a Brownian motion W and a random Poisson measure π

$$\begin{cases} Y_t &= g(X_T) + \int_t^T F(s, X_s, Y_s, Z_s, U_s(\cdot))ds - \int_t^T Z_s dW_s \\ & \quad - \int_t^T \int_E U_s(e)\tilde{\pi}(de, ds), \\ X_t &= x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\ & \quad + \int_0^t \int_E \beta(s, X_{s-}, e)\tilde{\pi}(de, ds). \end{cases}$$

Singular terminal condition and liquidation problem

We assume that the terminal condition $\xi = g(X_T)$ can be equal to infinity :

$$\mathbb{P}(\xi = +\infty) > 0.$$

This is useful to apply the following results to the liquidation problem where we minimize the functional

$$J(t,a) = \mathbb{E}\left(\int_t^T (\eta_s |a_s|^p + \gamma_s |A_s|^p) ds + \xi |A_T|^p \middle| \mathcal{F}_t \right)$$

where

$$A_s = x + \int_t^s a_u du, \quad a \in L^1(t, +\infty) \ a.s.$$

Singular terminal condition and liquidation problem

Indeed

$$\min J = J(t, a^*)$$

with

$$A_s^* = x_0 \exp\left(-\int_t^s \left(\frac{Y_u}{\eta_u}\right)^{q-1} du \right)$$

and (Y, Z) the minimal supersolution to the BSDE with the driver

$$F(t, x, y, z) = -(p-1)\frac{|y|^{q-1}y}{\eta_t^{q-1}} + \gamma_t.$$

But do we have

$$\liminf_{t \to T} Y_t = \xi \quad ?$$

to avoid an extra cost due to the liquidation constraint.

Theorem for the BSDE driven by a Brownian motion

Many assumptions about the parameters b, σ, g and F to obtain the continuity of the process Y.

Assumption 1

1. b bounded continuous, C^1 with respect to x, $\dfrac{\partial b}{\partial x_i}$ with polynomial growth.
2. σ bounded, continuous, C^2 with respect to x, $\dfrac{\partial \sigma}{\partial x_i}, \dfrac{\partial^2 \sigma\sigma^*}{\partial x_i \partial x_j}$ bounded, $\sigma\sigma$ λ-uniformly elliptic.

To obtain a Malliavin differentiable solution X and a control on its density.

Assumption 2

1. F continuous and χ-monotone with respect to y.
2. F C^1 and uniformly Lispchitz with respect to z.
3. $\exists \ell > 1, \forall \rho \in \mathbb{N}, \exists K_\rho \in \mathbb{R}_+^*, \forall t \in [0, T], x \in \mathbb{R}^m, z \in \mathbb{R}^d$,

$$\sup_{|y| \leq \rho} |F(t, x, y, z)| \leq K_\rho (1 + |x|^\ell + |z|).$$

To obtain solutions (Y^n, Z^n) to the truncated equations

$$Y_t^n = g(X_T) \wedge n + \int_t^T F_n(s, X_s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dW_s, \quad 0 \leq t \leq T,$$

with

$$F_n(s, x, y, z) = F(s, x, y, z) - F(s, x, 0, 0) + F(s, x, 0, 0) \wedge n.$$

Theorem for the BSDE driven by a Brownian motion

Assumption 3

1. g measurable C^1 Lipschitz on each $\{x \in \mathbb{R}^m, g(x) \leq n\}$.
2. $\mathcal{S} = \{x \in \mathbb{R}^m, g(x) = +\infty\}$ closed.
3. $g(X_T)1_{\mathcal{K}}(X_T) \in L^2(\Omega, \mathcal{F}_T)$ for all compact $\mathcal{K}$ of $\mathbb{R}^m \backslash \mathcal{S}$.

Assumption 4

1. $F(t, x, 0, 0) \geq 0$ for all $t \in [0, T], x \in \mathbb{R}^m$.
2. There exist $q > 1$ and $\eta : [0, T] \times \mathbb{R}^m \longrightarrow \mathbb{R}_+^*$ whose inverse is polynomial growth such that

$$\forall t \in [0, T], x \in \mathbb{R}^m, y \in \mathbb{R}_+, z \in \mathbb{R}^d,$$

$$F(t, x, y, z) - F(t, x, 0, z) \leq -\eta(t, x)|y|^q.$$

To obtain a minimal supersolution (Y, Z) to the equation with $Y = \lim_{n\to+\infty} Y^n$ and a suitable a priori estimate on Y^n.

Theorem for the BSDE driven by a Brownian motion

Assumption 5

1. F C^1 with respect to y, $\dfrac{\partial F}{\partial y}$ locally bounded.
2. F C^1 with respect to x and $\dfrac{\partial F}{\partial x_i}$ locally polynomial growth with respect to y.

To obtain Y^n Malliavin differentiable and to use

$$D_t Y_t^n = Z_t^n$$

and the Malliavin by parts integration to control the term in Z_t^n by using the control of the term in Y_t^n.

Theorem for the BSDE driven by a Brownian motion

First theorem

If we assume the previous assumptions and if $q \leq 3$, there exist $\alpha \in \left]0, \frac{2(q-1)}{q+1}\right[$ and $C \in \mathbb{R}_+^*$ such that

$$\forall s \in [0, T], x \in \mathbb{R}^m, z \in \mathbb{R}^d,$$

$$|F(s,x,0,z) - F(s,x,0,0)| \leq C(1 + |x|^\ell)|z|^\alpha$$

then $\mathbb{P}$-a.s.

$$\liminf_{t \to T} Y_t = \xi.$$

With the previous assumptions and different calculus

$$\mathbb{E}(\varphi(X_T)Y_T) = \lim_{t \to T} \lim_{n \to +\infty} \mathbb{E}(\varphi(X_t)Y_t^n)$$

for every function φ C^2 whose compact support is included in $\{g < +\infty\}$.

Counter-example for the BSDE with jumps

Can we have a similar theorem for the second equation ?

$$\begin{cases} Y_t &= g(X_T) + \int_t^T F(s, X_s, Y_s, Z_s, U_s(\cdot))ds - \int_t^T Z_s dW_s \\ & \quad - \int_t^T \int_E U_s(e)\tilde{\pi}(de, ds), \quad 0 \leq t \leq T \\ X_t &= x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\ & \quad + \int_0^t \int_E \beta(s, X_{s-}, e)\tilde{\pi}(de, ds), \quad 0 \leq t \leq T. \end{cases}$$

Counter-example for the second equation

Second theorem

With a Poisson process N, a simple process $X = N$, a quadratic driver

$$Y_t = g(X_T) - \int_t^T Y_s|Y_s|ds - \int_t^T U_s d\tilde{N}_s, \quad 0 \leq t \leq T$$

and a function g given by

$$g(x) = (+\infty)1_{\{x \geq x_0\}} + \varphi(x)1_{\{x < x_0\}}.$$

We have the solution

$$Y_t = \frac{1}{T-t}1_{\{t<T\}} + g(X_T)1_{\{t=T\}}, \quad 0 \leq t \leq T.$$

In particular $\lim_{t \to T} Y_t = +\infty > g(X_T)$.

Counter-example for the second equation

The truncated BSDEs

$$Y_t^n = g(X_T) \wedge n - \int_t^T Y_s^n |Y_s^n| ds - \int_t^T U_s^n d\tilde{N}_s, \quad 0 \leq t \leq T.$$

And the associated IPDEs

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)| & = & -\lambda u(t,x+1) \\ u(T,x) & = & g(x), \end{cases}$$

$$\begin{cases} \frac{\partial u^n}{\partial t}(t,x) - \lambda u^n(t,x) - u^n(t,x)|u^n(t,x)| & = & -\lambda u^n(t,x+1) \\ u^n(T,x) & = & g(x) \wedge n. \end{cases}$$

Counter-example for the BSDE with jumps

We show step by step that, for $x \geq x_0$ and $t \in [0, T[$,

$$u(t,x) = \frac{1}{T-t}, \quad u^n(t,x) = \frac{1}{T-t+\frac{1}{n}}.$$

Then, for $x \in [x_0 - 1, x_0[$,

$$\left\{ \begin{array}{rcl} \frac{\partial u^n}{\partial t}(t,x) - \lambda u^n(t,x) - u^n(t,x)|u^n(t,x)| & = & -\lambda \frac{1}{T-t+\frac{1}{n}} \\ u^n(T,x) & = & n. \end{array} \right. .$$

Thus, noting ψ_n an explicit auxiliary function,

$$u^n(t,x) = \frac{1}{T-t+\dfrac{1}{n}} - \psi_n(t,x) \underset{n \to +\infty}{\longrightarrow} \frac{1}{T-t} = u(t,x).$$

And finally we can reiterate the previous reasonning to obtain the theorem.

Associated Euler scheme

To understand the behavior of the solution $u^n(\cdot, x)$ for $x \in [x_0 - 1, x_0[$, we studied the convergence of the numerical scheme for the ODE

$$\begin{cases} u'(t) - \lambda u(t) - u(t)|u(t)| &= -\lambda \dfrac{1}{T-t} \\ u(T) &= \chi \in \mathbb{R}_+^*. \end{cases}$$

$$0 = t_0 < t_1 < ... < t_N = T, \quad h_N = \frac{T}{N}.$$

$$u_N(t_N) = \chi$$

and

$$u_N(t_{k+1}) = u_N(t_k) - h_N f(t_k, u_N(t_k))$$

with

$$f(t,u) = \lambda + u^2 - \lambda \frac{1}{T-t}.$$

Associated Euler scheme

Convergence

For all $\alpha \in \ ]0,1[$,

$$\max_{0 \leq k \leq \lfloor \alpha N \rfloor} \left| u_N(t_k) - \frac{1}{T - t_k} \right| \underset{N \to +\infty}{\longrightarrow} 0.$$

The implicit backward Euler scheme can be written explicit then we show that $\lim_{N \to +\infty} u_N(t_0) = \frac{1}{T}$ and conclude using the convergence results about the forward Euler scheme.

Associated Euler scheme

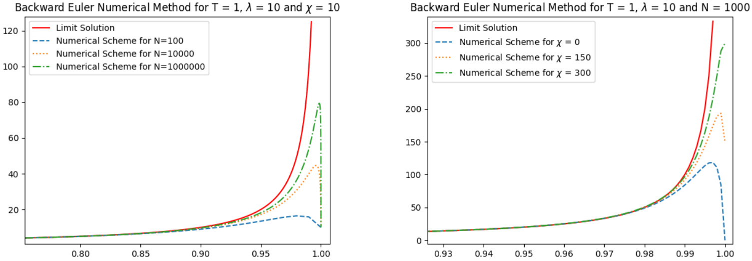

Figure: Backward Euler Numerical Method for $T = 1$ and $\lambda = 10$. On the left, $\chi = 10$; on the right $N = 1000$.